On homogenization of a quasilinear elliptic equation connected with heat conduction

Josef Malík

Division of Applied Mathematics

Institute of Geonics

Ostrava

malik@ugn.cas.cz

Electrical transformer

Introduction - model

Differential equation

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) = f(x) \qquad \text{in}\,\Omega$$

Differential equation

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) = f(x) \qquad \text{in } \Omega$$

Boundary conditions

$$u(x) = 0 \qquad \text{on } \partial\Omega$$

Introduction - model

Differential equation

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) = f(x) \qquad \text{in } \Omega$$

Boundary conditions

$$u(x) = 0 \qquad \text{on } \partial\Omega$$

Iterative method

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(x,u_n)\frac{\partial u_{n+1}}{\partial x_j}\right) = f(x)$$

Introduction - references

I. HLAVÁČEK, M. KŘÍŽEK, AND J. MALÝ, *On Galerkin approximations of a quasilinear elliptic problem of a nonmonotone type,* J. Math. Anal. Appl., 184 (1994), pp. 168-189.

Introduction - references

I. Hlaváček, M. Křížek, and J. Malý, *On Galerkin approximations of a quasilinear elliptic problem of a nonmonotone type,* J. Math. Anal. Appl., 184 (1994), pp. 168-189.

I. Hlaváček, *Reliable solution of quasilinear nonpotential elliptic problem of a nonmonotone type with respect to the uncertainty in coefficients,* J. Math. Anal. Appl., 212 (1997), pp. 452-466.

Introduction - references

I. Hlaváček, M. Křížek, and J. Malý, *On Galerkin approximations of a quasilinear elliptic problem of a nonmonotone type,* J. Math. Anal. Appl., 184 (1994), pp. 168-189.

I. Hlaváček, *Reliable solution of quasilinear nonpotential elliptic problem of a nonmonotone type with respect to the uncertainty in coefficients,* J. Math. Anal. Appl., 212 (1997), pp. 452-466.

L. Liu, M. Křížek, and P. Neittaanmäki, *Higher order finite element approximation of a quasilinear elliptic boundary value problem of a non-monotone type,* Applications of Mathematics, 41 (1996) pp. 467–478.

Weak formulation

Coefficients

$$a_{ij}(x,u) = a_{ji}(x,u), \qquad i,j = 1,\dots n$$

$$\nu\eta_i\eta_i \;\leq\; a_{ij}(x,u)\eta_i\eta_j$$

$$\mid a_{ij}(x,u_1) - a_{ij}(x,u_2) \mid \leq C_L \mid u_1 - u_2 \mid, \qquad i,j = 1,\dots n$$

Weak formulation

Coeffi cients

$$a_{ij}(x,u) = a_{ji}(x,u), \qquad i,j = 1,\dots n$$

$$\nu \eta_i \eta_i \ \le \ a_{ij}(x,u)\eta_i\eta_j$$

$$\mid a_{ij}(x,u_1) - a_{ij}(x,u_2) \mid \le C_L \mid u_1 - u_2 \mid, \qquad i,j = 1,\dots n$$

Variational equality

$$\int_\Omega a_{ij}(x,u)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx = \int_\Omega f u dx,$$

Weak formulation

Coeffi cients

$$a_{ij}(x,u) = a_{ji}(x,u), \qquad i,j = 1,\dots n$$

$$\nu\eta_i\eta_i \ \leq \ a_{ij}(x,u)\eta_i\eta_j$$

$$\mid a_{ij}(x,u_1) - a_{ij}(x,u_2) \mid \leq C_L \mid u_1 - u_2 \mid, \qquad i,j = 1,\dots n$$

Variational equality

$$\int_\Omega a_{ij}(x,u)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx = \int_\Omega fudx,$$

Functional spaces

$$f \in L^2(\Omega), \qquad u,v \in H_0^1(\Omega)$$

1D homogenization - setting up

$a(x,u)$ is measurable in x for all u, periodic in x with the period l

$$0 < \gamma \leq a(x,u) \leq C_M$$

$$|a(x,u_1) - a(x,u_2)| \leq C_L \, |u_1 - u_2|$$

$$a^{\epsilon}(x,u) = a\left(\frac{x}{\epsilon}, u\right)$$

1D homogenization - setting up

$a(x,u)$ is measurable in x for all u, periodic in x with the period l

$$0 < \gamma \leq a(x,u) \leq C_M$$
$$|a(x,u_1) - a(x,u_2)| \leq C_L\, |u_1 - u_2|$$
$$a^\epsilon(x,u) = a\left(\frac{x}{\epsilon}, u\right)$$

Variational equations

$$\int_c^d a^\epsilon(x,u)\frac{du^\epsilon}{dx}\frac{dv}{dx}dx = \int_c^d a^\epsilon f v dx,$$

where $u^\epsilon, v \in H_0^1(c,d)$.

1D homogenization - setting up

$a(x,u)$ is measurable in x for all u, periodic in x with the period l

$$0 < \gamma \le a(x,u) \le C_M$$
$$|a(x,u_1) - a(x,u_2)| \le C_L\,|u_1 - u_2|$$
$$a^\epsilon(x,u) = a\left(\frac{x}{\epsilon}, u\right)$$

Variational equations

$$\int_c^d a^\epsilon(x,u)\frac{du^\epsilon}{dx}\frac{dv}{dx}dx = \int_c^d a^\epsilon f v\,dx,$$

where u^ϵ, $v \in H_0^1(c,d)$.

Mean value

$$\langle w\rangle(u) = \frac{1}{l}\int_0^l w(x,u)dx\,.$$

1D homogenization - auxiliary lemma

Lemma 1 *Let the function* $a(x, u)$ *defined on* $R \times R$ *be bounded, measurable in* x *for all* u*, periodic in* x*, and satisfy the assumption above. Let* $u^\epsilon(x)$ *be a sequence of continuous functions defined on the founded interval* $\langle c, d\rangle$ *such that the limit*

$$u^\epsilon \longrightarrow u^0 \quad \text{in } C(\langle c, d\rangle) \qquad \text{as } \epsilon \longrightarrow 0$$

holds. Then the following limit

$$a^\epsilon(x, u^\epsilon(x)) \rightharpoonup \langle a\rangle(u^0(x)) \quad \text{in } L^2(c, d) \qquad \text{as } \epsilon \longrightarrow 0$$

folds true.

1D homogenization - main theorem

Theorem 1 *Let the function $a(x,u)$ defined on $R \times R$ be bounded, measurable in x for all u, periodic in x, and satisfy the assumption above. Let $u^\epsilon \in H_0^1(c,d)$ be the sequence of solutions . Then the limits*

$$u^\epsilon(x) \rightharpoonup u^0(x) \quad \text{in } H_0^1(c,d) \qquad \text{as } \epsilon \to 0\,,$$

$$\xi^\epsilon(x) \to \xi^0(x) \quad \text{in } C\left(\langle c,d\rangle\right) \qquad \text{as } \epsilon \to 0$$

hold, where u^0 is a solution to the variational equation

$$\int_c^d a^0(x)\frac{du_0}{dx}(x)v(x)dx = \int_c^d f v dx$$

which holds for all $v \in H_0^1(c,d)$.

$$a^0(u) = \left(\left\langle \frac{1}{a}\right\rangle(u)\right)^{-1},$$

$$\xi^\epsilon(x) = a^\epsilon\left(x, u^\epsilon(x)\right)\frac{du^\epsilon(x)}{dx}\,, \qquad \xi^0(x) = a^0\left(u^0(x)\right)\frac{du^0(x)}{dx}\,.$$

Some auxiliary results

$a_{ij}(x)$ are defined on a bounded domain with a Lipschitz boundary.

$$a_{ij}(x) = a_{ji}(x),$$

$$\nu \xi_i \xi_i \leq a_{ij}(x) \xi_i \xi_j$$

$$| \; a_{ij}(x) \; | \leq C_M \; , \quad i, j = 1, \ldots n$$

Some auxiliary results

$a_{ij}(x)$ are defined on a bounded domain with a Lipschitz boundary.

$$a_{ij}(x) = a_{ji}(x),$$
$$\nu\xi_i\xi_i \leq a_{ij}(x)\xi_i\xi_j$$
$$| \, a_{ij}(x) \, | \leq C_M \, , \quad i,j = 1, \ldots n$$

Linear variational equation

$$\int\limits_{\Omega} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx = \int\limits_{\Omega} fv dx \, ,$$

where $f \in L^2(\Omega)$ and $u, v \in H_0^1(\Omega)$.

Theorem 2 *Let $a_{ij}(x)$, $i, j, = 1, \dots n$ satisfy the assumptions above, Ω is a bounded domain with a Lipschitz boundary and $f \in L^p(\Omega)$, where $p > \frac{n}{2}$. Then the solution to the variational equation is continuous on $\bar{\Omega}$ and the inequality*

$$| u(x) - u(y) | \leq C_N \|f\|_{L^p(\Omega)} | x - y |^\lambda$$

holds for any $x, y \in \Omega$, where C_N is a positive constant which depends only on ν, C_M, Ω, n and λ satisfy the inequalities $0 < \lambda < 1$ and depends only ν, C_M, Ω, n as well.

Linear homogenization 1

Mean value

$$\langle g \rangle (u) = \frac{1}{\mid Y \mid} \int\limits_Y g(x,u)dx \,,$$

where $Y = \langle 0, l_1 \rangle \times \ldots \times \langle 0, l_n \rangle$ and $\mid Y \mid = l_1 \ldots l_n$.

Linear homogenization 1

Mean value

$$\langle g\rangle(u) = \frac{1}{\mid Y \mid} \int\limits_Y g(x,u)dx\,,$$

where $Y = \langle 0, l_1\rangle \times \ldots \times \langle 0, l_n\rangle$ and $\mid Y \mid = l_1 \ldots l_n$.

Auxiliary variational equations

$$\int\limits_Y a_{ij}(x)\left(\frac{\partial \chi_k}{\partial x_j} + \delta_{jk}\right)\frac{\partial \psi}{\partial x_i}dx = 0\,, \qquad k = 1,\ldots n\,,$$

where the functions ψ , $\chi_k \in H^1_{per}(Y)$.

Linear homogenization 1

Mean value

$$\langle g\rangle(u) = \frac{1}{\mid Y \mid} \int\limits_Y g(x,u)dx\,,$$

where $Y = \langle 0, l_1\rangle \times \ldots \times \langle 0, l_n\rangle$ and $\mid Y \mid = l_1 \ldots l_n$.

Auxiliary variational equations

$$\int\limits_Y a_{ij}(x)\left(\frac{\partial \chi_k}{\partial x_j} + \delta_{jk}\right)\frac{\partial \psi}{\partial x_i}dx = 0\,, \qquad k = 1, \ldots n\,,$$

where the functions ψ , $\chi_k \in H^1_{per}(Y)$.

Homogenized coeffi cients

$$a^0_{ij} = \langle a_{ij}\rangle + \left\langle a_{ik}\frac{\partial \chi_j}{\partial x_k}\right\rangle\,, \qquad i,j = 1, \ldots n$$

Linear homogenization 2

Let $a_{ij}^{z}(x)$, $z = 1, \dots m$, $i, j = 1, \dots n$ are periodic functions in x with the period $l = (l_1, \dots l_n)$, Ω be a domain which is divided into subdomains Ω_z, $z = 1, \dots m$.

$$\tilde{a}_{ij}^{\epsilon}(x) = a_{ij}^{z}\left(\frac{x}{\epsilon}\right) \qquad \text{as} \quad x \in \Omega_z\,,$$

$$\tilde{a}_{ij}^{0}(x) = a_{ij}^{z,0} \qquad \text{as} \quad x \in \Omega_z\,, \qquad i, j = 1, \dots n$$

Let $a^z_{ij}(x)$, $z = 1, \dots m$, $i, j = 1, \dots n$ are periodic functions in x with the period $l = (l_1, \dots l_n)$, Ω be a domain which is divided into subdomains Ω_z, $z = 1, \dots m$.

$$\tilde{a}^\epsilon_{ij}(x) = a^z_{ij}\left(\frac{x}{\epsilon}\right) \qquad \text{as} \quad x \in \Omega_z\,,$$

$$\tilde{a}^0_{ij}(x) = a^{z,0}_{ij} \qquad \text{as} \quad x \in \Omega_z\,, \qquad i, j = 1, \dots n$$

Variational equations

$$\int_\Omega \tilde{a}^\epsilon_{ij}(x) \frac{\partial u^\epsilon}{\partial x_i} \frac{\partial v}{\partial x_i} dx = \int_\Omega f v dx\,,$$

$$\int_\Omega \tilde{a}^0_{ij} \frac{\partial u^0}{\partial x_j} \frac{\partial v}{\partial x_i} dx = \int_\Omega f v dx\,,$$

where u^ϵ, u^0 belong to $H^1_0(\Omega)$ and f belongs to $L^2(\Omega)$.

Linear homogenization 3

Theorem 3 *Let u^ϵ, u^0 be solutions to the variational equations above, then the limits*

$$u^\epsilon \rightharpoonup u^0 \quad \text{in } H_0^1(\Omega) \qquad \text{as} \quad \epsilon \to 0\,,$$

$$\xi_i^\epsilon \rightharpoonup \xi_i^0 \quad \text{in } L^2(\Omega) \qquad \text{as} \quad \epsilon \to 0\,, \qquad i = 1, \dots n$$

hold, where

$$\xi_i^\epsilon = \tilde{a}_{ij}^\epsilon \frac{\partial u^\epsilon}{\partial x_j}, \quad \xi_i^0 = \tilde{a}_{ij}^0 \frac{\partial u^0}{\partial x_j}, \qquad i = 1, \dots n.$$

Homogenization - setting up

Study the following variational equations

$$\int_Y a_{ij}(x,u)\left(\frac{\partial \chi_k(x,u)}{\partial x_j} + \delta_{jk}\right)\frac{\partial \psi}{\partial x_i}dx = 0, \qquad k = 1,\dots n$$

which play the same role as the similar equation in linear homogenization.

Homogenization - setting up

Study the following variational equations

$$\int_Y a_{ij}(x,u)\left(\frac{\partial \chi_k(x,u)}{\partial x_j}+\delta_{jk}\right)\frac{\partial \psi}{\partial x_i}dx=0, \qquad k=1,\ldots n$$

which play the same role as the similar equation in linear homogenization.

To guarantee the uniqueness, we study the equation on the space

$$W=\{u\in H^1_{per}(Y)\mid \int_Y u dx=0\}$$

which is equipped with the classical norm on $H^1(Y)$.

Homogenization - setting up

Study the following variational equations

$$\int_Y a_{ij}(x,u)\left(\frac{\partial \chi_k(x,u)}{\partial x_j}+\delta_{jk}\right)\frac{\partial \psi}{\partial x_i}dx=0, \qquad k=1,\dots n$$

which play the same role as the similar equation in linear homogenization.

To guarantee the uniqueness, we study the equation on the space

$$W=\{u\in H^1_{per}(Y)\mid \int_Y udx=0\}$$

which is equipped with the classical norm on $H^1(Y)$.

Homogenized coeffi cients

$$a^0_{ij}(u)=\langle a_{ij}\rangle(u)+\left\langle a_{ik}\frac{\partial \chi_j}{\partial x_k}\right\rangle(u)\,, \qquad i,j=1,\dots n$$

Homogenization - main theorem

Theorem 4 *Let $a_{ij}(x,u)$, $i,j=1\dots n$ defined on $R^n \times R$ be bounded, measurable in x for all u, periodic in x, and satisfy the assumptions above. Let Ω be a bounded domain in R^n with a Lipschitz boundary and $f \in L^p(\Omega)$, where $p > \frac{n}{2}$ and $p \geq 2$. Let $u^\epsilon \in H_0^1(\Omega)$ be solutions to the quasilinear variational equation*

$$\int_\Omega a_{ij}^\epsilon(x,u^\epsilon)\frac{\partial u^\epsilon}{\partial x_j}\frac{\partial v}{\partial x_i}dx = \int_\Omega fv dx$$

and $u^0 \in H_0^1(\Omega)$ be a solution of the quasilinear variational equations

$$\int_\Omega a_{ij}^0(u^0)\frac{\partial u^0}{\partial x_j}\frac{\partial v}{\partial x_i}dx = \int_\Omega fv dx\,,$$

where variational equations are fulfilled for all $v \in H_0^1(\Omega)$. Then the limits

$$u^\epsilon \rightharpoonup u^0 \quad \text{in } H_0^1(\Omega) \qquad \text{as } \epsilon \to 0\,,$$

$$\xi_i^\epsilon \rightharpoonup \xi_i^0 \quad \text{in } L^2(\Omega) \qquad \text{as } \epsilon \to 0\,, \qquad i=1,\dots n$$

hold, where

Homogenization - laminated material

General case – $a_{ij}(x_1, u)$, $i, j = 1, \ldots n$ x_1 with the period l_1.

$$a^0_{11}(u) = \left\langle \frac{1}{a_{11}} \right\rangle (u)\,,$$

$$a^0_{1j}(u) = \left\langle \frac{a_{1j}}{a_{11}} \right\rangle (u)\,, \quad 2 \leq j \leq u\,,$$

$$a^0_{ij}(u) = \left\langle \frac{a_{1i}}{a_{11}} \right\rangle (u) \left\langle \frac{a_{1j}}{a_{11}} \right\rangle (u) + \left\langle a_{ij} - \frac{a_{1i}a_{1j}}{a_{11}} \right\rangle (u)\,, \quad 2 \leq i, j \leq n$$

Homogenization - laminated material

General case – $a_{ij}(x_1,u)$, $i,j=1,\dots n$ x_1 with the period l_1.

$$a^0_{11}(u) = \left\langle \frac{1}{a_{11}} \right\rangle (u)\,,$$

$$a^0_{1j}(u) = \left\langle \frac{a_{1j}}{a_{11}} \right\rangle (u)\,, \quad 2 \le j \le u\,,$$

$$a^0_{ij}(u) = \left\langle \frac{a_{1i}}{a_{11}} \right\rangle (u) \left\langle \frac{a_{1j}}{a_{11}} \right\rangle (u) + \left\langle a_{ij} - \frac{a_{1i}a_{1j}}{a_{11}} \right\rangle (u)\,, \quad 2 \le i,j \le n$$

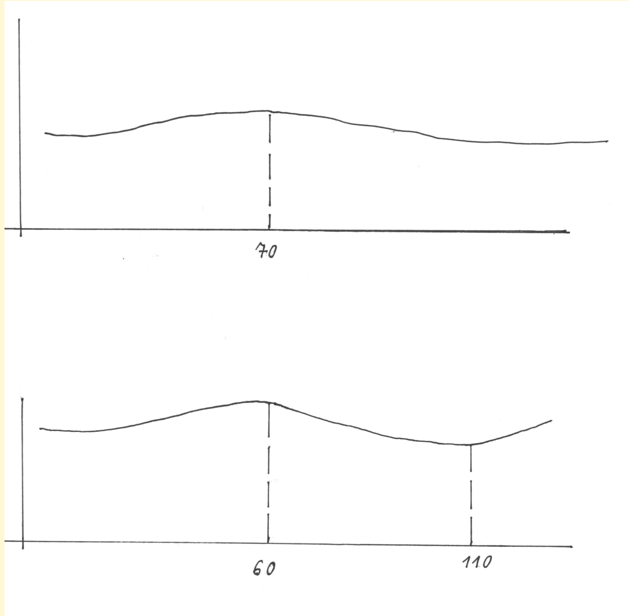
Isotopic materials $\quad a_{ij}(x_1,u) \;=\; 0 \quad$ if $\quad i \ne j$

$$a^0_{11}(u) = \left\langle \frac{1}{a_{11}} \right\rangle (u)\,,$$

$$a^0_{ii}(u) = \langle a_{ii} \rangle\,(u)\,, \qquad 2 \le i \le n\,,$$

$$a^0_{ij}(u) = 0\,, \qquad i \ne j\,.$$

Coefficients obtained experimentally



$$a_{ij}^0(u) = 0, \qquad i \neq j$$

$$a_{22}^0(u) = a_{33}^0(u)$$

Thank you