Algebraic Multilevel Methods with Aggregations: An overview

Radim Blaheta

Department of Applied Mathematics, Institute of Geonics AS CR
Studentská 1768, 70800 Ostrava-Poruba, Czech Republic
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Abstract. This paper deals with the numerical solution of elliptic boundary value problems by multilevel solvers with coarse levels created by aggregation. Strictly speaking, it deals with the construction of the coarse levels by aggregation, possible improvement of the simple aggregation technique and use of aggregations in multigrid, AMLI preconditioners and two-level Schwarz methods.

1 Introduction

This paper considers *multilevel solvers* for algebraic systems arising from the finite element approximation to selfadjoint elliptic problems. It is well known that nested finite element grids allow to introduce *two-level* and *multilevel* methods for solving the finite element systems. Multigrid methods [13], AMLI preconditioners [1] and Schwarz methods [24] are typical examples of numerical methods exploiting the grid hierarchy.

Algebraic multilevel methods avoid the necessity of nested triangulation of the problem domain and allow algebraic construction of coarser spaces by using mostly only information involved in the matrix of the solved problem. As multilevel iterative methods reduce the error by two complementary tools like relaxation on the fine grid and coarse grid correction for multigrid methods, the algebraic approach also enables to construct the coarse space with approximating properties that are necessary for efficiency of the complementary tool. In this way, the algebraic approach enhances the robustness of the multilevel solution methods.

In this paper, we outline the *aggregation techniques* that constitute one of possible approaches to algebraic multilevel methods. The aggregation technique originally appeared in the context of multigrid methods but can be exploited also in hierarchical algebraic multilevel preconditioners and the two-level Schwarz domain decomposition methods.

The paper provides an overview of applications of the aggregation technique, which undergo an important development during the last decade.

2 AMG with aggregation

Let us consider the solution of linear systems appearing from the finite element (FE) approximation of elliptic boundary value problems and let $\mathcal{T}_h$ be a FE triangulation which arises as a refinement of a coarser triangulation $\mathcal{T}_H$ of the problem domain Ω. Then the FE system corresponding to the fine triangulation $\mathcal{T}_h$,

$$A_h u_h = b_h, \quad u_h,\, b_h \in R^{n_h} \tag{1}$$

can be solved by the following iterative *two-grid method*. Its one iteration $u_h^{i+1} = TG(A_h,\, b_h,\, u_h^i)$ is described as follows

function TG($A_h,\, b_h,\, u_h^i = \bar{u}$)

$\nu_1 - times:\quad \bar{u} \leftarrow S(A_h,\, b_h,\, \bar{u})$	pre-smoothing
$r_H = I_h^H\,(b_h - A_h \bar{u})$	restriction of the residual
$v_H = A_H^{-1} r_H$	coarse grid correction
$\bar{u} = \bar{u} + I_H^h v_H$	prolongation of the correction
$\nu_2 - times:\quad \bar{u} \leftarrow S(A_h,\, b_h,\, \bar{u})$	post-smoothing

return ($u_h^{i+1} = \bar{u}$)

Above, the *smoothing* $\bar{u} \leftarrow S(A_h,\, b_h,\, \bar{u})$ represents one iteration of an inner iterative (relaxation) procedure like Jacobi, Gauss-Seidel etc. The *coarse grid correction* uses matrix A_H from FE discretization of the solved problem on the coarse grid, restriction I_h^H to the coarse FE space and prolongation I_H^h induced by the natural interpolation between the nested grids. The smoothing procedure should collaborate with the coarse grid correction. Usually S efficiently reduces oscillating error components and produces *smooth error* that can be reduced by the coarse grid correction. Therefore S is called the *smoother*. Note that the introduced two-level method can be naturally extended to the multilevel one.

For a broad class of problems, it can be shown that multigrid methods are highly efficient and even *optimal,* which means that the system (1) is solved in $O(n_h)$ operation. But application of multigrid methods can also meet two drawbacks: it can be difficult or impossible to produce a sequence of auxiliary coarser discretizations of the solved boundary value problems and it can be difficult to produce coarse discretizations collaborating well with the used smoother in the case of problems with certain anisotropy, singularity etc.

These difficulties motivate an interest in *algebraic multigrid methods* (AMG), which construct the prolongation, restriction and coarse matrices by using only the information included in the solved system or very little additional geometric information. In the AMG context, the system at a current level k ($k = 1$ is the finest level) is written as

$$A_k u_k = b_k, \quad u_k,\; b_k \in R^{n_k} \tag{2}$$

and the coarser level works with $n_{k+1} \times n_{k+1}$ matrix $A_{k+1} = I_k^{k+1}\, A_k\, I_{k+1}^k$ defined with the aid of a *prolongation* I_{k+1}^k and a *restriction* I_k^{k+1}. For symmetric positive definite (SPD) problems, we choose $I_k^{k+1} = \left(I_{k+1}^k\right)^T$ ensuring that A_k remains

also SPD. Note that in this case $\left\|e_k - I_{k+1}^k v_{k+1}\right\|_{A_k} = \min$ is equivalent to the coarse correction

$$(I_{k+1}^k)^T A_k I_{k+1}^k v_{k+1} = A_{k+1} v_{k+1} = (I_{k+1}^k)^T A_k e_k.$$

One iteration $u_k^{i+1} = MG_\mu(A_k,\, b_k,\, u_k^i)$ of the multilevel AMG method is recursively described as follows

function $MG_\mu(A_k,\, b_k,\, u_k^i = \bar{u})$
  $\nu_1 - times:\quad \bar{u} \leftarrow S(A_k,\, b_k,\, \bar{u})$ — pre-smoothing
  $r_{k+1} = I_k^{k+1}\,(b_k - A_k \bar{u})$ — residual restriction
  if $\quad k+1 = coarsest$ then — coarse grid correction
    $v_{k+1} = A_{k+1}^{-1} r_{k+1}$
  else
    $v_{k+1} = 0$
    $\mu - times:\quad v_{k+1} = MG_\mu(A_{k+1},\, r_{k+1},\, v_{k+1})$
  end
  $\bar{u} = \bar{u} + I_{k+1}^k v_{k+1}$ — correction prolongation
  $\nu_2 - times:\quad \bar{u} \leftarrow S(A_k,\, b_k,\, \bar{u})$ — post-smoothing
return$(u_k^{i+1} = \bar{u})$

For discrete PDE problems, AMG has the following advantages:

- there is no need for creating nested grids, it is possible to develop black box solvers,
- instead of seeking of smoothers adapted to the coarse problem, the coarse grid can be adapted to the smoother,
- the size of the coarse problem can be controlled e.g. for balancing the work load on many processors in the case of two-level Schwarz method.

AMG methods can be based on different ideas including the aggregation technique on which we focus our interest. From the other ideas, we can mention AMG based on an C-F decomposition and interpolation developed by A. Brandt, J.W. Ruge, K. Stüben and others, see e.g. [23].

2.1 Aggregation of unknowns

We shall restrict our attention to SPD problems, when AMG needs only to define the interpolation I_{k+1}^k. The simplest interpolation and restriction are in the form $I_{k+1}^k = \mathcal{R}^T$, $I_k^{k+1} = \mathcal{R}$ with $n_{k+1} \times n_k$ Boolean matrix $\mathcal{R}$ with just one unity in each column, e.g.

$$\mathcal{R} = \begin{bmatrix} 1 & & & & & & & 1 & & \\ & 1 & & & & & 1 & & & \\ & & 1 & & & 1 & & & & \\ & & & 1 & & & & & 1 & \\ & & & & 1 & & & & & 1 \\ & & & & & 1 & & & & \end{bmatrix}.$$

Definition of $\mathcal{R}$ is equivalent to the division of the set of n_k unknowns into n_{k+1} disjoint groups (*aggregation of unknowns*)

$$\{1, \ldots, n_k\} = \bigcup_{i=1}^{n_{k+1}} G_i, \text{ where } G_i = \{j : \mathcal{R}_{ij} = 1\}.$$

For 1D problems, the aggregation of unknowns can be easily defined by clustering of neighbouring nodes. This clustering can be easily generalized to regular grids in 2D and 3D domains, see e.g. Fig.1. More general aggregation on irregular grids will be discussed later in Subsection 2.4. For an application of the two-level and multilevel aggregation methods, it is important that the prolongation, restriction and construction of the coarse matrix can be efficiently implemented.

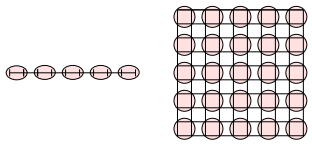

Fig. 1. An aggregation on regular 1D and 2D grids, regular clustering of 2 and 2×2 nodes, respectively.

The idea of aggregation in the context of iterative solution methods was already used in [22]. In the context of multigrid methods for solving elliptic boundary value problems, the aggregations were used e.g. in [2,3,4] and [9].

2.2 Overcorrection

Let us consider a model 1D or 2D Dirichlet problem for Poisson equation in an interval or a square (see Fig.1) and the linear finite element discretization of these problems on uniform meshes with mesh size h providing 3 and 5 point stencil, respectively. Then the aggregations can be constructed e.g. by regular clustering of 2 and 2×2 nodes, respectively. In these cases, it is easy to compute the coarser matrices and see that these matrices differ from matrices arising from discretization on the coarser uniform grids with mesh size $2h$ by the factor 2.

We can also consider a 1D Dirichlet problem for the equation $-u'' = f$ in $\langle 0,\, 1\rangle$, aggregation by regular clustering and approximation of a hat shape error in the energy norm, see Fig.2. The computed approximation from the coarse space created by aggregation indicates the possible improvement by scaling the correction by the factor 2.

For the 1D model problem, it is also possible to apply the Fourier analysis [2,4] to show that the smooth error components are only partly reduced by the correction from the aggregated space and that the efficiency of multi-level aggregation method can be substantially improved by the *overcorrection,*

$$\bar{x} = \bar{x} + \omega\, I_{k+1}^{k} v_{k+1} \quad \text{with the scaling factor } \omega > 1.$$

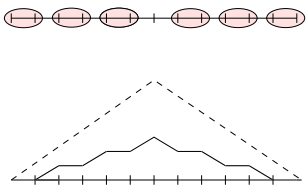

Fig. 2. Correction by aggregation (solid line) computed to a hat shape error (dashed line) for a 1D model problem.

The use of overcorrection was introduced in [3,4] and [9]. For more general problems, a variational computation of ω was suggested in [3,4] by using the following algorithm

$v_{k+1} = A_{k+1}^{-1} r_{k+1}$	corse grid correction
$\bar{v}_k = I_{k+1}^k v_{k+1}$	prolongate the correction
$\eta - times: \;\; \bar{v}_k \leftarrow S(A_k,\, r_k,\, \bar{v}_k)$	smooth the correction
$\omega = \langle \bar{v}_k,\, r_k \rangle \,/\, \langle \bar{v}_k,\, A_k \bar{v}_k \rangle$	compute the scaling factor
$\quad = \operatorname{argmin} \left\| A_k^{-1} b_k - (\bar{x} + \omega \bar{v}_k) \right\|_{A_k}$	
$\bar{x} = \bar{x} + \omega\, I_{k+1}^k v_{k+1}$ or $\;\bar{x} = \bar{x} + \omega \bar{v}_k$	perform the overcorrection

2.3 Smoothing

Above, we mentioned that the matrix created by aggregation is too stiff. This matrix is a Galerkin type matrix defined with the aid of basis functions in aggregation space, which are sums of basis functions in the original fine FE space or in a previous aggregation space. A difficulty is in a high energy of these aggregation basis functions. The improvement can be find in smoothing the aggregation basis functions, which produces new ones with a lower energy, see e.g. [25,26,27].

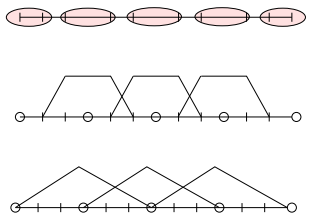

Fig. 3. 1D model problem, aggregation basis functions before and after smoothing.

For a 1D model problem in Fig. 3, the smoothing of the aggregation basis function ϕ^a by $\mathcal{S} = I - \frac{2}{3}A$ produces new basis function $\mathcal{S}\phi^a$ having a larger sup-

port but a lower energy. For this model problem, the new smoothed aggregations are even piecewise linear. But it is not true in more general cases.

The process of smoothing can be formalized as follows. Firstly, we define a prolongation $\mathcal{I}_{k+1}^k$ defined by aggregation (*tentative prolongation*). Then, a more efficient prolongation operator is constructed in the form

$$I_{k+1}^k = \mathcal{S}_k \mathcal{I}_{k+1}^k \text{ with } \mathcal{S}_k = I - \omega \Lambda_k^{-1} \bar{A}_k.$$

In [26], the *prolongation smoother* $\mathcal{S}$ has the components $\omega = \frac{2}{3}$, $\Lambda_k = \mathrm{diag}(A_k)$ and $\bar{A}_k = (\bar{a}_{ij})$ arises from $A_k = (a_{ij})$ by filtering,

$$\bar{a}_{ij} = \left\{ \begin{array}{cc} a_{ij} \text{ if } |a_{ij}| \geq \varepsilon \sqrt{a_{ii}} \sqrt{a_{jj}} \\ 0 \qquad \text{otherwise} \end{array} \right\}$$

for $i \neq j$, $\bar{a}_{ii} = a_{ii} - \sum_{i \neq j} (a_{ij} - \bar{a}_{ij})$. A heuristic choice of the parameter ε is $\varepsilon = 0.08\,(\frac{1}{2})^{k-1}$.

In [27], $\omega = \frac{4}{3\lambda_k}$, $\Lambda_k = (P_k^1)^T P_k^1$, where $P_k^1 = I_2^1 \cdots I_k^{k-1}$ and $\lambda_k \geq \rho(\Lambda^{-1} A_k)$, $\bar{A}_k = A_k$. A possible choice is $\lambda_k = 9^{k-1}\rho$, where $\rho \geq \rho(A_1)$. For this choice, the convergence factor q_{MG-L} of L level multigrid can be estimated as follows,

$$q_{MG-L} \leq 1 - \frac{1}{C(L)},$$

where $C(L)$ is a polynomial in L, see [27] for the proof.

2.4 Construction of aggregations

The construction of aggregations on general meshes with paying the attention to strong couplings between unknowns (smooth error character) can be node or element oriented. A standard node oriented algorithm for creating the aggregates is the following one:

preliminary phase: separate isolated points as individual aggregates,
phase I: repeat until all unaggregated nodes are adjacent to an aggregate:
  a) pick the root node not adjacent to any existing aggregate,
  b) define new aggregate as the root node plus all its neighbours,
phase II: sweep unaggregated nodes into existing aggregates (to which they are connected) or use them to form new aggregates.

Such algorithm can be found e.g. in [26] and has many variants. At first, some measures can be done for not leaving too many nodes for the phase II. Secondly, the connection and neighbourhood can be defined in a *strong* sense to create the coarse problem suitable for handling those error components, which can not be removed by the smoother. A typical strong coupling between the nodes (unknowns) i and j means that

$$|a_{ij}| \geq \varepsilon \sqrt{a_{ii}} \sqrt{a_{jj}}.$$

For elasticity problems, the aggregation of unknowns is restricted to aggregation of unknowns corresponding to the displacements in the same coordinate direction. Alternatively, we can still start with aggregation of the nodes and assign more degrees of freedom (DOF) to each aggregate (see also next subsection). The strength of coupling can be defined by means of blocks corresponding to nodal DOFs, see [16].

A further information about the character of the smooth error can be obtained from an auxiliary iterative solution of the homogeneous variant $A_k = 0$ of the solved problem (2). This information can be used for an improved construction of aggregations, see e.g. [11].

From the other algorithms for the construction of aggregations, we can mention subsequent pairing [9,19]. The algorithm can be described as follows

step I: repeat until all unaggregated nodes are classified as aggregated pair or singleton:

- ***** pick up a node i and find the node j with the strongest coupling to i. If this coupling is not strength enough classify i as singleton otherwise create a pair $\{i, j\}$.
- ***** aggregate the matrix

step II: apply the previous algorithm to aggregated nodes and aggregated matrix to create generalized quaternion aggregations, etc.

This algorithm creates aggregations similar to aggregations on a regular grid.

The aggregations can be also created by agglomeration of adjacent finite elements. Such approach is described e.g. in [15,12].

2.5 Enriched aggregations

For scalar boundary value problems, the aggregation of unknowns is equal to aggregation of nodes, i.e. one DOF is assigned to each aggregate of nodes. For elasticity problems or systems of equations, it is natural to aggregate separately displacements in different directions or unknowns corresponding to different phenomena. In other terminology, more DOFs are assigned to the aggregates. For elasticity, these DOFs can be two or three displacements per aggregate but an additional enhancement is also possible, e.g by adding the rotations [14]. For 2D elasticity, it gives 3 unknowns: the displacements u, v and rotation angle α per aggregate. If (x_T, y_T) is the barycentre of the aggregate, then the prolongation assign the displacement $(u - \alpha(y - y_T), v + \alpha(x - x_T))$ to any node (x, y) of the aggregate.

In the case of aggregation by agglomeration of finite elements, a further enrichment can be done by using low energy eigenvectors corressponding to the agglomeration matrices, see [15,12].

3 AMLI preconditioners with aggregation

Aggregation based AMG methods can be also used as preconditioners, a pioneering work in this respect is [9].

Beside multigrid preconditioners, there is also a class of hierarchical AMLI preconditioners, which use a space decomposition and work separately on the coarse space and its complement. These preconditioners can be also constructed with the aid of aggregation, see [18,19].

In the case of scalar boundary value problem and the system $Au = b,\ u, b \in R^n$, we start with creating the aggregations $\{G_i : i = 1, \ldots, m\}$ and selecting one node in each aggregation as a C-node. All remaining nodes are considered to be F-nodes. The F-C decomposition induces a decomposition of the matrix A,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & \\ & S_A \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

and a preconditioner

$$B = \begin{bmatrix} I & \\ A_{21}P_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} P_{11} & \\ & S \end{bmatrix} \begin{bmatrix} I & P_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

where $S \sim S_A$ and $P_{11} \sim A_{11}$. In [19], S is given by a scaled aggregation and P_{11} is realized by dynamically constructed MILU factorization. The dynamic feature means that F-nodes, which are problematic for the MILU factorization, are shifted among C-nodes. Multilevel preconditioners then arise by solution of the second pivot block by inner iterations (CG) with the same type of hierarchical preconditioner.

Alternatively, we can create a hierarchical basis (HB) with basis functions

$$\phi_i^{HB} = \sum_{k \in G_i} J_{ik}\, \phi_i^h \quad \text{if } i \in C \ \text{ and } \ \phi_i^{HB} = \phi_i^h \ \text{ if } i \in F,$$

where C and F denote the sets of C-nodes and F-nodes, respectively. The transformation between the standard and hierarchical bases is given by the matrix J, which can be written as follows,

$$J = (J_{ij}) = \begin{bmatrix} I_1 & 0 \\ I_{21} & I_2 \end{bmatrix} \begin{matrix} F \\ C \end{matrix}$$

where I_{21} is a Boolean matrix with one unity per column, I_1 and I_2 are identity matrices of proper dimensions.

The matrix A can be transformed to the hierarchical form A_{HB} and both matrices A and A_{HB} can be written in F-C, F-C ordering as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{HB} = J\, A\, J^T = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

where $A_{11} = H_{11}$ and H_{22} is the matrix arising from aggregation of A. This decomposition enables to define both additive and multiplicative preconditioners

$$B_A = J^{-1} \begin{bmatrix} \widetilde{H}_{11} & \\ & \widetilde{H}_{22} \end{bmatrix} J^{-T}, \quad B_A^{-1} = J^T \begin{bmatrix} \widetilde{H}_{11} & \\ & \widetilde{H}_{22} \end{bmatrix}^{-1} J,$$

$$B_M = \begin{bmatrix} I & \\ A_{21}\widetilde{H}_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \widetilde{H}_{11} & \\ & \widetilde{H}_{22} \end{bmatrix} \begin{bmatrix} I & \widetilde{H}_{11}^{-1}A_{12} \\ & I \end{bmatrix},$$

where $\widetilde{H}_{11} \sim H_{11}$ and $\widetilde{H}_{22} \sim H_{22}$. Some analysis and comparisons with standard AMLI can be done on the basis of the strengthened CBS inequalities, see [7].

4 Schwarz methods with aggregation

The algebraic coarse space created by aggregation can be also used in the framework of the *two-level additive and hybrid Schwarz preconditioners*. General form of these preconditioners is as follows

$$B_A \;=\; B_0 + B_{1L}, \quad B_{1L} = \sum_1^m B_k \,, \;\; B_k = R_k^T A_k^{-1} R_k \tag{3}$$

where A_k $(k = 1, \ldots, m)$ are FE matrices of local subproblems and A_0 is a coarse matrix created by aggregation. More details will be provided later. We shall also consider nonsymmetric hybrid preconditioner defined by

$$B_H \;=\; B_0 + B_{1L}(I - AB_0) \tag{4}$$

and its symmetrized version

$$B_{SH} \;=\; B_0 + (I - B_{1L}A)B_0(I - AB_{1L}), \tag{5}$$

More details about the Schwarz preconditioners can be found e.g. in [24].

Now, let us solve the system (1) arising from a finite element discretization of an elliptic boundary value problem in Ω. Let $\mathcal{T}_h$ be a FE triangulation of the domain Ω and V_h be a corressponding FE space. The triangulation $\mathcal{T}_h$ can be divided into m parts in two steps: firstly $\mathcal{T}_h$ is divided into nonoverlapping sets $\mathcal{T}_k^0$, which are consequently extended to overlapping sets $\mathcal{T}_k^\delta$. We shall denote

$$\Omega_k^0 = \cup\{E : \, E \in \mathcal{T}_k^0\}, \qquad \Omega_k^\delta = \cup\{E : \, E \in \mathcal{T}_k^\delta\}.$$

Now, we can define the local FE spaces $V_k \subset V_h$ of admissible functions on Ω which vanish outside Ω_k, matrices A_k and restrictions R_k. Let A_k^δ be the FE matrix arising from assembling the element matrices A_E for $E \in \mathcal{T}_h^\delta$. Then A_k will be the matrix arising from A_k^δ by incorporating homogeneous Dirichlet type boundary conditions on the inner boundary $\partial\Omega_k^\delta \setminus \partial\Omega$. The boundary conditions on the outer boundary $\partial\Omega_k^\delta \cap \partial\Omega$ are given from the solved boundary value problem.

The decomposition $V_h = V_1 + \ldots + V_m$ can be enriched by a coarse space V_0 created algebraically by aggregation, which ensures the numerical scalability with respect to the number of the subdomains. If $G_1, \ldots, G_N$ be the aggregations and $V_h = \mathrm{span}\{\phi_1^h, \ldots, \phi_n^h\}$, where ϕ_i^h are basis functions, then it is possible to define aggregated basis functions ψ_k and the space $V_0 \subset V$ as follows,

$$\psi_k \;=\; \sum_{i \in G_k} \phi_i^h \,, \qquad V_0 \;=\; \mathrm{span}\{\psi_1, \ldots, \psi_N\} \,.$$

We shall assume that the aggregations are regular, i.e. there is a constant $\bar{\beta}$ such that each $\mathrm{supp}\, \psi_k$ contains a ball with diameter $\bar{\beta} H$, where

$$H \;\sim\; \max_k \mathrm{diam}(\mathrm{supp}\, \psi_k).$$

Such construction gives again a stable decomposition $V = V_0 + V_1 + \ldots + V_m$ resulting in numerically scalable preconditioners B_A and B_{SH}. For more details see [10,17,6,21].

We shall conclude this section with some numerical examples. The efficiency of various preconditioners arising from implementation of the described ideas can be compared by solving two boundary value problems in $\Omega = \langle 0, 2\rangle \times \langle 0, 3\rangle$ with pure homogeneous Dirichlet boundary conditions ($\partial\Omega_D = \partial\Omega$). The first problem is for the Poisson equation, the second one is a model elasticity (plane deformation) problem with the elasticity modulus $E = 1$ and Poisson ratio $\nu = 0.3$. The right hand side is a linear function in both cases.

The problems are discretized by linear triangular FE on a uniform grid with the mesh size $h = 1/30$. The local problems are given on subdomains $\Omega_k = \langle 0, 2\rangle \times \langle x_k, x_{k+1}\rangle$ with overlap $\delta = 2h$. The subproblems are solved exactly.

The required numbers of iterations for the accuracy $\varepsilon = 10^{-3}$ and various additive (AP) and hybrid (HP) Schwarz preconditioners can be seen in Tables 1 and 2. The hybrid preconditioners are used in nonsymmetric form in combination with a generalized conjugate gradient method GPCG[1], see [5]. The coarse problem uses either the nested coarse triangular grid with the mesh size $H = 2h$ or the aggregations with clustering 2×2 square macroelements (3×3 nodes). The smoothing was done by $S = I - \frac{2}{3} diag(A)^{-1} \bar{A}$ where $\bar{A}$ was equal to A in both cases. For the elasticity, we test also sparser $\bar{A}$ given by the separate displacement component part of A but both choices give the same results.

5 Conclusions

In this paper, we provide an overview of possible applications of the aggregation technique in multilevel methods. Additionally, we can mention application of the aggregation technique in a nonoverlapping Schwarz method with interfaces on the coarse grid, see [6], or a specific aggregations for construction of AMLI preconditioners for nonconforming Crouzeix-Raviart finite elements [8].

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Overlap 2h, #subdomains:	4	16	24
c-grid H=3h, AP	7	7	8
c-grid H=3h, HP	6	6	6
aggreg. 2h, AP	13	17	17
aggreg. 2h, HP	10	11	11
smooth. aggreg. 2h, AP	10	11	11
smooth. aggreg. 2h, HP	7	7	8

Table 1. Poisson equation problem. Numbers of iterations for $\varepsilon = 10^{-3}$. AP=additive preconditioner, HP=hybrid preconditioner + GPCG[1].

Overlap 2h, #subdomains:	4	16	24
c-grid H=3h, AP	8	8	9
c-grid H=3h, HP	6	7	8
aggreg. 2h, AP	17	20	20
aggreg. 2h, HP	12	13	14
aggreg. 2h-rotat, AP	16	18	19
aggreg. 2h-rotat, HP	11	12	12
smooth. aggreg. 2h, AP	12	14	14
smooth. aggreg. 2h, HP	9	10	10

Table 2. Elasticity problem. Numbers of iterations for $\varepsilon = 10^{-3}$. AP=additive preconditioner, HP=hybrid preconditioner + GPCG[1].

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