INDEFINITE PRECONDITIONING OF SYMMETRIC INDEFINITE SYSTEMS

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OUTLINE

1. SYMMETRIC INDEFINITE SYSTEM + POSITIVE DEFINITE PRECONDITIONER: CG OR MINRES (OR SYMMLQ) METHOD

2. SYMMETRIC INDEFINITE SYSTEM + INDEFINITE PRECONDITIONER:
   SIMPLIFIED BI-CG OR QMR METHOD $\Longleftrightarrow$ PRECONDITIONED CG METHOD

3. **INDEFINITE SADDLE POINT PROBLEMS + INDEFINITE (CONSTRAINT OR PROJECTION) PRECONDITIONERS: PRECONDITIONED CG METHOD**

SYMMETRIC INDEFINITE SYSTEM, SYMMETRIC POSITIVE DEFINITE PRECONDITIONER

$$Ax = b$$

A symmetric indefinite, M positive definite

$$M^{-1/2}AM^{-1/2}\tilde{x} = M^{-1/2}b, \qquad x = M^{-1/2}\tilde{x}$$

$\tilde{A}\tilde{x} = \tilde{b}$, $\quad \tilde{A}$ symmetric (indefinite)!

ITERATIVE SOLUTION OF PRECONDITIONED (SYMMETRIC INDEFINITE) SYSTEM

PRECONDITIONED MINRES (MR), SYMMLQ (ME) - VARIOUS IMPLEMENTATIONS

CG APPLIED TO SYMMETRIC BUT INDEFINITE SYSTEM:

CG iterate exists at least at every second step (tridiagonal form T_n is nonsingular at least at every second step)

Paige, Saunders, 1975

peak/plateau behavior:
CG converges fast $\rightarrow$ MINRES is not much better than CG
CG norm increases (peak) $\rightarrow$ MINRES stagnates (plateau)

Greenbaum, Cullum, 1996

THE CG METHOD WITH POSITIVE DEFINITE PRECONDITIONING

CG($\tilde{A}$, $\tilde{b}$): $\tilde{x}_k \rightarrow \tilde{x}$

CG($M^{-1/2} A M^{-1/2}$, $M^{-1/2} b$, (.)): $x_k = M^{-1/2} \tilde{x}_k$

$\Longleftrightarrow$ CG($M^{-1} A$, $M^{-1} b$, $(\; . \;)_M$): $x_k \rightarrow x$

$\Longleftrightarrow$ PCG(A, b, M^{-1}, (.))

e.g. books of Saad, Greenbaum, ...

THE MINRES METHOD WITH POSITIVE DEFINITE PRECONDITIONING

MINRES($\tilde{A}$, $\tilde{b}$): $\tilde{x}_k \longrightarrow \tilde{x}$

MINRES($M^{-1/2}AM^{-1/2}$, $M^{-1/2}b$, (.)): $x_k = M^{-1/2}\tilde{x}_k$

$\Longleftrightarrow$ MINRES($M^{-1}A$, $M^{-1}b$, $(\ . \ )_M$): $x_k \rightarrow x$

$\Longleftrightarrow$ PMINRES(A, b, M^{-1}, $(\ . \ )_M$)

$\neq$ PMINRES(A, b, M^{-1}, (.))!

Paige, Saunders, 1975, Battermann, 1996

SYMMETRIC INDEFINITE SYSTEM, INDEFINITE PRECONDITIONER

M symmetric indefinite

$$M = M_1 M_2 = M_2^T M_1^T = M^T, \quad M_2 \neq M_1^T$$

M_1, M_2 can be nonsymmetric

$$M_1^{-1} A M_2^{-1} y = M_1^{-1} b, \quad x = M_2^{-1} y$$

$\tilde{A}\tilde{x} = \tilde{b}$, $\quad \tilde{A}$ nonsymmetric!

ITERATIVE SOLUTION OF (INDEFINITELY) PRECONDITIONED (NONSYMMETRIC) SYSTEM

$$\tilde{A} = M_1^{-1} A M_2^{-1} \;, \qquad \mathcal{J} = M_1^T M_2^{-1}$$

$$\Longrightarrow$$

$$\tilde{A}^T \mathcal{J} = \mathcal{J} \tilde{A}$$

$\mathcal{J}$-SYMMETRIC VARIANT OF (NONSYMMETRIC) LANCZOS PROCESS

Freund, Nachtigal, 1995

SIMPLIFIED $\mathcal{J}$-SYMMETRIC LANCZOS PROCESS AND QMR

$$\tilde{A}^T \mathcal{J} = \mathcal{J} \tilde{A}$$

$$AV_n = V_{n+1} T_{n+1,n}, \; A^T W_n = W_{n+1} \tilde{T}_{n+1,n}$$

$$W_n^T V_n = I \Longrightarrow W_n = \mathcal{J} V_n$$

$\mathcal{J}$-SYMMETRIC VARIANT OF Bi-CG
$\mathcal{J}$-SYMMETRIC VARIANT OF QMR

Freund, Nachtigal, 1995

ITERATIVE SOLUTION OF PRECONDITIONED SYSTEM WITH SIMPLIFIED LANCZOS PROCESS

QMR-from-BiCG:

$\mathcal{J}$-symmetric Bi-CG + QMR-smoothing
$\Longrightarrow$ $\mathcal{J}$-symmetric QMR

Freund, Nachtigal, 1995

Walker, Zhou 1994

peak/plateau behavior:
QMR does not improve the convergence of Bi-CG (BiCG converges fast $\rightarrow$ QMR is not much better, Bi-CG norm increases $\rightarrow$ quasi-residual of QMR stagnates)

Greenbaum, Cullum, 1996

FINITE PRECISION ARITHMETIC:

smoothing does not improve but also does not deteriorate the rate of primary (unsmoothed Bi-CG) methods (FP analogues for the peak/ plateau property)

smoothing does not improve the final accuracy of the primary method (Bi-CG/QMR on the same level)

Gutknecht, R, 2001

focus on implementation: coupled two-term recursions over the three-term recurrences

Gutknecht, Strakoš, 2000

two-term QMR implementation or two-term Bi-CG + QMR smoothing

SIMPLIFIED Bi-CG ALGORITHM $\Longleftrightarrow$
PRECONDITIONED CG ALGORITHM

$\mathcal{J}$-symmetric Bi-CG algorithm
(classical two-term BI-CG (BIOMIN))

is nothing but

classical (Hestenes-Stiefel) CG algorithm

preconditioned with (indefinite) matrix $\mathcal{J}$!

INDEFINITE SYSTEM, INDEFINITE PRECONDITIONER + CLASSICAL CONJUGATE GRADIENTS METHOD

PCG applied to (symmetric) indefinite system $Ax = b$ with (symmetric) indefinite preconditioner M

is in fact

CG applied to nonsymmetric (and often to non-normal) preconditioned system with AM^{-1} and bilinear form $(,)_M$

Nevertheless, it frequently works in practice. Theoretical results?

SADDLE POINT PROBLEM AND INDEFINITE CONSTRAINT PRECONDITIONER

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

$$M = \begin{pmatrix} I & B \\ B^T & 0 \end{pmatrix}$$

PCG method starting with particular right-hand side or initial guess: $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, r_0 = \begin{pmatrix} s_0 \\ 0 \end{pmatrix}$

R, Simoncini, 2001, Gould et al., 2000

SADDLE POINT PROBLEM AND INDEFINITE PRECONDITIONER

$$\begin{pmatrix} A & B \\ B^T & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

$$M = \begin{pmatrix} A & B \\ B^T & -C + B^T A^{-1} B - D \end{pmatrix}$$

PCG method starting with particular right-hand side and initial guess: $\begin{pmatrix} x_0 \\ 0 \end{pmatrix}, r_0 = \begin{pmatrix} 0 \\ s_0 \end{pmatrix}$

Durazzi, Ruggiero, 2001