Application of Hierarchical Decomposition: Preconditioners and Error Estimates for Conforming and Nonconforming FEM

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Abstract. A successive refinement of a finite element grid provides a sequence of nested grids and hierarchy of nested finite element spaces as well as a natural hierarchical decomposition of these spaces. In the case of numerical solution of elliptic boundary value problems by the conforming FEM, this sequence can be used for building both multilevel preconditioners and error estimates. For a nonconforming FEM, multilevel preconditioners and error estimates can be introduced by means of a hierarchy, which is constructed algebraically starting from the finest discretization.

1 Introduction

Let us consider a model elliptic boundary value problem in $\Omega \subset R^2$,

$$\text{find} \quad u \in V \ : \ a(u,v) = b(v) \qquad \forall v \in V, \tag{1}$$

where $V = H_0^1(\Omega)$, $b(v) = \int_\Omega f v dx$ for $f \in L_2(\Omega)$ and

$$a(u,v) = \int_\Omega \sum_{ij}^{2} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx \,. \tag{2}$$

Above $K = (k_{ij})$ is a symmetric and uniformly bounded positive definite matrix.

This type of boundary value problems are most frequently solved by the finite element method (FEM). A successive refinement of a finite element grid provides a sequence of nested grids and hierarchy of nested finite element spaces as well as a natural hierarchical decomposition of these spaces. This sequence can be used for building both multilevel preconditioners and error estimates. In Section 2, we describe such hierarchy for conforming Courant type finite elements. We also mention the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality, which is important for characterization of the hierarchical decomposition. In Section 3, we show that the hierarchical decomposition allows to construct preconditioners and error estimates. Section 4 is devoted to hierarchical decompositions constructed algebraically for nonconforming Crouzeix-Raviart FEM. We show that this decomposition allows again to introduce both preconditioners and error estimates.

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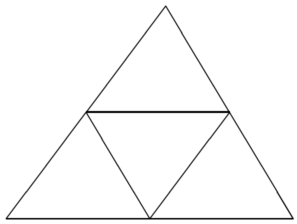

Fig. 1. A regular decomposition of a triangle

2 Hierarchical Decomposition for Conforming FEM

Let us consider a coarse triangular finite element grid $\mathcal{T}_H$ in Ω and a fine grid $\mathcal{T}_h$, which arises by a refinement of the coarse elements, see Fig. 1 for the most typical example. We assume that $\Omega = \bigcup\{E : \ E \in \mathcal{T}_H\}$.

By $\overline{\mathcal{N}}_H$ and $\overline{\mathcal{N}}_h$, we denote the sets of nodes corresponding to $\mathcal{T}_H$ and $\mathcal{T}_h$, respectively. Further, $\mathcal{N}_H = \{x \in \overline{\mathcal{N}}_H, \ x \notin \partial\Omega\}$, $\mathcal{N}_h = \{x \in \overline{\mathcal{N}}_h, \ x \notin \partial\Omega\}$. Naturally, $\mathcal{N}_h = \mathcal{N}_H \cup \mathcal{N}_H^+$, where $\mathcal{N}_H^+$ is the complement of $\mathcal{N}_H$ in $\mathcal{N}_h$.

Now, we can introduce the finite element spaces V_H and V_h ($V_H \subset V_h$) of functions which are continuous and linear on the elements of the triangulation $\mathcal{T}_H$ and $\mathcal{T}_h$, respectively.

The space V_h allows a natural hierarchical decomposition. Let $\{\phi_i^H\}$ and $\{\phi_i^h\}$ be the standard nodal finite element bases of V_H and V_h, i.e. $\phi_i^H(x_j) = \delta_{ij}$ for all $x_j \in \mathcal{N}_H$, $\phi_i^h(x_j) = \delta_{ij}$ for all $x_j \in \mathcal{N}_h$. Then V_h can be also equipped with a *hierarchical basis* $\{\bar{\phi}_i^h\}$, where

$$\bar{\phi}_i^h = \begin{cases} \phi_i^h & \text{if } x_i \in \mathcal{N}_H^+, \\ \phi_i^H & \text{if } x_i \in \mathcal{N}_H. \end{cases}$$

It gives a *natural hierarchical decomposition* of the space V_h,

$$V_h = V_H \oplus V_H^+ , \quad V_H^+ = \text{span}\ \{\phi_i^h, \ x_i \in \mathcal{N}_H^+\}. \tag{3}$$

The decomposition (3) is characterized by the strengthened CBS inequality with the constant $\gamma = \cos(V_H, V_H^+)$, which is defined as follows:

$$\begin{aligned} \gamma &= \cos(V_H, V_H^+) \\ &= \sup \left\{ \frac{|\ a(u,v)\ |}{\|u\|_a \ \|v\|_a} : u \in V_H,\ u \neq 0,\ v \in V_H^+,\ v \neq 0 \right\} . \end{aligned} \tag{4}$$

Above $\|u\|_a = \sqrt{a(u,u)}$ is the energy norm. If $\mathcal{T}_h$ arises from $\mathcal{T}_H$ by a regular division of the coarse grid triangles into 4 congruent triangles (see Fig. 1) and if the coefficients $K = (k_{ij})$ are constant on the coarse grid elements then $\gamma < \sqrt{3/4}$ for general anisotropic coefficients and arbitrary shape of the coarse grid elements. For more details, see [1] and the references therein.

3 Hierarchical Preconditioners and Error Estimates

The decomposition (3) can be used for construction of preconditioners for the FE matrices A_h and $\bar{A}_h$,

$$\langle A_h \mathbf{u},\, \mathbf{v}\rangle = \langle \bar{A}_h \bar{\mathbf{u}},\, \bar{\mathbf{v}}\rangle = a(u,\, v) \tag{5}$$

for $u = \sum \mathbf{u}_i \phi_i^h = \sum \bar{\mathbf{u}}_i \bar{\phi}_i^h$ and $v = \sum \mathbf{v}_i \phi_i^h = \sum \bar{\mathbf{v}}_i \bar{\phi}_i^h$. Both nodal and hierarchical basis FE matrices A_h and $\bar{A}_h$ then have a hierarchic decomposition

$$A_h = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} \mathcal{N}_H^+ \\ \mathcal{N}_H \end{matrix} \qquad \text{and} \quad \bar{A}_h = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{matrix} \mathcal{N}_H^+ \\ \mathcal{N}_H \end{matrix} \,. \tag{6}$$

Note that the diagonal blocks A_{11}, A_{22} of A_h carry only the local information. On the opposite, the diagonal blocks $\bar{A}_{11} = A_{11}$ and $\bar{A}_{22} = A_H$ of $\bar{A}_h$ carry both local and global information on the discretized problem.

Note also that the relation between A_h and $\bar{A}_h$ implies the identity between the Schur complements,

$$S_h = \bar{S}_h, \quad S_h = A_{22} - A_{21}A_{11}^{-1}A_{12}, \quad \bar{S}_h = \bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12}\,.$$

The standard *hierarchic multiplicative preconditioner* then follows from an approximate factorization of A_h with Schur complement S_h replaced by $\bar{A}_{22}$,

$$B_h = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & \\ & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} \,. \tag{7}$$

Note that getting efficient preconditioners assumes that

- A_{11} is approximated for a cheaper computation. The simplest approximation is the diagonal of A_{11}, see [2], more accurate approximation can use incomplete factorization or a locally tridiagonal element-by-element approximation of A_{11}, see [3],
- $\bar{A}_{22} = A_H$ is also approximated. A natural way how to do it is to use hierarchical decomposition recursively and to solve the system with $\bar{A}_{22}$ by a few inner iterations with a proper hierarchical preconditioner. In a multilevel setting, we can get an optimal preconditioner, see [4,5,6].

Another application of the hierarchical decomposition is in error estimation, see [7,10] and the references therein. If $u \in V$ is the exact solution, $u_H \in V_H$ and $u_h \in V_h$ are the finite element solutions of the problem (1) in V_H and V_h, respectively, and if there is a constant $\beta < 1$ such that

$$\| u - u_h \|_a \le \beta \, \| u - u_H \|_a \,, \tag{8}$$

(saturation condition) then the Galerkin orthogonality allows to show that

$$\| w_h \|_a \le \| u - u_H \|_a \le \frac{1}{1-\beta^2} \, \| w_h \|_a \,, \tag{9}$$

where $w_h = u_h - u_H$, see [7]. Thus $\eta = \| w_h \|_a$ can serve as an *efficient and reliable error estimator.*

A cheaper error estimator $\bar{\eta}$ can be computed via the hierarchical decomposition (3). Let $\bar{\eta} = \| \bar{w}_h \|_a$, where

$$\bar{w}_h \in V_H^+ : \quad a(\bar{w}_h, v_h) = b(v_h) - a(u_h, v_h) \qquad \forall v_h \in V_H^+ \tag{10}$$

then

$$\| \bar{w}_h \|_a \leq \| u - u_H \|_a \leq \frac{1}{(1-\beta^2)(1-\gamma^2)} \| \bar{w}_h \|_a \tag{11}$$

where γ is the CBS constant from (4).

Algebraically,

$$\bar{\eta} = \langle A_{11}\mathbf{w}_1, \mathbf{w}_1 \rangle^{1/2}, \tag{12}$$

where

$$\mathbf{w}_1 : \; A_{11}\mathbf{w}_1 = \mathbf{b}_1 - \bar{A}_{12}\mathbf{w}_2, \tag{13}$$

$$\mathbf{w}_2 : \; \bar{A}_{22}\mathbf{w}_2 = \mathbf{b}_2. \tag{14}$$

A still cheaper estimators can be computed by using the approximations of A_{11}. In this respect, the locally tridiagonal approximation introduced by Axelsson and Padiy [3], which is robust with respect to anisotropy and element shape, is a good candidate for obtaining a cheap reliable and efficient hierarchic error estimator. The multiplicative preconditioner of A_{11} has more than two times better κ.

4 Nonconforming Finite Elements

Let $\mathcal{T}_h$ be a triangulation of Ω, $\mathcal{M}_h$ be the set of midpoints of the sides of triangles from $\mathcal{T}_h$, $\mathcal{M}_h^0$ and $\mathcal{M}_h^1$ consist of those midpoints from $\mathcal{M}_h$, which lie inside Ω and on the boundary $\partial\Omega$. Then the Crouzeix-Raviart finite element space V_h is defined as follows

$$V_h = \{ v \in U_h : \quad v(x) = 0 \quad \forall x \in \mathcal{M}_h^1 \}, \tag{15}$$

$$U_h = \{ v \in L_2(\Omega) : \quad v|_e \in P_1 \quad \forall e \in \mathcal{T}_h, \quad [v](x) = 0 \quad \forall x \in \mathcal{M}_h^0 \}, \tag{16}$$

where $[v](x)$ denotes the jump in $x \in \mathcal{M}_h^0$.

The finite element solution $u_h \in V_h$ of (1) is now defined as

$$u_h \in V_h : \quad a_h(u_h, v_h) = b(v_h) \qquad \forall v_h \in V_h \tag{17}$$

where a_h is the broken bilinear form,

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{ij} k_{ij} \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} dx \, . \tag{18}$$

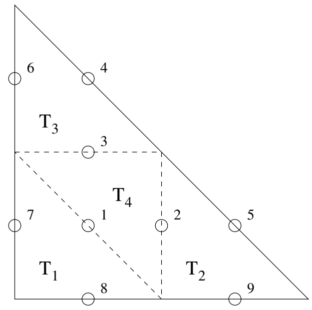

Fig. 2. A macroelement E with 9 midpoint nodes m_i

If $\mathcal{T}_H$ and $\mathcal{T}_h$ are two nested triangulations and V_H and V_h are the corresponding Crouzeix-Raviart spaces then $V_H \nsubseteq V_h$ and it is impossible to repeat the constructions of Section 2. But still there is a possibility to introduce a hierarchical basis in V_h algebraically, one such possibility, the DA splitting, is described in [8]. The construction is associated with the coarse triangles $E \in \mathcal{T}_H$ considered as macroelements composed from four congruent triangles $T \in \mathcal{T}_h$, see Fig. 2.

Let $\phi_1^h, \dots, \phi_9^h$ be the nodal basis functions of the macroelement E, i.e. $\phi_i^h(m_j) = \delta_{ij}$. Then a hierarchical basis on E can be created from the following basis functions,

$$\begin{aligned} \bar{\phi}_l^h &= \phi_i^h \quad \text{for} \quad li = 11,\ 22,\ 33 \\ \bar{\phi}_l^h &= \phi_i^h - \phi_j^h \quad \text{for} \quad lij = 445, 567, 689 \\ \bar{\phi}_l^h &= \phi_i^h + \phi_j^h + \phi_k^h \quad \text{for} \quad lijk = 7145, 8267, 9389 \end{aligned} \tag{19}$$

The last triple will be called aggregated basis functions.

The hierarchical basis on a macroelement can be extended to a hierarchical basis in the whole space V_h. Using this hierarchical basis, the space V_h can be decomposed as follows

$$V_h = V_A \oplus V_A^+ ,$$

where V_A is spanned on the aggregated basis functions and V_A^+ is spanned on the remaining basis functions. For this decomposition, $\gamma = \cos(V_A, V_A^+) = \sqrt{3/4}$, see [8].

In [8,9], it is shown that the decomposition can be used for defining optimal order hierarchical preconditioners.

Now, we shall investigate the use of the DA hierarchical decomposition for the hierarchical error estimation in the case of nonconforming Crouzeix-Raviart FEM.

Let $u_H,\ u_h$ be the nonconforming finite element solutions of (1) in V_H and V_h, respectively, and let us define

$$u_A \in V_A : \ a_h(u_A, v_h) = b(v_h) \quad \forall v_h \in V_A , \tag{20}$$

$$w_A \in V_A^+ : \ a_h(w_A, v_h) = b(v_h) - a(u_A, v_h) \quad \forall v_h \in V_A^+ . \tag{21}$$

Algebraically, let $\bar{A}_h u_h = \bar{b}_h$ be the algebraic version of (17) in the introduced hierarchical basis and the hierarchical decomposition of this system gives the following block form,

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} \tag{22}$$

with the first and second block corresponding to V_A^+ and V_A, respectively. Then

$$u_A \sim \mathbf{w}_2 = \bar{A}_{22}^{-1}\mathbf{b}_2, \tag{23}$$

$$w_A \sim \mathbf{w}_1 = \bar{A}_{11}^{-1}(\mathbf{b}_1 - \bar{A}_{12}\mathbf{w}_2). \tag{24}$$

We shall also consider the algebraic system

$$A_H \mathbf{u}_H = \mathbf{b}_H \tag{25}$$

corresponding to V_H and the broken energy norms

$$\| v_H \|_H = \sqrt{a_H(v_H, v_H)}, \qquad \| v_h \|_h = \sqrt{a_h(v_h, v_h)}$$

for $v_H \in V_H$ and $v_h \in V_h$, respectively.

Now, our aim is to investigate if

$$\eta = \| w_A \|_h = \sqrt{\langle \bar{A}_{11}\mathbf{w}_1, \mathbf{w}_1 \rangle} \tag{26}$$

is again a possible error estimator. We shal do it in three steps.

1. First, under the assumption that the saturation condition is valid, i.e. there is a $\beta < 1$,

$$\| u - u_h \|_h \leq \beta \| u - u_H \|_H,$$

it is possible to use $\| u_h - u_H \|_h$ as an error estimator for $\| u - u_H \|_H$, because

$$\frac{1}{1+\beta} \| u_h - u_H \|_h \leq \| u - u_H \|_H \leq \frac{1}{1-\beta} \| u_h - u_H \|_h \ . \tag{27}$$

Note that (27) follows from the triangle inequality. It is not possible to use the Galerkin orthogonality as it was done for (9).

2. Second, we shall investigate a relation between $\| u_h - u_H \|_h$ and $\| u_h - u_A \|_h$. For example, if f is constant on the elements $E \in \mathcal{T}_H$, then the vector $\mathbf{b}_2$ from (23) and $\mathbf{b}_H$ from (25) are equal. From [8], we have $\bar{A}_{22} = 4A_H$, thus $\mathbf{w}_2 = \frac{1}{4}\mathbf{u}_H$. Then

$$\| u_h - u_A \|_h^2 = \| u_h \|_h^2 - \| u_A \|_h^2 = \| u_h \|_h^2 - \frac{1}{4} \| u_H \|_h^2$$

$$\| u_h - u_H \|_h^2 = \| u_h \|_h^2 - \| u_H \|_h^2 + c_h$$

where c_h is the consistency term, $c_h = b(u_H) - a_h(u_h, u_H)$. It can be proved [11] that $c_h \to 0$ for $h \to 0$. Thus, for h sufficiently small

$$\| u_h - u_H \|_h \leq \| u_h - u_A \|_h \ \text{(but not } \| u_h - u_H \|_h \sim \| u_h - u_A \|_h \text{)}.$$

3. Third, the norm $\| u_h - u_A \|_h$ can be estimated by $\| w_A \|_h$. It holds, that

$$\| w_A \|_h^2 \leq \| u_h - u_A \|_h^2 \leq (1-\gamma^2)^{-1} \| w_A \|_h^2 ,$$

where $\gamma = \sqrt{3/4}$ is the strengthened CBS constant for the DA splitting.

The proof is simple. First,

$$\| w_A \|_h^2 = a_h(u_h - u_A, w_A) \leq \| u_h - u_A \|_h \| w_A \|_h .$$

Next, let $u_h = \hat{u}_A + \hat{w}_A$, $\hat{u}_A \in V_A$, $\hat{w}_A \in V_A^+$. Then

$$\begin{aligned} \| u_h - u_A \|_h^2 &= a_h(u_h - u_A,\, u_h - u_A) = a_h(u_h - u_A,\, \hat{u}_A - u_A + \hat{w}_A) \\ &= a_h(u_h - u_A,\, \hat{w}_A) = a_h(w_A, \hat{w}_A) \leq \| w_A \|_h \| \hat{w}_A \|_h \\ \| u_h - u_A \|_h^2 &= \| \hat{u}_A - u_A + \hat{w}_A \|_h^2 \\ &\geq \| \hat{u}_A - u_A \|_h^2 + \| \hat{w}_A \|_h^2 - 2 \mid a_h(\hat{u}_A - u_A, \hat{w}_A) \mid \\ &\geq (1-\gamma^2) \| \hat{w}_A \|_h^2 \end{aligned}$$

Consequently,

$$\begin{aligned} & (1-\gamma^2) \| \hat{w}_A \|_h^2 \leq \| u_h - u_A \|_h^2 = a_h(w_A, \hat{w}_A) \leq \| w_A \|_h \| \hat{w}_A \|_h \\ \text{i.e.} \quad & \left(1-\gamma^2\right) \| \hat{w}_A \|_h \leq \| w_A \|_h \end{aligned}$$

and

$$\| u_h - u_A \|_h^2 \leq (1-\gamma^2)^{-1} \| w_A \|_h^2 .$$

5 Conclusions

The first aim of the paper is to show that the progress in construction and analysis of the hierarchical multilevel preconditioners, as e.g. the mentioned locally tridiagonal approximation [3] to the pivot block can be exploited also for development of hierarchical error estimates.

The second aim is to extend the hierarchical error estimate concept to nonconforming finite elements with the aid of an auxiliary algebraic subspace V_A, $V_H \sim V_A$, $V_A \subset V_h$. This extension could provide an evaluation of the error and its distribution on an early stage of multilevel iterations and gives a chance to improve the discretization in the case of insufficient accuracy.

We have shown that a crucial point for this extension will be the approximation property of the algebraic space V_A. For the DA construction, the approximation property is not sufficient and we can get error estimator which is reliable but not efficient. A possible remedy could be in the use of the generalized DA decompositions, see [12,13]. In this respect, a further investigation is required.

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