Chapter 1

Solution of boundary variational inequalities by combining fast quadratic programming algorithms with symmetric BEM

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Abstract. *Efficient algorithms for the solution of elliptic boundary variational inequalities are presented. A model variational inequality is first reduced to minimization of the energy functional defined on the boundary subject to inequality constraints. The problem is then discretized by the Galerkin method using the well known results on representation of the Steklov-Poincaré operator. Using the duality theory, we finally get well a conditioned, strictly convex quadratic programming problem with either bound or bound and equality constraints. The resulting problem is then solved by new efficient algorithms with the rate of convergence in the spectral condition number of the Hessian. Both theoretical results and numerical experiments indicate efficiency of the algorithms presented.*
Keywords *Boundary elements, variational inequality, efficient algorithm.*

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1.1 Introduction

We shall be concerned with the elliptic boundary variational inequalities such as those describing the equilibrium of a system of elastic bodies in mutual contact. Such problems are difficult to solve because their boundary conditions involve the inequalities which make them strongly nonlinear. Moreover, the classical Dirichlet and Neumann boundary conditions are known only after the solution has been found, so that the analysis should not rely on the ellipticity of the related

quadratic forms. Translated into the language of mechanics, it follows that any effective solver of such problems must be able not only to identify an a priori unknown contact interface, but also to deal with singular matrices describing the response of "floating" bodies to given traction.

The discretization of contact problems may be based both on the finite element discretization [2] or on the boundary element discretization [6, 3], each approach having its own merits. For example, our development of boundary element based method has been motivated by possible applications in shape optimization, where the finite element method requires remeshing of the interior of the body with adverse effect on the precision of shape gradients [5]. Using appropriate discretization methods, it is possible to achieve that the discretized problems have similar structure, so that they may be solved by the same algorithms.

In this paper, we describe application of the quadratic programming algorithms [4, 1] that were used to develop scalable methods (e.g. [2, 1]) for variational inequalities. A model variational inequality is first reduced to minimization of the energy functional defined on the boundary subject to inequality constraints. The problem is then discretized by the Galerkin method using the well known results on representation of the Steklov-Poincaré operator. Using the duality theory, we finally reduce the problem to rather well conditioned, strictly convex quadratic programming problem with either bound constraints or bound and equality constraints so that our efficient algorithms [4, 1] may be applied. The unique feature of these algorithms is the rate of convergence in terms of the spectral condition number of the Hessian of the related quadratic form. Both theoretical results and numerical experiments indicate that there are problems which may be solved efficiently by the algorithms presented.

1.2 A model problem.

To simplify our exposition, we shall restrict our presentation to a simple model problem. Let $\Omega \subset \mathbb{R}^2$ denote a bounded domain with the Lipschitz boundary Γ which consists of three mutually disjoint parts Γ_u, Γ_f and Γ_c, where we admit $\Gamma_u = \emptyset$, but always assume that Γ_c has nonzero measure. Let $f \in L^2(\Omega)$, where $L^2(\Omega)$ denote the space of the square integrable functions on Ω in the sense of Lebesgue, and let $g \in L^2(\Gamma_c)$. We shall consider the problem to find u which satisfies the classical Poisson equation and boundary conditions

$$-\triangle u = f \quad \in \Omega, \quad u = 0 \ \text{ on } \ \Gamma_u \quad \text{ and } \quad \frac{\mathrm{d}u}{\mathrm{d}n} = 0 \ \text{ on } \ \Gamma_f, \tag{1.1}$$

where n denotes the outer unit normal vector defined on Γ, and the Signorini conditions

$$u \geq g, \quad \frac{\mathrm{d}u}{\mathrm{d}n} \geq 0 \quad \text{ and } \quad \frac{\mathrm{d}u}{\mathrm{d}n}(u - g) = 0. \tag{1.2}$$

The solution u may be interpreted as a vertical displacement of a membrane which is placed over the boundary obstacle Γ_c, fixed on Γ_u and pressed down by the traction f.

To give the weak formulation of the problem (1.1) - (1.2), let

$$H_0^1(\Omega, \Gamma_u) = \left\{ v \in H^1(\Omega) : \ Tv = 0 \text{ on } \Gamma_u \right\} \quad \text{and} \quad \mathcal{K} = \left\{ v \in H_0^1(\Omega, \Gamma_u) : \ Tv \geq g \text{ on } \Gamma_c \right\}$$

denote the closed subspace of the Sobolev space $H^1(\Omega)$ and the closed convex subset of $H_0^1(\Omega, \Gamma_u)$, respectively, where $Tv \in L^2(\Gamma)$ denotes the trace of a function v, and let us define a continuous symmetric bilinear form a and a continuous linear functional ℓ by

$$a : \ \left(H_0^1(\Omega, \Gamma_u)\right)^2 \mapsto \mathbb{R}, \qquad a(u, v) = \int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x \qquad \ell : \ H_0^1(\Omega, \Gamma_u) \mapsto \mathbb{R}, \ \ell(v) = \int_\Omega f \cdot v \ \mathrm{d}x.$$

Then we call the function $u \in \mathcal{K}$ satisfying

$$a(u, v - u) \geq \ell(v - u) \qquad \text{for all } v \in \mathcal{K} \tag{1.3}$$

a *weak solution* of the problem (1.1) - (1.2). If a is $H_0^1(\Omega,\Gamma_u)$-elliptic, then there is the unique solution $u \in \mathcal{K}$ of (1.3) which is also the unique minimizer of the energy functional

$$J:\ H_0^1(\Omega,\Gamma_u) \mapsto \mathbb{R}, \qquad J(v) = \frac{1}{2}a(v,v) - \ell(v)$$

on $\mathcal{K}$. It can be shown that J is coercive if either Γ_u has a positive measure or $\int_\Omega f\ \mathrm{d}x < 0$. In what follows, we assume that J is coercive so that (1.1) - (1.2) has the unique solution.

1.3 Reduction to the boundary and discretization

The fundamental solution of the Laplace operator in 2D is given by

$$v:\ \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}, \qquad v(x,y) = -\frac{1}{2\pi}\log\|x-y\|.$$

If both the solution u of the partial differential equation (1.1) and its normal derivative $\frac{\mathrm{d}u}{\mathrm{d}n}$ are known on the whole boudary Γ, then we can compute the solution u in any interior point $x \in \Omega$ by the Green representation formula

$$u(x) = \int_\Gamma \left(\frac{\mathrm{d}u}{\mathrm{d}n}(y)v(x,y) - u(y)\frac{\mathrm{d}v}{\mathrm{d}n_y}(x,y)\right)\ \mathrm{d}s_y + \int_\Omega f(y)v(x,y)\ \mathrm{d}y. \tag{1.4}$$

Let us now introduce the operators V, K, K', D and the Newton potentials N_0, N_1 by

$$\begin{aligned}
&V: H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma), && V\left(\frac{\mathrm{d}u}{\mathrm{d}n}\right)(x) = \int_\Gamma \frac{\mathrm{d}u}{\mathrm{d}n}(y)v(x,y)\ \mathrm{d}s_y,\\
&K: H^{1/2}(\Gamma) \mapsto H^{1/2}(\Gamma), && K(u)(x) = \int_\Gamma u(y)\frac{\mathrm{d}v}{\mathrm{d}n_y}(x,y)\ \mathrm{d}s_y,\\
&K': H^{-1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma), && K'\left(\frac{\mathrm{d}u}{\mathrm{d}n}\right)(x) = \int_\Gamma \frac{\mathrm{d}u}{\mathrm{d}n}(y)\frac{\mathrm{d}v}{\mathrm{d}n_x}(x,y)\ \mathrm{d}s_y,\\
&D: H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma), && D(u)(x) = -\frac{\mathrm{d}}{\mathrm{d}n_x}\int_\Gamma u(y)\frac{\mathrm{d}v}{\mathrm{d}n_y}(x,y)\ \mathrm{d}s_y,\\
&N_0 f(x) = \int_\Omega f(y)v(x,y)\ \mathrm{d}y, && N_1 f(x) = \int_\Omega f(y)\frac{\mathrm{d}v}{\mathrm{d}n_x}(x,y)\ \mathrm{d}y.
\end{aligned}$$

Here $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ denote the trace space of $H^1(\Omega)$ with the norm

$$\|u\|_{H^{1/2}(\Gamma)} = \inf_{v\in H^1(\Omega):\ Tv=u} \|v\|_{H^1(\Omega)}$$

and its dual space, respectively. We call V a single layer operator, K a double layer operator, K' an adjoint to K and D a hypersingular operator. It is known (see [11]) that the operators V, K, K' and D are linear and continuous. Moreover, V is also symmetric, and if $\mathrm{diam}(\Omega)$ is sufficiently small, then it is also $H^{-1/2}(\Gamma)$-elliptic. Finally, operator D is symmetric, $H^{1/2}(\Gamma)$-semielliptic and if Γ_u has a positive measure, then it is also $H_0^{1/2}(\Gamma,\Gamma_u)$-elliptic, where

$$H_0^{1/2}(\Gamma,\Gamma_u) = \left\{v \in H^{1/2}(\Gamma):\ v = 0 \text{ on } \Gamma_u\right\}.$$

Letting $\tilde{x} \in \Omega$ pass to $x \in \Gamma$, we can derive from (1.4) the system of integral equations

$$\frac{1}{2}u = V\left(\frac{\mathrm{d}u}{\mathrm{d}n}\right) - K(u) + N_0 f, \tag{1.5}$$

$$\frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}n} = K'\left(\frac{\mathrm{d}u}{\mathrm{d}n}\right) + D(u) + N_1 f \tag{1.6}$$

for u and $\frac{\mathrm{d}u}{\mathrm{d}n}$ on Γ. Using simple manipulations with (1.5) and (1.6), we get

$$\frac{\mathrm{d}u}{\mathrm{d}n}(x) = S(u)(x) - N(f)(x) \quad \text{for } x \in \Gamma, \tag{1.7}$$

where S denotes the Steklov-Poincaré operator

$$S:\ H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma), \qquad S(u) = \left[\left(\frac{1}{2}I + K'\right) V^{-1} \left(\frac{1}{2}I + K\right) + D\right](u)$$

and $N:\ L^2(\Omega) \mapsto H^{-1/2}(\Gamma)$ is defined by $N(f) = V^{-1}N_0 f$.

The Steklov-Poincaré operator S is known to be linear, continuous, symmetric and $H^{1/2}(\Gamma)$-semielliptic. Again, if Γ_u has a positive measure, then S is $H_0^{1/2}(\Gamma, \Gamma_u)$-elliptic. Using the Green's formula and (1.7), we get that the problem of finding the weak solution of the model problem (1.1) - (1.2) is equivalent to the problem to

$$\text{find } u \in \mathcal{K} \text{ such that } \int_\Gamma S(u)\cdot(v-u)\ \mathrm{d}s \geq \int_\Gamma N(f)\cdot(v-u)\ \mathrm{d}s \quad \text{for all } v \in \mathcal{K}. \tag{1.8}$$

Using suitable grid points $z_i \in \Gamma$ and test functions $\varphi_i, \varphi_i(z_i) = \delta_{ij}, i, j = 1, \ldots, n_0$, we can use the Galerkin method to obtain the discretized formulation of (1.8). The boundary element matrices V_h, K_h and D_h can be evaluated using analytical integration in combination with numerical integration schemes. For evaluation of the vector N_{0_h}, it is necessary to compute the Newton potential $N_0 f$. This can be done by an indirect computation approach using FEM (see [10]). We can exploit the symmetry of the matrices V_h and D_h and equations

$$\frac{1}{2}\sum_{j=1}^{n_0} M_h(i,j) = -\sum_{j=1}^{n_0} K_h(i,j) \quad \text{and} \quad \sum_{j=1}^{n_0} D_h(i,j) = 0 \qquad \text{for } i = 1, \ldots, n_0, \tag{1.9}$$

which follow from (1.5) and (1.6). Assuming a suitable numbering of nodes and eliminating v_i that correspond to $z_i \in \Gamma_u$, we shall get the discrete energy functional

$$J(v) = \frac{1}{2}v^{\mathrm{T}}\tilde{S}_h v - \tilde{R}_h^{\mathrm{T}} v \quad \text{for } \ v \in \mathbb{R}^{n_1} \tag{1.10}$$

and the approximation $\mathcal{K}_h = \{v \in \mathbb{R}^{n_1} : v_i \leq g(z_i) \text{ for } i = q, \ldots, n\}$, so that the discretized formulation of (1.8) may be give in the form

$$J(v) \longrightarrow \min \quad \text{subject to } \ Bv \leq c, \tag{1.11}$$

where $B = [-I, O] \in \mathbb{R}^{n \times n_1}$, n is the number of the nodes on Γ_c, and $c_i = -g(z_i)$. Since we assume that the Dirichlet conditions are enhanced in J, the matrix $\tilde{S}_h$ is positive definite if there is $z_i \in \Gamma_u$. Otherwise $\tilde{S}_h$ is only positive semidefinite.

Our final step is elimination of primal variables by duality in order to improve conditioning of our problem. This step would reduce more general constraints, such as those arising in multibody problems, to the bound constraints. We first assume that the problem is coercive, so that the stiffness matrix $\tilde{S}_h$ is positive definite. We get that the problem (1.11) is equivalent to

$$\begin{aligned} &\Theta(\lambda) \ \longrightarrow \min \quad \text{subject to } \lambda \geq 0, \\ \Theta:\ \mathbb{R}^n \mapsto \mathbb{R}, \qquad &\Theta(\lambda) \ = \frac{1}{2}\lambda^{\mathrm{T}} F \lambda - \lambda^{\mathrm{T}} b, \quad F = B\tilde{S}_h^{-1}B^{\mathrm{T}}, \quad b = \left(B\tilde{S}_h^{-1}\tilde{R}_h - c\right). \end{aligned} \tag{1.12}$$

The solution $u \in \mathbb{R}^n$ on the boundary is then determined by the unique solution $\overline{\lambda}$ of (1.12):

$$u = \tilde{S}_h^{-1}\left(\tilde{R}_h - B^{\mathrm{T}}\overline{\lambda}\right).$$

Now we shall assume that $\Gamma_u = \emptyset$ so that the matrix $\tilde{S}_h$ is positive semidefinite, but we shall assume that the energy functional (1.10) is coercive due to the linear term. Let us denote by $\tilde{S}_h^+$ any matrix that satisfies $\tilde{S}_h^+ = \tilde{S}_h \tilde{S}_h^+ \tilde{S}_h$ and by R any matrix whose columns span the null space of $\tilde{S}_h$. Then (see [3]) the problem (1.11) is equivalent to the problem:

$$\begin{aligned} \Theta(\lambda) &\longrightarrow \min \quad \text{subject to } \lambda \geq 0 \text{ and } G\lambda = e, \\ \Theta(\lambda) &= \frac{1}{2}\lambda^{\mathrm{T}} F \lambda - \lambda^{\mathrm{T}} b,\ F = B\tilde{S}_h^+ B^{\mathrm{T}},\ b = \left(B\tilde{S}_h^+ \tilde{R}_h - c\right),\ G = R^T B^T,\ e = R^{\mathrm{T}} \tilde{R}_h. \end{aligned} \tag{1.13}$$

Again, the discrete boundary solution $u \in \mathbb{R}^n$ may be recovered from the solution $\overline{\lambda}$ of (1.13) by

$$u = \tilde{S}_h^+ \left(\tilde{R}_h - B^{\mathrm{T}}\overline{\lambda}\right) + R\left(R^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{B}R\right)^{-1} R^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\left(\tilde{c} - \tilde{B}\tilde{S}_h^+\left(\tilde{R}_h - B^{\mathrm{T}}\overline{\lambda}\right)\right),$$

where $(\tilde{B}, \tilde{c})$ is formed by the rows of (B, c) that correspond to the positive entries of $\overline{\lambda}$. It may be useful to homogenize the equality constraints in (1.13) by using any $\tilde{\lambda}$ which satisfies $G\tilde{\lambda} = e$ and substituting $\lambda = \tilde{\lambda} + \mu$ into (1.13). The details and tanalysis of the effect may be found in the reference [1].

1.4 Algorithms

To simplify the description of our working set based algorithm [4] for the solution of (1.12), let $\mathcal{N} = \{1, \ldots, n\}$ and let $g = g(\lambda) = \nabla\Theta(\lambda)$ denote the gradient of Θ at $\lambda \in \mathbb{R}^n$. The unique solution $\overline{\lambda}$ of (1.12) is fully determined by the Karush-Kuhn-Tucker (KKT) optimality conditions

$$\overline{\lambda}_i = 0 \ \text{ implies } \ \overline{g}_i \geq 0, \quad \text{and } \overline{\lambda}_i > 0 \ \text{ implies } \overline{g}_i = 0. \tag{1.14}$$

The set of all indexes $i \in \mathcal{N}$ for which $\lambda_i = 0$ is called an *active set* of λ. We denote it by $\mathcal{A}(\lambda) = \{i \in \mathcal{N} : \lambda_i = 0\}$. The complement $\mathcal{F}(\lambda) = \mathcal{N} \setminus \mathcal{A}(\lambda)$ of $\mathcal{A}(\lambda)$ will be called a *free set* of λ. To enable an alternative reference to the KKT conditions (1.14), let $\varphi(\lambda)$ be the *free gradient* of λ and let $\beta(\lambda)$ be the *chopped gradient* of λ defined by

$$\varphi_i(\lambda) = \begin{cases} g_i(\lambda), & \text{for } i \in \mathcal{F}(\lambda) \\ 0, & \text{for } i \in \mathcal{A}(\lambda) \end{cases} \quad \text{and} \quad \beta_i(\lambda) = \begin{cases} 0, & \text{for } i \in \mathcal{F}(\lambda) \\ g_i^-(\lambda), & \text{for } i \in \mathcal{A}(\lambda) \end{cases}$$

Thus the KKT conditions (1.14) are satisfied if and only if the *projected gradient* $g^P(\lambda) = \varphi(\lambda) + \beta(\lambda)$ is equal to zero. We call λ *feasible* if $\lambda_i \geq 0$ for $i \in \mathcal{N}$. The projection P_+ to the set of feasible vectors is defined for any n-vector λ by $P_+(\lambda)_i = \max\{\lambda_i, 0\}$. Our algorithm [4] for the solution of (1.12) uses a test to decide about leaving the face and three types of steps to generate a sequence of iterates $\{\lambda^k\}$ that approximate the solution of (1.12).

The *expansion step* is defined by

$$\lambda^{k+1} = P_+\left(\lambda^k - \overline{\alpha}\varphi(\lambda^k)\right) \tag{1.15}$$

with the steplength $\overline{\alpha} \in (0, \|F\|^{-1}]$, F denoting the Hessian of Θ_c. This step may expand the current active set. We may describe it without P_+ by introducing the *reduced free gradient* $\widetilde{\varphi}(\lambda)$ with the entries $\widetilde{\varphi}_i = \widetilde{\varphi}_i(\lambda) = \min\{\lambda_i/\overline{\alpha}, \varphi_i\}$ for $i \in \mathcal{N}$, so that $P_+(\lambda - \overline{\alpha}\varphi(\lambda)) = \lambda - \overline{\alpha}\widetilde{\varphi}(\lambda)$. If the inequality

$$||\beta(\lambda^k)||^2 \leq \Gamma^2 \widetilde{\varphi}(\lambda^k)^{\top}\varphi(\lambda^k) \tag{1.16}$$

holds, then we call the iterate λ^k *strictly proportional*. The test (1.16) is used to decide which component of the projected gradient $\nu(\lambda^k)$ will be reduced in the next step.

The *proportioning step* is defined by

$$\lambda^{k+1} = \lambda^k - \alpha_{cg}\beta(\lambda^k).$$

The steplength α_{cg} is chosen to minimize $f(\lambda^k - \alpha\beta(\lambda^k))$ with respect to α. The purpose of the proportioning step is to remove indexes from the active set.

The *conjugate gradient step* is defined by

$$\lambda^{k+1} = \lambda^k - \alpha_{cg}p^k \tag{1.17}$$

where p^k is the conjugate gradient direction which is constructed recurrently. The conjugate gradient steps are used to carry out the minimization in the face $\mathcal{W}_{\mathcal{I}} = \{\lambda : \lambda_i = 0 \text{ for } i \in \mathcal{I}\}$ given by $\mathcal{I} = \mathcal{A}(\lambda^s)$ efficiently. The algorithm that we use may now be described as follows.

Algorithm MPRGP for bound constrained problems
Step 0. {*Initialization of parameters*}
Let λ^0 be a feasible n-vector, $\overline{\alpha} \in (0, \|F\|^{-1}]$, and $\Gamma > 0$ be given. For $k \geq 0$ and λ^k known, choose λ^{k+1} by the following rules:
Step 1. {*Test.*}
If $\|g^P(\lambda^k)\|$ is small, then $\lambda^{k+1} = \lambda^k$.
Step 2. {*Proportional iteration*}
If λ^k is strictly proportional and $g^P(\lambda^k) \neq 0$, try to generate λ^{k+1} by the conjugate gradient step. If $\lambda_i^{k+1} \geq 0$ for $i \in \mathcal{N}$, then accept it, else generate λ^{k+1} by the expansion step.
Step 3. {*Proportioning*}
If λ^k is not strictly proportional, define λ^{k+1} by proportioning.

More details about the implementation of Algorithm MPRGP may be found in [4]. This algorithm was proved to converge for any set of initial parameters that satisfy the prescribed inequalities. Its unique feature is the R-linear rate of convergence of both $\|\lambda^k\|$ and $\|g^P(\lambda^k)\|$ in terms of the condition number of the Hessian of Θ_c [4].

To exploit MPRGP for the solution of (1.13), we shall introduce Lagrange multipliers μ for the equality constraints so that the augmented Lagrangian for (1.13) and its gradient read

$$L(\lambda, \mu, \rho) \;=\; \frac{1}{2}\lambda^T(PFP + \rho Q)\lambda \;-\; \lambda^T Pd \;+\; \mu^T G\lambda, g(\lambda, \mu, \rho) \;=\; PFP\lambda - Pd + G^T(\mu + \rho G\lambda).$$

Algorithm SMALBE for bound and equality constrained problems.
Step 0. {*Initialization of parameters*}
Given $\eta > 0,\ \beta > 1,\ M > 0, \rho_0 > 0$ and μ^0, set $k = 0$.
Step 1. {*Inner iteration with adaptive precision control.*}
Find λ^k such that $||g^P(\lambda^k, \mu^k, \rho_k)|| \leq \min\{M\|G\lambda^k\|, \eta\}$.
Step 2. {*Stopping criterion.*}
If $||g^P(\lambda^k, \mu^k, \rho_k)||$ and $||G\lambda^k||$ are sufficiently small, then λ^k is the solution.
Step 3. {*Update of the Lagrange multipliers.*}
$\mu^{k+1} = \mu^k + \rho_k G\lambda^k$
Step 4. {*Update the penalty parameter.*}
If $k > 0$ and $L(\lambda^k, \mu^k, \rho^k) < L(\lambda^{k-1}, \mu^{k-1}, \rho_{k-1}) + \rho_k\|C\lambda^k\|^2/2$
then $\rho_{k+1} = \beta\rho_k$
else $\rho_{k+1} = \rho_k$
end if.
Step 5. Increase k and return to Step 1.

The implementation of Step 1 is carried out by the minimization of the augmented Lagrangian L subject to $\lambda \geq 0$ by means of the MPRGP algorithm. The SMALBE algorithm was proved to converge for any set of parameters that satisfy the relations prescribed in [1]. Moreover, it was shown that the number of iterations necessary to achieve the prescribed relative feasibility error may be bounded independently of the conditioning of the constraints. The results are true even for dependent constraints [1].

1.5 Numerical experiments

In this section, we illustrate the performance of our algorithm on problems defined by

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 < 1\}, \qquad f(x,y) = -1 \ \text{ for } \ (x,y) \in \Omega,$$
$$\Gamma_c = \{(x,y) \in \Gamma : \ |x| \leq \frac{\sqrt{2}}{2} \text{ and } y < 0\}, \quad g(x,y) = \sqrt{1/2 - x^2} - \sqrt{2}/2 - 0.6.$$

We used the the stopping criterion $\left\|g^P(\lambda^k)\right\| \leq 10^{-5}$, $\Gamma = 1$ and $\overline{\alpha} = 0.5\|F\|^{-1}$. We started from λ^0 equal to the vector of ones. For solving (1.13), we used for feasibility the criterion $\|G\lambda^k\| < 10^{-5}$, the parameters $\eta = 10^{-5}$, $\beta = 10^2$, $M = 1$ and initiations $\rho^0 = 10^2$ and $\mu^0 = 0$. The solutions of the coercive (1.12) and semicoercive (1.13) problems discretized by 128 nodes we obtained after 9 and 45 conjugate gradient iterations, respectively. The solutions are in Fig. (1.5).

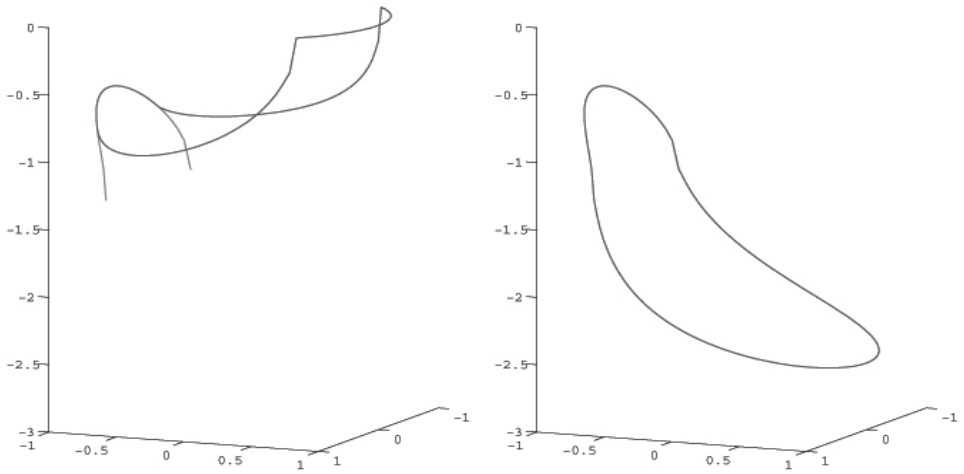

Figure 1.1: Solution of the coercive (1.12) (left) and semicoercive (1.13) (right) problems.

1.6 Comments and conclusions

The boundary element discretization of elliptic boundary variational inequality has been described which complies with recently proposed algorithms for bound (and equality) constrained quadratic programming. If applied to a multidomain problem, our algorithms may also be classified as a duality based domain decomposition method closely related to FETI [2]. Moreover, using the technique developed in domain decomposition methods, it is even possible to prove a kind of optimality results, namely that a system of similar bodies may be solved to a given relative precision in a number of iterations which is independent of the number of subdomains. The algorithms may be useful also for the solution of problems that are discretized partly by the

finite element method and partly by the boundary element method. The algorithm may also be modified to comply with the BETI methods [8].

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