Schwarz Methods for Simulation of THM Processes

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Abstract. In this paper, we first recall a previous favourable experience with the Schwarz type domain decomposition methods applied to problems of heat flow and elasticity [9, 11, 12] and then discuss a possible use of these methods for solving Darcy flow problems.

1 Introduction

This paper is devoted to the numerical solution of PDE problems by overlapping Schwarz methods. The work is motivated by the assessment of nuclear waste repository projects, which demands modelling and numerical simulations with the finite element analysis of problems on stress-strain changes, heat transfer and Darcy flow in the rock mass. Large scale aspect of these simulations, which is given by multiphysics, large scale 3D domains of interest, multiscale character etc. imposes an interest in an iterative solution of the arising algebraic problems by methods, which are efficient and suitable for implementation on parallel computers.

The plan of this paper is as follows. First, we recall standard analysis of Schwarz methods for both elliptic problems (see e.g. [26]) and parabolic problems (see [14, 15]). The standard analysis can be extended to the case of coarse space constructed algebraically by aggregation, see [18] for elliptic problems. We show that an extension to the algebraic coarse problem in the parabolic case is then straightforward. Second, we describe numerical experiments related to a nuclear waste repository assessment to show a real behaviour of the described methods and show good efficiency and scalability (as also indicated by the theory) of the one-level Schwarz preconditioner for computation of heat evolution with time steps adequate to a good approximation of the solution. Third, we consider Darcy flow problems, which brings two new aspects: nonlinear model for the case of nonsaturated porous media flow and discretization by a mixed finite element method for more accurate approximation of fluxes and obtaining locally conservative approximate solution. The way of application of the Schwarz type methods for solving such problems is not uniquely determined. From several possible approaches, we shall discuss more thoroughly an approach using H(div) preconditioner.

2 Schwarz preconditioners for elliptic and parabolic problems

Let us consider a general *elliptic* boundary value problem

$$\text{find u=u(x):} \quad \begin{aligned} Lu \;&=\; f \quad \text{in} \quad \Omega \\ u \;&=\; 0 \quad \text{on} \;\; \partial\Omega \end{aligned} \tag{1}$$

and a *parabolic* initial-boundary value problem

$$\text{find u=u(x,t):} \quad \begin{aligned} \frac{\partial u}{\partial t} + Lu \;&=\; f \quad \text{in} \quad \Omega \times \langle 0, T\rangle \\ u(x,t) \;&=\; 0 \quad \text{on} \;\; \partial\Omega \times \langle 0, T\rangle \\ u(x,0) \;&=\; u_0 \quad \text{in} \quad \Omega \end{aligned} \tag{2}$$

where $\Omega \subset R^d$ $(d = 2, 3)$ is a space domain and $\langle 0, T\rangle$ is a time interval. L is an elliptic differential operator. For simplicity, we restrict the analysis to homogeneous Dirichlet boundary conditions, but our conclusions can be extended to more general cases.

The variational formulation, followed by a space discretization by some finite element method, leads to the following form of the above problems

$$\text{find } u \in V: \quad a(u,v) \;=\; l(v) \quad \forall\, v \in V, \tag{3}$$

$$\text{find } u(\cdot,t) \in V: \; \begin{aligned} (\frac{\partial u}{\partial t}, v) + a(u,v) \;&=\; l(v,t) \qquad \forall\, v \in V \\ (u(\cdot,0) - u_0, v) \;&=\; 0 \qquad\qquad\; \forall\, v \in V \end{aligned} \tag{4}$$

where V is a finite element space, which is finite dimensional subspace of the Sobolev space $\mathcal{V} = H_0^1(\Omega)$, a is a bounded symmetric positive definite (SPD) bilinear form on $\mathcal{V}$, $(\cdot,\cdot)$ is the inner product in $L_2(\Omega)$ and l or $l(\cdot,t)$ is a bounded linear functional on $\mathcal{V}$.

To be more specific, we consider elliptic and parabolic problems with the bilinear form

$$a(u,v) \;=\; \int_\Omega \sum_{i,j=1}^{d} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \tag{5}$$

defined in the Sobolev space $H^1(\Omega)$ equipped with the seminorm $|\cdot|_{H^1(\Omega)}$ and the norm $\|\cdot\|_{H^1(\Omega)}$. We assume that $\mathcal{K} = (k_{ij})$ is a symmetric positive definite $d \times d$ matrix, which guarantees the existence of positive constants κ_1, κ_2, such that

$$\kappa_1\, |v|^2_{H^1(\Omega)} \;\le\; a(v,v) \;\le\; \kappa_2\, |v|^2_{H^1(\Omega)} \qquad \forall\, v \in H^1(\Omega). \tag{6}$$

Due to the assumed boundary conditions, there is also a constant κ_0, such that

$$\kappa_0\, \|v\|^2_{H^1(\Omega)} \;\le\; a(v,v) \qquad \forall\, v \in \mathcal{V}. \tag{7}$$

Problem (4) can be further discretized in time by replacing the time derivative by a finite difference. If we divide the time interval $\langle 0, T\rangle$ as $0 = t_0 < t_1 <$

$\ldots < t_j < \ldots < t_m = T$, $\Delta_i = t_i - t_{i-1}$ and $\theta \in \langle 0,\, 1\rangle$ is a parameter, then $u^k(x) \sim u(x,\, t_k)$ can be computed from

$$\begin{aligned} u^k \in V: \quad ((u^k - u^{k-1})/\triangle_k) + \theta a(u^k, v) + (1-\theta)a(u^{k-1}, v) \; &= \\ = \theta l(v, t_k) + (1-\theta)l(v, t_{k-1}) \qquad &\forall\, v \in V, \end{aligned} \tag{8}$$

which holds for $k = 1, \ldots, m$. For $k = 0$, we have $(u^0 - u_0, v) = 0 \;\forall\, v \in V$. The discretization formula (8) is known as θ-method. It is equivalent to explicit and implicit Euler and Crank-Nicolson methods for the values $\theta = 0,\, 1,\, 1/2$, respectively.

Using an inner product $\langle\,,\rangle$ in V, the problems (3) and (4) can be rewritten into the following operator forms

$$\text{find } u \in V: \qquad Au \;=\; b \tag{9}$$

$$\begin{aligned} \text{find } u^k \in V: \quad Mu^0 \;&=\; z^0 \\ S_k u^k \;&=\; z^k \qquad \text{for } k = 1, \ldots, m \end{aligned} \tag{10}$$

where A, M, $S_k \colon V \to V$ and b, $z^k \in V$ are determined by the identities

$$\begin{aligned} \langle Au, v\rangle \;&=\; a(u,v) && \forall\, u, v \in V \\ \langle Mu, v\rangle \;&=\; (u,v) && \forall\, u, v \in V \\ S_k \;&=\; M + \triangle_k \theta\, A \\ \langle b, v\rangle \;&=\; l(v) && \forall\, v \in V \\ \left\langle z^k, v\right\rangle \;&=\; \left\langle (M - \triangle_k(1-\theta)A) z^{k-1}, v\right\rangle \\ &\quad + \theta l(v, t_k) + (1-\theta) l(v, t_{k-1}) && \forall\, v \in V, \\ \left\langle z^0,\, v\right\rangle \;&=\; (u_0,\, v) \qquad \forall\, v \in V. \end{aligned}$$

3 Schwarz methods

The numerical solution of elliptic and parabolic problems then requires to solve algebraic systems, which represent the operator equations in (9) and (10). In many applications (see also the next section), these systems are very large and the use of iterative methods and suitable preconditioners is crucial for their efficient solution.

A class of parallelizable iterative methods and preconditioners, so called *Schwarz type methods and preconditioners*, can be based on a decomposition of the space V,

$$V \;=\; V_1 + \ldots + V_s \tag{11}$$

where V_k $(k = 1, \ldots, s)$ are subspaces of V, which are not necessarily linearly independent. To define Schwarz preconditioners for both operators A and $S_\tau = M + \tau A$ appearing in (9) and (10), we consider a general case of a linear operator $X : V \to V$ defined by a SPD bilinear form χ, $\langle Xu, v\rangle = \chi(u, v) \;\forall\, u, v \in V$. For each subspace V_k, we introduce

- operator $X_k\colon\ V_k \to V_k$ defined by $\langle X_k u, v\rangle \;=\; \chi(u,v) \quad \forall\, u, v \in V_k$,
- prolongation operator $I_k\colon\ V_k \to V$ given by the inclusion $V_k \subset V$,
- restriction operator $R_k\colon\ V \to V_k$ given by orthogonal projection to V_k, i.e. $\forall\, v \in V\colon R_k v \in V_k$ and $\langle v - R_k v, w\rangle \;=\; 0 \;\; \forall w \in V_k$.

Note that I_k and R_k are adjoint with respect to $\langle\,,\rangle$ and $X_k = R_k X I_k$.

The decomposition (11) leads to an additive Schwarz preconditioner to X in the form

$$G_A \;=\; G_A^{1L} \;=\; \sum_{k=1}^{s} B_k, \qquad B_k \;=\; I_k X_k^{-1} R_k\,. \tag{12}$$

This preconditioner is symmetric and positive definite with respect to $\langle\,,\rangle$. Moreover,

$$G_A^{1L} X \;=\; \sum_{k=1}^{s} P_k, \qquad P_k \;=\; I_k X_k^{-1} R_k X, \tag{13}$$

where P_k are projections $V \to V_k$, which are orthogonal with respect to the inner product induced by X, $\langle u, v\rangle_X \;=\; \langle Xu, v\rangle$.

In the following, we shall also enrich the decomposition (11) by an additional (coarse) space V_0 and consider the two-level additive preconditioner

$$G_A = G_A^{2L} \;=\; \sum_{k=1}^{s} B_k + B_0, \qquad B_0 \;=\; I_0 X_0^{-1} R_0, \tag{14}$$

and the nonsymmetric and symmetric hybrid preconditioners

$$G_H \;=\; B_0 + G_A^{1L}(I - AB_0),\;\; G_{SH} \;=\; B_0 + (I - B_0 A) G_A^{1L}(I - AB_0). \tag{15}$$

As concerns the computational work, beside one solution of the subdomain problems and one or two solutions of the coarse grid problem, there is zero, one and two multiplication with the system matrix A involved in the application of G_A^{2L}, G_H and G_{SH}, respectively. Therefore, the nonsymmetric preconditioners G_H or G_H^T are substantially cheaper than G_{SH}. Their use within the PCG method can be justified in the case of a special initial guess $u_0 = B_0 r$, where r is the right hand side of the solved problem, see [26] for more details. Another possibility is their use within a generalized PCG method, see [6].

A suitable space decomposition (3) can be constructed via overlapping decomposition of the computational domain Ω. Now, we describe and analyse a typical model situation. We solve the boundary value problem (3) in $\Omega \subset R^d$ by the *finite element method with linear triangular or tetrahedral elements.* Let $\mathcal{T}_h$ be a regular finite element division of Ω, $h \sim \max\{\mathrm{diam}(e),\, e \in \mathcal{T}_h\}$. Further, let us assume that there is a division of Ω into s non-overlapping subdomains $\Omega_1^0,\, \ldots,\, \Omega_s^0$. Let Ω_k^δ be subdomains of Ω aligned with the division $\mathcal{T}_h$ and such that $\Omega_k^\delta \;\supset\; \{x \in \Omega :\ \mathrm{dist}(x, \Omega_k^0) \;\le\; \delta/2\}$. Let us also denote $H_s \sim \max\{\mathrm{diam}(\Omega_k^\delta),\, k = 1, \ldots, s\}$ and assume that each point of Ω belongs to at most m_c subdomains, see Fig. 1.

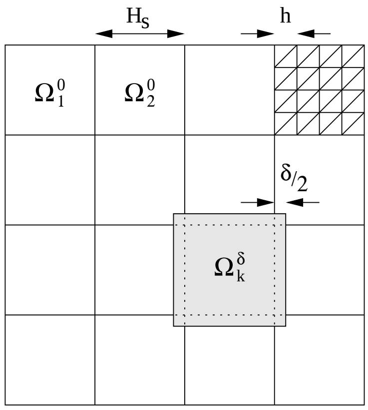

Fig. 1. Decomposition of Ω, subdomains Ω_k^0, Ω_k^δ, fine triangulation $\mathcal{T}_h$, $m_c = 4$.

The overlapping domain decomposition

$$\Omega = \Omega_1^\delta \cup \ldots \cup \Omega_s^\delta \tag{16}$$

now induces a decomposition (11) of the finite element space $V = V_h$ into the subspaces $V_1, \ldots, V_s$,

$$V_k = \{v \in V : \quad v = 0 \text{ in } \Omega \setminus \Omega_k^\delta\}. \tag{17}$$

The following theorem gives a characterization of this decomposition.

Theorem 1. *Consider the above construction of the decomposition $V = V_1 + \ldots + V_s$ with the local finite element spaces (17). Then for the operators A, S_τ from the previous section, we get*

$$\begin{aligned} K_0(A) &= C(1+\delta^{-2}), \\ K_0(S_\tau) &= C(1+\tau\delta^{-2}) \\ K_1(A) &= K_1(S_\tau) = m_c. \end{aligned} \tag{18}$$

Remark 1. For elliptic problems, the assertion of Theorem 1 can be found in many references, see e.g. [26]. The extension to parabolic problems is done in [14, 15].

Conclusion 1 *If the overlap δ is kept proportional to the subdomain size H_s, i.e. $\delta = \beta H_s$, where β is a proportionality constant, then for discrete elliptic problems* $\text{cond}(G_A A)$ *deteriorates with an increasing number of subdomains ($H_s \to 0$ for $s \to \infty$). A remedy can be found by adding an auxiliary coarse FE space V_0 to the space decomposition with local FE spaces, see the next section. However, for the discrete parabolic problems, this deterioration can be also removed by taking* sufficiently small time steps.

4 Two-level Schwarz preconditioners

Let us consider the extended decomposition

$$V = V_0 + V_1 + \ldots + V_s, \tag{19}$$

where $V = V_h$ is the FE space corresponding to the FE division $\mathcal{T}_h$, $V_1, \ldots, V_m$ are the local FE spaces (17) and $V_0 = V_H$ is the FE space corresponding to a coarser FE division $\mathcal{T}_H$ of the domain Ω. We assume that $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, which guarantees that $V_H \subset V_h$. Moreover, we assume that $H \leq H_s$.

Theorem 2. *Under the above assumptions, we get*

$$K_0 = C(1 + H^2\delta^{-2}), \qquad K_1 = m_c + 1 \tag{20}$$

for both operators A and S_τ.

If $\mathcal{T}_h$ arises as a refinement of a coarser FE division $\mathcal{T}_H$, then the use of $V_0 = V_H$ is fully natural. In other cases, it may be impractical and costly to construct an extra division $\mathcal{T}_H$ together with the interpolation I_H^h and coarse grid operator $A_0 = A_H$. For these cases, it may be advantageous to use another construction of the auxiliary global space V_0 by aggregation.

Let $V = V_h = \mathrm{span}\{\phi_1^h, \ldots, \phi_n^h\}$ and let the index set $\{1, \ldots, n\}$ be decomposed into groups $G_1, \ldots, G_N$. Then it is possible to define an aggregated basis functions ψ_k and the space $V_0 \subset V$ as follows,

$$\psi_k = \sum_{i \in G_k} \phi_i^h, \qquad V_0 = \mathrm{span}\{\psi_1, \ldots, \psi_N\}. \tag{21}$$

We shall assume that the aggregations are regular, i.e. there is a constant $\bar{\beta}$ such that each $\mathrm{supp}\,\psi_k$ contains a ball with diameter $\bar{\beta}H$, where

$$H \sim \max_k \mathrm{diam}(\mathrm{supp}\,\psi_k) \leq H_s.$$

As a consequence, there are positive constants C_1, C_2 such that

$$C_1 H^d \leq |\mathrm{supp}\,\psi_k| \leq C_2 H^d.$$

The space V_0 from (21) together with the local FE spaces $V_1, \ldots, V_s$ from (17) create a new decomposition

$$V = V_0 + V_1 + \ldots + V_s. \tag{22}$$

Theorem 3. *Under the above assumptions, we get*

$$K_0 = C(1 + h^{-1}H + \delta^{-2}H^2), \qquad K_1 = m_c + 1. \tag{23}$$

for both operators A and S_τ.

Note that the space V_0 created by aggregation has been initially introduced in the multigrid context, see e.g. [5]. The properties of the basis functions $\{\psi_k\}$ can be improved by smoothing and the smoothed aggregations can be again used in two-level Schwarz preconditioners, see [8] for a further discussion of this subject.

5 Solution of a T-M problem

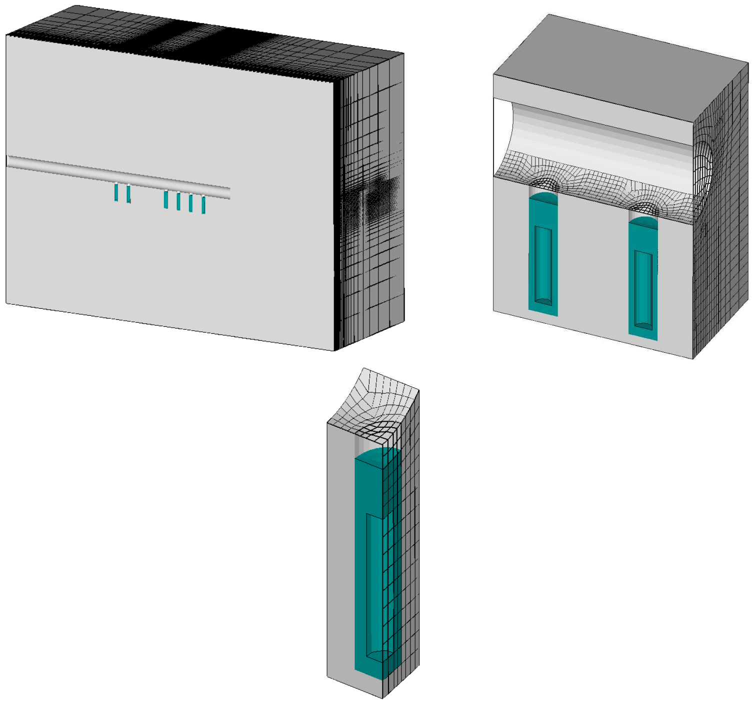

Fig. 2. Large scale KBS-3 model. 158×57×115 m, 391×63×105 nodes.

Further we show, that one-level Schwarz preconditioner is sufficiently efficient for reasonable size of time steps, is valid also for a large-scale problem, which represents a heat conduction in rocks for a prototype nuclear waste repository located at the Äspö Underground Rock Laboratory in Sweden. The computational domain has dimensions 200 × 200 × 100 m. The model is discretized by linear tetrahedral FE with 15 088 320 tetrahedra, *2 586 465 DOF for heat conduction.* Note that the problem has also an elastic part with 7 759 397 DOF, see [10] for more details.

For the Äspö heat conduction analysis, Table 5 shows numbers of iterations required for the solution of the system arising in one time step up to the accuracy $\varepsilon = 10^{-6}$. One level additive Schwarz preconditioner is used with inexact solution (by incomplete factorization) of the subproblems.

#P \ Δt	10^{-4}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}
1	12	12	17	27	39	61	110
4	14	14	17	25	40	68	137
8	16	18	22	25	40	84	167
16	16	18	22	25	41	99	228
24	16	18	22	26	42	97	262

Table 1. Numbers of iterations for accuracy $\varepsilon = 10^{-6}$ in dependence on the time step size Δt and the number of subdomains (processors) #P.

6 Darcy flow problems

The simplest form of the Darcy or porous media flow problem, which corresponds to linearized flow model in a fully saturated porous media, can be again described in the form (1) or (2).

More exactly, a fully saturated time dependent flow can be described by the equation

$$C_S \frac{\partial p}{\partial t} + \nabla \cdot u \, = \, f \qquad \text{in} \quad \Omega \subset R^d \ (d = 2, \, 3), \tag{24}$$

$$u \, = \, -K \nabla (p + z) \qquad \text{in} \quad \Omega \tag{25}$$

where p is the pressure head, u is the Darcy velocity, C_S is the specific storage, $\mathcal{K}$ is the hydraulic conductivity and f is a source/sink term. The equations (24), (25) have to be completed by proper boundary and initial conditions.

The fully saturated problem can be again discretized by finite elements in space and finite differences in time as described in the previous sections and the resulting linear systems can be solved by the Schwarz type domain decomposition methods.

More accurate porous media flow models require some improvements of the model itself and the discretization scheme.

An improvement of the Darcy flow model concerns the flow in both saturated and nonsaturated regions. In this case, both specific storage and the hydraulic conductivity will depend on the pressure p. See eg. [21, 19, 18] for a further discussion.

The arising nonlinear boundary value problem can be again discretized by finite elements in space and the finite differences in time. In each time step, we have to solve a nonlinear algebraic system. The solution can be done by Newton–type method with inexact iterative solution of the arising linear systems. These systems can be again efficiently solved by two–level Schwarz method with coarse grid constructed by aggregation as described in the previous sections, see [19, 18].

7 Mixed formulation

The mixed formulation of the porous media flow means that both pressure p and Darcy velocity u are considered as independent unknowns. This formulation is an origin for the mixed finite element methods with two big advantages over the standard approach: better approximation of fluxes and preservation of the local mass conservation for the approximate solution.

In the simplest case of stationary saturated flow with zero flux through the boundary, the mixed variational formulation has the form:

$$\text{find} \quad (u,p) \in H_0(\text{div},\, \Omega) \times L_2(\Omega),$$

$$m(u,v) + b(v,p) = 0 \qquad \forall v \in H_0(\text{div}, \Omega), \tag{26}$$

$$b(u,q) = -F(q) \qquad \forall q \in L_2(\Omega), \tag{27}$$

where

$$m(u,v) = \int_\Omega \left\langle \mathcal{K}^{-1} u, v \right\rangle \, dx, \tag{28}$$

$$b(u,q) = -\int_\Omega \text{div}(u) q \, dx, \tag{29}$$

$$F(q) = \int_\Omega f q \, dx \tag{30}$$

and

$$H_0(\text{div},\, \Omega) = \{ v \in L_2(\Omega)^d : \;\; \text{div}(v) \in L_2(\Omega),\, v \cdot \nu = 0 \quad \text{on } \partial\Omega \}, \tag{31}$$

where ν denotes the unit outward normal vector to $\partial\Omega$.

The simplest discretization of the above mixed problems uses decomposition of the domain Ω into a set $\mathcal{T}_h$ of triangles or tetrahedra and the lowest order Raviart–Thomas–Nedelec finite element spaces $X_h \subset H_0(\text{div}\,, \Omega)$, $Y_h \subset L_2(\Omega)$,

$$X_h = \{ v \in H_0^1\,(\text{div},\, \Omega) : \;\; v\,|e \; = \begin{pmatrix} \alpha_1 + \beta x_1 \\ \dots \;\; \dots \\ \alpha_d + \beta x_d \end{pmatrix} \quad \forall e \in \mathcal{T}_h \}, \tag{32}$$

$$Y_h = \{ q \in L_2(\Omega) : \;\; q\,|e \; = \; \text{const.} \quad \forall e \in \mathcal{T}_h \}\,. \tag{33}$$

Note that $\alpha_1, \dots, \alpha_d$, β are defined independetly on each $e \in \mathcal{T}_h$. Using an inner products in both X_h and Y_h , e.g. $L_2(\Omega)^d$ and $L_2(\Omega)$ inner products (,), the mixed FE formulation of the model Darcy flow problem gets the following form:

$$\text{find a pair of functions} \quad (u_h, p_h) \in X_h \times Y_h\,,$$

$$\mathcal{A}\begin{bmatrix} u_h \\ p_h \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathcal{F} \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} \mathcal{M} & \mathcal{B}^* \\ \mathcal{B} & 0 \end{bmatrix} \tag{34}$$

where

$$\begin{aligned} (\mathcal{M}v, w) &= m(v, w) & \forall v, w \in X_h \\ (\mathcal{B}v, q) &= b(v, q) & \forall v \in X_h, q \in Y_h \\ (\mathcal{B}^* q, v) &= b(v, q) & \forall v \in X_h, q \in Y_h \\ (\mathcal{F}, q) &= F(q) & \forall q \in Y_h \,. \end{aligned}$$

The system (34) can be rewritten into a matrix form by using proper finite element bases in (32), (33).

In the rest of the paper, we are interested in an efficient and parallelizable solution of the system (34) or more precisely its matrix form

$$A \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ -\varphi \end{pmatrix}, \quad A = \begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix}. \tag{35}$$

For the solution of (35), we would like to apply again a Schwarz domain decomposition method and exploit as much of the experience described in the previous sections as possible.

In this respect, we describe first an approach based on $H(\mathrm{div})$ preconditioner and then discusse some orther approaches.

8 Schwarz method based on $H(\mathrm{div})$ preconditioner

The system (35), which is symmetric, indefinite and regular, can be solved e.g. by the minimum residual (MINRES) or GCG methods with a proper preconditioning. The MINRES [28, 24, 25] and GCG [4] algorithms include matrix–vector multiplication, inner products, vector operations and application of preconditioner. Excluding the preconditioning, all these operations are naturally parallelizable, so that our aim is concentrated in finding a preconditioner, which will be both efficient and parallellizable.

In this respect, we describe a parallelizable preconditioner including a Schwarz domain decomposition technique. This preconditioner seems to be a natural extension of our investigation of Schwarz methods in the previous section. Later, we shall briefly mention some other possible approaches.

The $H(\mathrm{div})$–Schwarz preconditioner can be constructed in two steps. First, the system (35) can be rewritten as

$$A_\eta \begin{bmatrix} u \\ p \end{bmatrix} = - \begin{bmatrix} \eta^{-1} B^T \varphi \\ \varphi \end{bmatrix}, \quad A_\eta = \begin{bmatrix} M_\eta & B^T \\ B & 0 \end{bmatrix} \tag{36}$$

where

$$M_\eta = M + \eta^{-1} B^T B. \tag{37}$$

This reformulation is a basis for introduction and analysis block diagonal $H(\mathrm{div})$ preconditioner C_η for A_η,

$$C_\eta = \begin{bmatrix} M_\eta & 0 \\ 0 & \eta_1 I \end{bmatrix} . \tag{38}$$

This preconditioner is introduced and analysed in [27], where we can found a proof of h-independent spectral equivalence between C_η and A_η. Note that η_1 is a scaling parameter, its standard choice can be $\eta_1 = \eta$.

The second step consists in use of a Schwarz type preconditioner for M_η, i.e.

$$M_\eta^{-1} \sim G_\eta = \sum_{i=(0)1}^{s} R_i^T M_{\eta,i}^{-1} R_i, \qquad C_\eta^{-1} \sim \begin{bmatrix} G_\eta & 0 \\ 0 & \eta_1^{-1} I \end{bmatrix} . \tag{39}$$

The construction of the Schwarz preconditioner M_η is described e.g. in [3]. The considerations start with pointing out that $M_\eta g = r$ represents the finite element representation of the problem: find $g_h \in X_h$ such that

$$\Lambda(g_h, w_h) = (\mathcal{K}^{-1} g_h, w_h) + \eta^{-1}(\mathrm{div}(g_h), \mathrm{div}(w_h)) \qquad \forall w_h \in X_h \tag{40}$$

where (,) denotes again L_2 products, g_h, r_h, w_h are finite element functions isomorphic with the algebraic vectors g, r, w. For completness, (40) corresponds to the differential operator

$$\mathcal{K}^{-1} - \eta^{-1} \mathrm{grad\ div}. \tag{41}$$

After this preliminary consideration, the construction of a Schwarz preconditioner is straightforward. Similarly to the case of elliptic problems, we shall use a coarse grid $\mathcal{T}_H$, fine grid $\mathcal{T}_h$ and decomposition of the domain into nonoverlapping subdomains Ω_i, which are further extended into overlapping ones Ω_i^δ with overlap δ. Then X_h can be decomposed as follows

$$X_h = X_0 + X_1 \ldots + X_s, \tag{42}$$

$$X_0 = X_H, \qquad X_i = \{v \in X_h : w = 0 \text{ in } \Omega \setminus \bar{\Omega}_i^\delta\} \tag{43}$$

and the restriction operators R_i and subproblem matrices $M_{\eta,\,i}$ can be constructed according to this decomposition. In [3], it is shown that for $\mathcal{K} = I$, $\eta = 1$ and the two-level Schwarz preconditioner, G_η^{-1} is spectrally equivalent to M_η in the H(div) inner product independently on the parameters H and h. Results from analysis of the one-level Schwarz method can be found e.g. in [26]

9 Other approaches

Let us briefly describe some other approaches.

- Instead of the H(div) preconditioner $C_\eta = C_{div}$, it is possible to consider a Schur complement preconditioner in the form

$$C_{schur} = \begin{bmatrix} M & 0 \\ 0 & BM^{-1}B^T \end{bmatrix} \quad \text{or} \quad C_{schur} = \begin{bmatrix} M_\eta & 0 \\ 0 & BM_\eta^{-1}B^T \end{bmatrix}. \tag{44}$$

  The application of this preconditioner is a little more complicated. On the other hand, this type of preconditioner can be advantageously combined with the mixed-hybrid form (see e.g. [20, 25]) of the finite element discretization.
- Instead of solving the saddle point system (35), it is possible to eliminate velocity and solve an SPD system for the Schur complement

$$BM^{-1}B^T p = -\varphi. \tag{45}$$

  In a contrary to a realization of the Schur preconditioner (44), the inverse M^{-1} or M_η^{-1} should be done here more accurately [4] as it contributes not to the preconditioner but to the matrix of the solved problem.
- It is also possible to eliminate the pressure variable by constructing the null space null(B) and expressing the solution of $Bu = \varphi$ as a particular solution u_0 plus the null space contribution Zw, where Z is a matrix with columns defining a basis of null(B). Then

$$Mu_0 + MZw + B^T p = 0$$

  which implies

$$Z^T MZw = -Z^T Mu_0. \tag{46}$$

  The construction of Z can be done either analytically [25] or algebraically [1, 2]. The system (46) is again SPD and can be solved also by Schwarz type methods [23, 16, 25, 17, 26].

10 Concluding remarks

We described applications of Schwarz type methods for solving T-H-M problems. In Sections 2-5, these applications concern elliptic and parabolic problems of heat flow and elasticity discretizad by standatd finite element methods. The application of Schwarz methods to these problems is now more less standard. Our contribution can be seen in algebraically constructed coarse grid by aggregation and investigation of inexact solution of subproblems including inner CG iterations for solving coarse grid problem. Note that the algebraic coarse grid allows to develop a parallel black box like solver and that such type of solver was successfully implemnted in our in-house FEM software.

The second part of the paper concerns application of Schwarz type methods to Darcy flow problems discretized by a mixed FE method. Our research in this field is in the beginning and will continue e.g. by using aggregations or investigation of suitable choices of parameters, which among others can again increase efficiency of the one-level Schwarz method.

Acknowledgement: The work is supported by the project 1M4674788502 of the Ministry of Education of the Czech Republic and the grant 1ET400300415 of the Academy of Sciences of the Czech Republic.

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