A Scalable FETI–DP Algorithm for a Semi–coercive Variational Inequality

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Abstract

We develop an optimal algorithm for the numerical solution of semi-coercive variational inequalities by combining dual-primal FETI algorithms with recent results for bound and equality constrained quadratic programming problems. The discretized version of the model problem, obtained by using the FETI–DP methodology, is reduced by the duality theory of convex optimization to a quadratic programming problem with bound and equality constraints, which is solved by a new algorithm with a known rate of convergence given in terms of the spectral condition number of the quadratic problem. We present convergence bounds that guarantee the scalability of the algorithm. These results are confirmed by numerical experiments.

1 Introduction

The Finite Element Tearing and Interconnecting (FETI) method was originally proposed by Farhat and Roux [27] as a parallel solver for problems described by elliptic partial differential equations. The computational domain is decomposed (teared) into non-overlapping subdomains that are “glued” by Lagrange multipliers. After eliminating the primal variables by solving possibly singular local problems, the original problem is reduced to a small, relatively well conditioned, typically equality constrained quadratic programming problem that is solved iteratively. Observing that the equality constraints define a so-called “natural coarse grid”, Farhat, Mandel and Roux [26] modified the basic FETI method to obtain a numerically scalable algorithm. The efficiency of the FETI method was further improved by preconditioning [31,42]. To improve the convergence of the FETI method for plate and shell problems, Farhat, Mandel and Tezaur [25,38] suggested to

1 Corresponding author: Zdeněk Dostál. The first two authors were supported by Grant 101/04/1145 of the GA CR, by Grant S3086102 of GA CAS, and by Projects 1ET400300415 and ME641 of the Ministry of Education of the Czech Republic. The third author was supported by the National Science Foundation Grant NSF-DMS-0103588 and by the Research Foundation of the City University of New York Awards PSC-CUNY 665463-00 34 and 66529-00 35.
project the Lagrange multipliers in each iteration onto an auxiliary space to enforce continuity of the primal solutions at the crosspoints.

A similar effect was achieved by the Dual-Primal FETI method (FETI-DP) introduced by Farhat et al. [24]; see also [32]. The continuity of the primal solution at crosspoints is implemented directly into the formulation of the primal problem by considering one degree of freedom at each crosspoint shared by more than two adjacent subdomains. Across the rest of the subdomain interfaces, the continuity of the primal solution is once again enforced by Lagrange multipliers. After eliminating the primal variables, the problem is reduced to a small, unconstrained, relatively well conditioned strictly convex quadratic programming problem that is solved iteratively. An attractive feature of FETI-DP is that the local problems that are solved to eliminate the primal variables are nonsingular. Moreover, the conditioning of the resulting quadratic programming problem may be further improved by preconditioning [40]. Recently, close connections between the FETI-DP method and the balancing domain decomposition method were uncovered [36,37].

The FETI method is also successful for solving variational inequality problems, since duality transforms the general inequality constraints into non-negativity constraints, and efficient algorithms that employ inexpensive projections may be used. Numerical scalability was shown experimentally by Dureisseix and Farhat [23] for algorithms using a coarse grid initial approximation, and by Dostál et al. [12,13,15] for algorithms based on the application of special solvers [6,11]. Scalability was also proved theoretically for an algorithm that combined FETI with optimal dual penalty [17,18].

A new Lagrange multipliers algorithm, FETI–C, based on FETI–DP and on active set strategies with additional planning steps, was introduced by Farhat et al. [1,23,41,47]. The scalability of FETI–C was established experimentally. Using recent algorithms for the solution of bound constrained problems [7,22], the present authors proved the numerical scalability for a FETI–DP algorithm for coercive problems [19] and extended the result to include mortar discretizations [20].

In this paper, we use the FETI–DP method to develop a scalable algorithm for the numerical solution of a semi-coercive variational inequality. We recall that the variational inequality describing the equilibrium of a system of bodies in contact is semicoercive if floating bodies exist. In this case, the application of FETI–DP methodology results in a convex quadratic programming problem with a positive semidefinite Hessian matrix and bound and equality constraints. For contact problems, the kernel of the Hessian matrix is spanned by the rigid body motions of the floating bodies. We solve this problem by using a recently proposed semi-monotonic augmented Lagrangian algorithm [8,9]. The rate of convergence of this algorithm can be estimated in terms of the bounds on the spectrum of the regular part of the Hessian of the quadratic cost function. We derive such bounds in terms of the decomposition parameter $H$ and of the discretization parameter $h$, therefore establishing the scalability of our algorithm.

We also obtain a bound on the number of conjugate iterations required for finding the solution of the discretized variational inequality to a given precision. This bound is independent of both the decomposition of the computational domain and the discretization,
provided that we the ratio $H/h$ is kept fixed. We report numerical results that are in agreement with the theory and confirm the numerical scalability of our algorithm.

Several extensions of our method are possible and work on these problems is currently in progress. For example, a faster algorithm might be obtained by using a preconditioner in the conjugate gradient step; see recent work by Dostal and Lesoinne [21]. Another possibility is to use a mortar finite element discretization of the computational domain [3, 48]; see also, e.g., [1, 45, 46]. To minimize the inherent extra computational effort required to compute the mortar conditions, every subdomain would be discretized with continuous elements. The mesh would be unstructured only across the contact interface, where mortar conditions will be required. In [20], a FETI-DP algorithm using mortar discretizations was proven to be numerically scalable when solving coercive problems.

We note that the effort to develop scalable solvers for variational inequalities was not restricted to FETI-type methods. For example, multigrid ideas were used early on by Mandel [35], Kornhuber, Krause and Wohlmuth [33, 34, 49] introduced an algorithm based on monotone multigrid with scalable solution of auxiliary linear problems. Combining multigrid ideas and approximate projections, Schöberl [43, 44] introduced an algorithm for which linear complexity was established.

2 Model problem

The computational domain for our model problem is $\Omega = \Omega^1 \cup \Omega^2$, where $\Omega^1 = (0, 1) \times (0, 1)$ and $\Omega^2 = (1, 2) \times (0, 1)$, with boundaries $\Gamma^1$ and $\Gamma^2$, respectively. We denote by $\Gamma_u^i$, $\Gamma_f^i$, and $\Gamma_c^i$ the fixed, free, and potential contact parts of $\Gamma^i$, $i = 1, 2$; see Figure 1a. Note that $\Gamma_u^2 = \emptyset$. Let $\Gamma_c = \Gamma_c^1 \cup \Gamma_c^2$.

The Sobolev space of the first order on $\Omega^i$ is denoted by $H^1(\Omega^i)$ and the space of Lebesgue square integrable functions is denoted by $L^2(\Omega^i)$. Let $V = V^1 \times V^2$, with

$$V^i = \{ v^i \in H^1(\Omega^i) : v^i = 0 \quad \text{on} \quad \Gamma_u^i \}, \quad i = 1, 2.$$

Let $\mathcal{K} \subset V$ be a closed convex subset of $\mathcal{H} = H^1(\Omega^1) \times H^1(\Omega^2)$ defined by

$$\mathcal{K} = \{ (v^1, v^2) \in V : v^2 - v^1 \geq 0 \quad \text{on} \quad \Gamma_c \}.$$

We define the symmetric bilinear form $a(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to R$ by

$$a(u, v) = \sum_{i=1}^{2} \int_{\Omega^i} \left( \frac{\partial u^i}{\partial x_1} \frac{\partial v^i}{\partial x_1} + \frac{\partial u^i}{\partial x_2} \frac{\partial v^i}{\partial x_2} \right) dx.$$

Let $f \in L^2(\Omega)$ be a given function and $f^i \in L^2(\Omega^i)$, $i = 1, 2$, be the restrictions of $f$ to $\Omega^i$, $i = 1, 2$. We define the linear form $l(\cdot) : \mathcal{H} \to R$ by

$$l(v) = \sum_{i=1}^{2} \int_{\Omega^i} f^i v^i dx.$$
and consider the following problem:

\[
\text{Find } \min \frac{1}{2} a(u, u) - \ell(u) \quad \text{subject to } u \in K. \tag{1}
\]

![Fig. 1a: Semi-coercive model problem](image)
![Fig. 1b: Decomposition: $H = .5, H/h = 3$](image)

The solution of the model problem may be interpreted as the displacement of two membranes under the traction $f$. The left membrane $\Omega^1$ is fixed on the left edge as in Figure 1a. The left edge of $\Omega^2$ is not allowed to penetrate below the right edge of $\Omega^1$. The energy function minimized in (1) is coercive if and only if

\[
\int_{\Omega^2} f \, dx < 0. \tag{2}
\]

We assume that $f$ satisfies condition (2). A well known result on the existence and uniqueness of the minimum of convex coercive functionals, see, e.g., [29], guarantees that problem (2) has a unique solution.

### 3 FETI–DP discretization of the problem

The first step in our domain decomposition method is to partition each domain $\Omega^i, i = 1, 2$, on a rectangular grid into subdomains of diameter of order $H$. Let $W$ be the finite element space whose restrictions to $\Omega^1$ and $\Omega^2$ are $Q_1$ finite element spaces of comparable mesh sizes of order $h$, corresponding to the subdomain grids in $\Omega^1$ and $\Omega^2$. Note that the subdomain grids do not necessarily match across the potential contact interface $\Gamma_c$. We call a crosspoint either a corner that belongs to four subdomains, or a corner that belongs to two subdomains and is located on $\partial \Omega^1 \setminus \Gamma^1_u$ or on $\partial \Omega^2$. The nodes corresponding to the end points of $\Gamma_c$ are not crosspoints; see Figure 1b. An important feature for developing FETI–DP type algorithms is that a single global degree of freedom is considered at each crosspoint, while two degrees of freedom are introduced at all the other matching nodes across subdomain edges.

Let $v \in W$. The continuity of $v$ in $\Omega^1$ and $\Omega^2$ is enforced at every interface node that is not a crosspoint. For simplicity, we also denote by $v$ the nodal values vector of $v \in W$. The discretized version of problem (1) has the form

\[
\min \frac{1}{2} v^T K v - v^T f \quad \text{subject to } B_1 v \leq 0 \quad \text{and} \quad B_\mathcal{E} v = 0, \tag{3}
\]
where the full rank matrices $B_I$ and $B_E$ describe the non-penetration (inequality) conditions and the gluing (equality) conditions, respectively, and $f$ represents the discrete analog of the linear form $\ell(\cdot)$. In (3), $K = \text{diag}(K_1, K_2)$ is the block diagonal positive semidefinite stiffness matrix corresponding to the model problem (1). The block $K_1$ corresponding to $\Omega^1$ is nonsingular, due to the Dirichlet boundary conditions enforced on $u^1$. The block $K_2$ corresponding to $\Omega^2$ is singular and its kernel is made of a vector $e$ with all entries equal to 1. Therefore, the kernel of $K$ is spanned by the matrix $R$ defined by

$$R = \begin{bmatrix} 0 \\ e \end{bmatrix}. \quad (4)$$

Even though $R$ is a column vector for our model problem, we will regard $R$ as a matrix whose columns span the kernel of $K$. Moreover, we assume that the nodes in $\Omega^1$ and $\Omega^2$ are numbered contiguously, so that $K_i$, $i = 1, 2$ are block diagonal matrices.

We partition the nodal values of $v \in W$ into crosspoint nodal values, denoted by $v_c$, and remainder nodal values, denoted by $v_r$. The continuity of $v$ at crosspoints is enforced by using a global vector of degrees of freedom $v_c^g$ and a global-to-local map $L_c$ with one nonzero entry equal to 1 in each row, i.e., we require that $v_c = L_c v_c^g$. Therefore,

$$v = \begin{bmatrix} v_r \\ v_c \end{bmatrix} = \begin{bmatrix} v_r \\ L_c v_c^g \end{bmatrix}.$$

Let $f_c$ and $f_r$ be the parts of the right hand side $f$ corresponding to the corner and remainder nodes, respectively. We reorder the matrices $K$, $B_I$, $B_E$, and $R$ to comply with the above splitting of the nodal values of $v \in W$. The Lagrangian associated with problem (3) can be expressed using the above splitting of $v$ and Lagrange multipliers $\lambda_E$ and $\lambda_I$ to enforce the inequality and redundancy constraints as follows:

$$L(v_r, v_c^g, \lambda_E, \lambda_I) = \frac{1}{2} \begin{bmatrix} v_r^T, (L_c v_c^g)^T \end{bmatrix} K \begin{bmatrix} v_r \\ L_c v_c^g \end{bmatrix} - [v_r^T, (L_c v_c^g)^T] \begin{bmatrix} f_r \\ f_c \end{bmatrix}$$

$$+ v_r^T B_E^T \lambda_E + v_c^T B_I^T \lambda_I. \quad (5)$$

Let $B_{I,r}$ and $B_{I,c}$ be the matrices made of the columns of $B_I$ corresponding to $v_r$ and $v_c$, respectively; define $B_{E,r}$ and $B_{E,c}$ similarly. Then $B_I = [B_{I,r} \ B_{I,c}]$ and $B_E = [B_{E,r} \ B_{E,c}]$. We can also group the parts of the Lagrange multiplier matrices $B_I$ and $B_E$ together with respect to the corner and remainder nodes as follows:

$$B_r = \begin{bmatrix} B_{I,r} \\ B_{E,r} \end{bmatrix} \quad \text{and} \quad B_c = \begin{bmatrix} B_{I,c} \\ B_{E,c} \end{bmatrix}.\quad (5)$$
Let

\[ \lambda = \begin{bmatrix} \lambda_I \\ \lambda_E \end{bmatrix}. \]

Then

\[ v^T B^T_E \lambda_E + v^T B^T_I \lambda_I = v^T B^T_r \lambda + (v^g) L^T_c B^T_c \lambda. \]

Let \( K_{rr} \), \( K_{rc} \), and \( K_{cc} \) denote the blocks of \( K \) corresponding to the decomposition of \( v \) into \( v_r \) and \( v_c \). Note that \( K_{cc} \) has small size and \( K_{rr} \) is block diagonal. Any system corresponding to the matrix \( K_{rr} \) can be solved efficiently. To minimize \( L(v_r, v^g, \lambda_E, \lambda_I) \) over \( v_r \), we rewrite (5) as

\[
L(v_r, v^g, \lambda_E, \lambda_I) = \frac{1}{2} \left( v^T R v_r + 2v^T K_{rc} L_c v^g + (v^g) L^T_c K_{cc} L_c v^g \right) \\
- v^T f_r - (v^g) L^T_c f_c + v^T B^T_r \lambda + (v^g) L^T_c B^T_c \lambda
\]

and obtain that \( v_r \) is a solution of

\[ K_{rr} v_r + K_{rc} L_c v^g - f_r + B^T_c \lambda = 0. \]

Since \( K_{rr} \) is nonsingular, we arrive at the following Lagrangian to minimize over \( v^g \):

\[
L_c(v^g, \lambda_E, \lambda_I) = \frac{1}{2} (v^g)^T L^T_c K_{cc} L_c v^g - (v^g)^T L^T_c f_c + (v^g)^T L^T_c B^T_c \lambda \\
- \frac{1}{2} \left( f_r - K_{rc} L_c v^g - B^T_c \lambda \right)^T K^{-1}_{rr} \left( f_r - K_{rc} L_c v^g - B^T_c \lambda \right) \\
= \frac{1}{2} (v^g)^T K^*_c v^g - (v^g)^T \left( \tilde{F}^T_{r,c} \lambda + f^*_c \right) - \frac{1}{2} \left( f_r - B^T_c \lambda \right)^T K^{-1}_{rr} \left( f_r - B^T_c \lambda \right),
\]

where we used the following notations related to those from [24]:

\[
F_{r,c} = B_r K^{-1}_{rr} B^T_r; \\
\tilde{F}_{r,c} = B_r K^{-1}_{rr} K_{rc} - B_c L_c; \\
K^*_c = L^T_c (K_{cc} - K^T_{rc} K^{-1}_{rr} K_{rc}) L_c; \\
f^*_c = L^T_c (f_r - K^T_{rc} K^{-1}_{rr} f_r).
\]

The solution \( v^g_c \) to the minimization of \( L_c(v^g, \lambda_E, \lambda_I) \) over \( v^g_c \) must satisfy

\[ K^*_c v^g_c - \tilde{F}^T_{r,c} \lambda - f^*_c = 0. \]

Note that \( K^*_c \) is singular so that problem (7) is solvable if and only if \( \tilde{F}^T_{r,c} \lambda + f^*_c \) belongs to the range of \( K^*_c \). Since \( K^*_c \) is symmetric, this is equivalent to requiring that

\[ \tilde{F}^T_{r,c} \lambda + f^*_c \perp \text{Ker} K^*_c. \]

Let \( R_c \) be the matrix whose columns span the kernel of the Schur complement matrix \( K_{cc} - K^T_{rc} K^{-1}_{rr} K_{rc} \). Note that \( R_c \) and \( R \), the matrix spanning the kernel of the stiffness
matrix $K$, cf. (4), are connected by

$$ R = \begin{bmatrix} -K_{rr}^{-1}K_{rc} \\ I_c \end{bmatrix} R_c, $$

where $I_c$ is an identity matrix. Then, the kernel of $K^*_c$ is spanned by

$$ \tilde{R}_c = (L_c^T L_c)^{-1} L_c^T R_c. $$

The condition for the solvability of (7) can be written as

$$ \tilde{R}_c^T (\tilde{F}_{Irc}^T \lambda + f^*_c) = 0. \quad (8) $$

If (8) is satisfied, the solution $\tilde{v}_c^0$ of (7) has the form

$$ \tilde{v}_c^0 = (K^*_c)^\dagger (\tilde{F}_{Irc}^T \lambda + f^*_c) + \tilde{R}_c \alpha, $$

where $(K^*_c)^\dagger$ denotes a pseudoinverse of $K^*_c$, i.e., a matrix satisfying $K^*_c (K^*_c)^\dagger K^*_c = K^*_c$, and $\alpha$ is an arbitrary vector of appropriate size. The corresponding minimal value of $L_c(\tilde{v}_c^0, \lambda_E, \lambda_I)$ is

$$ L_\lambda(\lambda_E, \lambda_I) = -\frac{1}{2} (f^*_c + \tilde{F}_{Irc}^T \lambda)^T (K^*_c)^\dagger (f^*_c + \tilde{F}_{Irc}^T \lambda) - \frac{1}{2} (f_r - B_r^T \lambda)^T K_{rr}^{-1} (f_r - B_r^T \lambda). $$

Maximizing $L_\lambda$ over $\lambda_I \geq 0$ is equivalent to finding

$$ \min \Theta(\lambda) \quad \text{s.t.} \quad \lambda_I \geq 0 \quad \text{and} \quad \tilde{R}_c^T (\tilde{F}_{Irc}^T \lambda + f^*_c) = 0 \quad (9) $$

where

$$ \Theta(\lambda) = \frac{1}{2} \lambda^T F \lambda - \lambda^T \tilde{d} $$

with

$$ F = F_{Irr} + \tilde{F}_{Irc} (K^*_c)^\dagger \tilde{F}_{Irc}^T; \quad \tilde{d} = \tilde{F}_{Irc} (K^*_c)^\dagger f^*_c - B_r K_{rr}^{-1} f_r. \quad (11) $$

4 Modifications

Even though problem (9) is much more suitable for computations than (1) and was used for solving discretized variational inequalities efficiently [10], further improvement may be achieved as follows. Let $\tilde{G} = \tilde{R}_c^T \tilde{F}_{Irc}^T$ and denote by $\tilde{T}$ a nonsingular matrix that defines the orthonormalization of the rows of $\tilde{G}$ such that the matrix $G = \tilde{T} \tilde{G}$ has orthonormal rows. Let $e = -\tilde{T} \tilde{R}_c^T f^*_c$. Then, problem (9) reads

$$ \min \frac{1}{2} \lambda^T F \lambda - \lambda^T \tilde{d} \quad \text{s.t.} \quad \lambda_I \geq 0 \quad \text{and} \quad G \lambda = e. \quad (12) $$

Next, we transform the problem of minimization on the subset of the affine space to a minimization problem on the subset of a vector space. Let $\tilde{\lambda}$ be an arbitrary feasible vector
such that $G\tilde{\lambda} = e$. We look for the solution $\lambda$ of (9) in the form $\lambda = \mu + \tilde{\lambda}$. Since

$$
\frac{1}{2} \lambda^T F \lambda - \lambda^T d = \frac{1}{2} \mu^T F \mu - \mu^T (\tilde{d} - F\tilde{\lambda}) + \frac{1}{2} \tilde{\lambda}^T F \tilde{\lambda} - \tilde{\lambda}^T \tilde{d},
$$

problem (12) is, after returning to the old notation by replacing $\mu$ by $\lambda$, equivalent to

$$
\min \frac{1}{2} \lambda^T F \lambda - d^T \lambda \quad \text{s.t.} \quad G\lambda = 0 \quad \text{and} \quad \lambda_I \geq -\tilde{\lambda}_I,
$$

(13)

with $d = \tilde{d} - F\tilde{\lambda}$. Our final step is based on the observation that the augmented Lagrangian for problem (13) may be decomposed by the orthogonal projectors

$$
Q = G^T G \quad \text{and} \quad P = I - Q
$$
on the image space of $G^T$ and on the kernel of $G$, respectively. Since $P\lambda = \lambda$ for any feasible $\lambda$, problem (13) is equivalent to

$$
\min \frac{1}{2} \lambda^T PFP \lambda - \lambda^T Pd \quad \text{s.t.} \quad G\lambda = 0 \quad \text{and} \quad \lambda_I \geq -\tilde{\lambda}_I.
$$

(14)

The Hessian $H = PFP + \rho Q$ of the augmented Lagrangian

$$
L(\lambda, \mu, \rho) = \frac{1}{2} \lambda^T (PFP + \rho Q) \lambda - \lambda^T Pd + \lambda^T G^T \mu
$$

(15)
is decomposed by the projectors $P$ and $Q$ whose image spaces are invariant subspaces of $H$. Let $a, b > 0$ such that the non-zero eigenvalues of $F$ restricted to $\text{Im}P$ are located in the interval $[a, b]$. Then,

$$
\sigma(H) \subseteq [a, b] \cup \{\rho\}.
$$

By the analysis of Axelsson [2], the number $k$ of conjugate gradient iterations necessary to reduce the gradient of the augmented Lagrangian (15) by $\epsilon$ satisfies

$$
k \leq \frac{1}{2} \text{int} \left( \sqrt{\frac{b}{a}} \ln \left( \frac{2}{\epsilon} \right) + 3 \right).
$$

In the next section, we find bounds on $a$ and $b$ in terms of the decomposition and the discretization parameters $H$ and $h$ that guarantee the optimality of our algorithm.

5 Bounds on the spectrum

The proof of the numerical scalability of the algorithm described in the Section 7 is based in part on spectral bounds for the operator $F$ restricted to $\text{Im}P$. To simplify our analysis, we first derive an alternative expression for $F$ by repeating the analysis of the previous section for the shortened vectors

$$
\varpi = \begin{bmatrix} v_r [v_g] [v_c] \end{bmatrix} \in \mathcal{W}
$$
without caring about implementation. The Lagrangian (6) can be expressed as

\[ L(\varpi, \lambda) = \frac{1}{2} \varpi^T K \varpi - \varpi^T f + \varpi^T B T \lambda, \]

with

\[
K = \begin{bmatrix} K_{rr} & K_{rc} L_c \\ L_c^T K_{rc}^T & L_c^T K_{cc} L_c \end{bmatrix}, \quad B = [B_r B_c L_c], \quad \text{and} \quad f = \begin{bmatrix} f_r \\ L_c^T f_c \end{bmatrix}.
\]

To maximize \( L(\varpi, \lambda) \) over \( \varpi \), we set the gradient \( \nabla_{\varpi} L(\varpi, \lambda) \) equal to 0, i.e.,

\[
K \varpi - f + B^T \lambda = 0. \tag{16}
\]

Problem (16) is solvable if and only if \( f - B^T \lambda \in \text{Im} K \). Let \( R \) be the matrix of column vectors spanning the kernel of \( K \) and let \( K^\dagger \) be a suitable generalized inverse of \( K \). If

\[
R^T B^T \lambda = R^T f,
\]

we obtain that \( \varpi = K^\dagger (f - B^T \lambda) + K \alpha \) and \( \Theta(\lambda) \) can be written as

\[
\Theta(\lambda) = \frac{1}{2} \lambda^T B K^\dagger B^T \lambda - \lambda^T B K^\dagger f.
\]

Thus, the operator \( F \) of (11) can be expressed as

\[
F = B K^\dagger B^T. \tag{17}
\]

For our model problem, the matrix \( R \) is made of one vector with entries equal to 0 corresponding to the nodes of \( \Omega^1 \), and with entries equal to 1 corresponding to the nodes of \( \Omega^2 \). It may be checked that all entries of \( R^T B^T \) are negative and, using the condition of solvability (2), that \( R^T f < 0 \), so that there is \( \lambda \) such that

\[
R^T B^T \lambda = R^T f \quad \text{and} \quad \lambda_i > 0.
\]

**Theorem 5.1.** The following spectral bounds hold:

\[
\lambda_{\text{max}}(F|\text{Im} P) \leq \|F\| \leq C \left( \frac{H}{K} \right)^2; \quad \lambda_{\text{min}}(F|\text{Im} P) \geq C. \tag{18}
\]

**Proof:** Using the formula (17), we find that

\[
\langle F \lambda, \lambda \rangle = \langle K^\dagger B^T \lambda, B^T \lambda \rangle = \|K^\dagger 1/2 B^T \lambda\|^2. \tag{19}
\]

If \( \varpi \in W \) such that \( \varpi \perp \text{Ker} K \), then \( \varpi \in \text{Im} K \) and there exists \( \overline{\varpi} \perp \text{Ker} K \) such that \( \varpi = K^{1/2} \overline{\varpi} \). From (19), we find that
\[
\langle F, \lambda \rangle = \sup_{\pi \perp \text{Ker} K} \frac{(\mathbf{B}^T \lambda, (K^\dagger)^{1/2} \mathbf{w})^2}{\langle \mathbf{w}, \mathbf{w} \rangle} \leq \sup_{\pi \perp \text{Ker} K} \frac{(\mathbf{B}^T \lambda, (K^\dagger)^{1/2} \mathbf{w})^2}{\langle \mathbf{w}, \mathbf{w} \rangle} = \sup_{\pi \perp \text{Ker} K} \frac{(\mathbf{B}^T \lambda, (K^\dagger)^{1/2} \mathbf{w})^2}{\langle \mathbf{w}, \mathbf{w} \rangle}.
\]

Let
\[
\mathbf{w} = \begin{bmatrix} w_r \\ w_c \end{bmatrix} \in \mathbb{W} \quad \text{and} \quad w = \begin{bmatrix} w_r \\ w_c \end{bmatrix} \in \mathbb{W}.
\]

Then \(\langle K\mathbf{w}, \mathbf{w} \rangle = \langle K\mathbf{w}, \mathbf{w} \rangle\). From inverse inequalities and Poincaré’s inequality, it follows that

\[
\frac{C}{H} \|w\|_{L^2}^2 \leq \langle K\mathbf{w}, \mathbf{w} \rangle \leq \frac{C}{h^2} \|w\|_{L^2}^2, \quad \forall \mathbf{w} \perp \text{Ker} K.
\]

Here, and throughout this section, \(C\) is a generic constant independent of \(h\) and \(H\).

Let \(||\cdot||_{L^2}\) denote the Euclidean norm. Recall that \(L_c\) has all entries equal to 0 except for one entry equal to 1 in each row. Since every crosspoint belongs to at most four subdomains, we find that

\[
||\mathbf{w}||_{L^2}^2 \leq ||w||_{L^2}^2 \leq 4||\mathbf{w}||_{L^2}^2.
\]

Since each row of \(\mathbf{B}\) is 0 except for at most two entries which are either 1 or −1, the Euclidean norm of \(\mathbf{B}^T \lambda\) is equivalent to the Euclidean norm of \(\lambda\), i.e.,

\[
||\mathbf{B}^T \lambda||_{L^2}^2 \approx ||\lambda||_{L^2}^2.
\]

Recall that \(w \in \mathbb{W}\) is the nodal values vector of a \(Q_1\) finite element function. Then,

\[
||w||_{L^2}^2 \approx h^2||w||_{L^2}^2.
\]

To show that \(||F|| \leq C \left(\frac{H}{h}\right)^2\), we use (20) and (21) to obtain

\[
\langle F, \lambda \rangle \leq C \frac{h^2}{H} \sup_{\pi \perp \text{Ker} K} \frac{||\mathbf{B}^T \lambda||_{L^2}^2}{||w||_{L^2}^2} \leq C \frac{h^2}{H} \sup_{\pi \perp \text{Ker} K} \frac{\|w\|_{L^2}^2}{||w||_{L^2}^2} \approx \frac{C}{H} \sup_{\pi \perp \text{Ker} K} \frac{||\mathbf{w}||_{L^2}^2}{||w||_{L^2}^2}.
\]

For any \(\mathbf{w} \perp \text{Ker} K\) there exists \(\mathbf{y}\) such that \(\mathbf{w} = (K^{1/2}) \mathbf{y}\), and therefore

\[
\|\mathbf{w}\|_{L^2}^2 = ||(K^{1/2} \mathbf{y})||_{L^2}^2 = ||\mathbf{y}^T K^{1/2} \mathbf{K}^\dagger \mathbf{y}||_{L^2}^2 = ||\mathbf{y}^T \mathbf{K}^{1/2} \mathbf{K} \mathbf{y}||_{L^2}^2 = ||\mathbf{y}^T \mathbf{w}||_{L^2}^2 \leq ||w||_{L^2}^2,
\]

where (22) was used for the last inequality. We conclude, using also (23) and (24), that
\[
\langle F, \lambda \rangle \leq CH^2 \|B^T \lambda\|_2^2 \sup_{\pi \perp \text{Ker}K} \frac{\|K^{1/2}K^{1/2}w\|_2^2}{\|w\|_2^2}\]
\[
= CH^2 \|B^T \lambda\|_2^2 \sup_{\pi \perp \text{Ker}K} \frac{\|w\|_2^2}{\|w\|_2^2} \leq C \left( \frac{H\epsilon}{\kappa} \right)^2 \|B^T \lambda\|_2^2 \]
\[
\leq C \left( \frac{H\epsilon}{\kappa} \right)^2 \|\lambda\|_2^2.
\]

To prove the lower bound \(\lambda_{\text{min}}(F|\text{Im}P) \geq C\), let \(\lambda \in \text{Im}P\) and \(\bar{w}_0 = \overline{B^T} \lambda\). Since \(\lambda \in \text{Im}P\) is equivalent to \(Q\lambda = 0\), it follows that \(G\lambda = 0\) and therefore that \(\overline{R^T} \overline{B^T} \lambda = 0\). In other words, \(\bar{w}_0 = \overline{B^T} \lambda \perp \text{Ker}K\) and \(\overline{B^T} \lambda \in \text{Im}K\).

Let \(K^\#\) denote the Moore-Penrose pseudoinverse of \(K\). For any \(\bar{w} \in W\) there exists a vector \(\bar{w} \in \text{Ker}K\) such that \(K^{1/2}w = K^\#(w + \bar{w})\). Since \(\overline{B^T} \lambda \perp \text{Ker}K\), from (17), we find that
\[
\langle F, \lambda \rangle = \langle K^{1/2}B^T \lambda, B^T \lambda \rangle = \langle K^\#B^T \lambda, B^T \lambda \rangle = \|K^\#\|_2 \|B^T \lambda\|_2^2.
\]

to (25)

Moreover, since the Moore-Penrose pseudoinverse is a matrix function, \((K^{1/2})^\# = (K^\#)^{1/2}\). Thus, for any \(\overline{w} \in \text{Im}K\), there exists \(\overline{w} \in \text{Im}K\) such that \(\overline{w} = (K^{1/2})^\# \overline{w}\) and \(\overline{w} = K^{1/2} \overline{w}\). We can use (25) to show, similarly as above, that
\[
\langle F, \lambda \rangle = \sup_{\pi \perp \text{Ker}K} \frac{\langle (K^{1/2})^1/2B^T \lambda, \overline{w} \rangle^2}{\langle \overline{w}, \overline{w} \rangle} = \sup_{\pi \perp \text{Ker}K} \frac{\langle B^T \lambda, (K^{1/2})^{1/2} \overline{w} \rangle^2}{\langle \overline{w}, \overline{w} \rangle} = \sup_{\pi \perp \text{Ker}K} \frac{\langle B^T \lambda, \overline{w} \rangle^2}{\langle \overline{w}, \overline{w} \rangle}.
\]

Recall that \(\overline{w}_0 = \overline{B^T} \lambda\). Using (26), (21), (22), and (23), we find
\[
\langle F, \lambda \rangle = \sup_{\pi \perp \text{Ker}K} \frac{\langle B^T \lambda, \overline{w} \rangle^2}{\langle \overline{w}, \overline{w} \rangle} \geq \frac{\langle B^T \lambda, \overline{w}_0 \rangle^2}{\langle \overline{w}_0, \overline{w}_0 \rangle} = \frac{\langle \overline{w}_0, \overline{w}_0 \rangle^2}{\langle \overline{w}_0, \overline{w}_0 \rangle} \geq Ch^2 \frac{\|\overline{w}_0\|_2^4}{\|w_0\|_2^2}
\]
\[
\geq Ch^2 \frac{\|w_0\|_2^2}{\|w_0\|_2^2} \approx C\|w_0\|_2^2 = C\|\overline{B^T} \lambda\|_2^2 \approx C\|\lambda\|_2^2.
\]

6 Solvers to Bound and Equality Constrained Problems

To develop a numerically scalable algorithm for the solution of (1), we use our recent algorithms [8,22] which can be applied to solve minimization problems of the same type as (14) to the prescribed relative error in a uniformly bounded number of steps [9]. These algorithms are based on well established methods [5–7,11,28] that were modified to achieve optimality. We briefly review some of these algorithms here. If a new Lagrange multiplier
vector $\mu$ is used for the equality constraints, the augmented Lagrangian for problem (14) can be written as

$$L(\lambda, \mu, \rho) = \frac{1}{2} \lambda^T (PFP + \rho Q) \lambda - \lambda^T Pd + \lambda^T G^T \mu.$$  

Recall that $Q = G^T G$. The gradient of $L(\lambda, \mu, \rho)$ is given by

$$g(\lambda, \mu, \rho) = PFP\lambda - Pd + G^T (\rho G \lambda + \mu).$$

Let $\mathcal{I}$ denote the set that contains the indices of the bound constrained entries of $\lambda$ from problem (14). Let $g_i^- = \min(g_i, 0)$. The projected gradient $g^P = g^P(\lambda, \mu, \rho)$ of $L$ at $\lambda$ is given component wise by

$$g^P_i = \begin{cases} 
    g_i, & \text{if } \lambda_i > -\bar{\lambda}_i \text{ or } i \notin \mathcal{I} \\
    g_i^-, & \text{if } \lambda_i = -\bar{\lambda}_i \text{ and } i \in \mathcal{I} 
\end{cases}$$

The algorithm we propose here is a variant of the algorithm proposed by Conn, Gould and Toint [5] for identifying stationary points of more general problems. Its modification by Dostál, Friedlander and Santos [11] was used by Dostál and Horák to develop a scalable FETI based algorithm, as shown experimentally in [15]. The key to proving optimality results is to combine the adaptive precision control of the auxiliary problems in Step 1 with the new update rule for the penalty parameter $\rho$ in Step 4. All the necessary parameters are listed in Step 0, and typical values of these parameters for our model problem are given in the right brackets.

**Algorithm 6.1.** Semi-monotonic augmented Lagrangian method for bound and equality constrained problems (SMALBE).

**Step 0.** **Initialization of parameters**

Given $\eta > 0$ [$\eta = \|Pd\|$], $\beta > 1$ [$\beta = 10$], $M > 0$ [$M = 1$], $\rho_0 > 0$ [$\rho_0 = 100$], and $\mu^0$ [$\mu^0 = 0$], set $k = 0$.

**Step 1.** **Inner iteration with adaptive precision control.**

Find $\lambda^k$ such that

$$||g^P(\lambda^k, \mu^k, \rho_k)|| \leq \min\{M\|G\lambda^k\|, \eta\}.$$  

**Step 2.** **Stopping criterion.**

If $||g^P(\lambda^k, \mu^k, \rho_k)||$ and $||G\lambda^k||$ are sufficiently small, then $\lambda^k$ is the solution.

**Step 3.** **Update of the Lagrange multipliers.**

$$\mu^{k+1} = \mu^k + \rho_k G\lambda^k$$

**Step 4.** **Update the penalty parameter.**

If $k > 0$ and $L(\lambda^k, \mu^k, \rho_k) < L(\lambda^{k-1}, \mu^{k-1}, \rho_{k-1}) + \rho_k \|C\lambda^k\|^2/2$ then $\rho_{k+1} = \beta \rho_k$

else $\rho_{k+1} = \rho_k$

end if.

**Step 5.** Increase $k$ and return to Step 1.
Note that Step 1 is carried out by the minimization of the augmented Lagrangian \( L \) subject to \( I \) by means of an algorithm to be described later. The unique solution \( \lambda = \bar{\lambda}(\mu, \rho) \) of this auxiliary problem satisfies the Karush-Kuhn-Tucker conditions

\[
g^P(\lambda, \mu, \rho) = 0. \tag{27}\]

The salient feature of this algorithm is that it deals with each type of constraint completely separately and accepts inexact solutions for the auxiliary box constrained problems in Step 1. For parallel implementation, it is necessary to keep the factors that form \( F_P = PFP \), since \( F_P \) is only needed for matrix-vector multiplications. The action of \( K^\dagger \) may be evaluated by means of a Cholesky decomposition. The matrix \( G \) is generated by a \( QR \) decomposition of \( \bar{G} \).

The Algorithm 6.1 was proved to converge for any set of parameters that satisfy the relations prescribed in [8]. Moreover, it was shown that the penalty parameter is never increased if \( \rho_k \) exceeds a relatively small bound, and that the number of iterations necessary to achieve the prescribed relative feasibility error may be bounded independently of the conditioning of the constraints. The results are true even for dependent constraints. More results concerning the algorithm may be found in [8]. The results that we need depend on implementation of Step 1 of Algorithm 6.1.

In the rest of this section we review some results on applying an active set strategy to solving the bound constrained quadratic programming problems for implementation of Step 2 in a form that is relevant for our problem. To simplify our exposition, assume that \( \mu \) and \( \rho \) are fixed, denote \( \Theta(\lambda) = L(\lambda, \mu, \rho) \), and let \( H \) be the Hessian of \( \Theta \). It is well known that the solution to the problem

\[
\min \Theta(\lambda) \quad \text{s. t.} \quad \lambda_I \geq -\bar{\lambda}_I \tag{28}
\]

always exists, and it is necessarily unique [4]. Let \( n \) be the dimension of the argument of \( \Theta \), and denote, for any \( n \)-vector \( \lambda \), the gradient of \( \Theta \) at \( \lambda \) by

\[
g = g(\lambda) = \nabla \Theta(\lambda). \tag{29}\]

Then the unique solution \( \bar{\lambda} \) of (14) is fully determined by the Karush-Kuhn-Tucker (KKT) optimality conditions [4]. To describe them in more detail, let \( \mathcal{N} = \{1, 2, \ldots, n\} \) denote the set of indices of dual variables \( \lambda_i \) which is decomposed into two disjoint sets \( \mathcal{I} \) and \( \mathcal{E} \), with \( \mathcal{I} \) denoting the indices of the inequality constrained entries and \( \mathcal{E} \) denoting the set of indices of the unconstrained entries of \( \lambda \), \( \mathcal{E} = \mathcal{N} \setminus \mathcal{I} \). Thus

\[
\bar{\lambda}_i = -\bar{\lambda}_i \quad \text{and} \quad i \in \mathcal{I} \implies \bar{g}_i \geq 0, \quad \text{and} \quad \bar{\lambda}_i > -\bar{\lambda}_i \quad \text{or} \quad i \in \mathcal{E} \implies \bar{g}_i = 0. \tag{30}\]

The set of all indexes \( i \in \mathcal{I} \) for which \( \lambda_i = -\bar{\lambda}_i \) is called an active set of \( \lambda \). We denote it by \( \mathcal{A}(\lambda) = \{i \in \mathcal{I} : \lambda_i = -\bar{\lambda}_i\} \). The complement \( \mathcal{F}(\lambda) = \mathcal{N} \setminus \mathcal{A}(\lambda) = \{i : \lambda_i > -\bar{\lambda}_i \text{ or } i \notin \mathcal{I}\} \) of \( \mathcal{A}(\lambda) \) will be called a free set of \( \lambda \). To enable an alternative reference to the KKT conditions (30), let \( \varphi(\lambda) \) be the free gradient of \( \lambda \) and let \( \beta(\lambda) \) be the chopped gradient of
\[ \varphi_i(\lambda) = \begin{cases} g_i(\lambda), & \text{for } i \in \mathcal{F}(\lambda) \\ 0, & \text{for } i \in \mathcal{A}(\lambda) \end{cases} \quad \text{and} \quad \beta_i(\lambda) = \begin{cases} 0, & \text{for } i \in \mathcal{F}(\lambda) \\ g_i^-(\lambda), & \text{for } i \in \mathcal{A}(\lambda) \end{cases} \]

Thus, the KKT conditions (30) are satisfied if and only if the projected gradient \( \nu(\lambda) = \varphi(\lambda) + \beta(\lambda) \) is equal to zero. We call \( \lambda \) feasible if \( \lambda_i \geq -\lambda_i \) for \( i \in \mathcal{I} \). The projection \( P_+ \) to the set of feasible vectors is defined for any \( n \)-vector \( \lambda \) by

\[ P_+(\lambda)_i = \max\{\lambda_i, -\lambda_i\} \quad \text{for} \quad i \in \mathcal{I}, \quad P_+(\lambda)_i = \lambda_i \quad \text{for} \quad i \in \mathcal{E}. \]

Let us briefly describe the algorithm [22] for the solution of (9) that combines the proportioning algorithm [6] with gradient projections [43]. We use a test to decide about leaving the face and three types of steps to generate a sequence of iterates \( \{\lambda^k\} \) that approximate the solution of (9).

The **expansion step** is defined by

\[ \lambda^{k+1} = P_+ \left( \lambda^k - \overline{\alpha} \varphi(\lambda^k) \right) \] (31)

with the steplength \( \overline{\alpha} \in (0, \|H\|^{-1}] \). This step may expand the current active set. To describe it without \( P_+ \), let \( \tilde{\varphi}(\lambda) \) be the reduced free gradient for any feasible \( \lambda \), with entries

\[ \tilde{\varphi}_i = \tilde{\varphi}_i(\lambda) = \min\{\lambda_i / \overline{\alpha}, \varphi_i\} \quad \text{for} \quad i \in \mathcal{I}, \quad \tilde{\varphi}_i = \varphi_i \quad \text{for} \quad i \in \mathcal{E} \]

such that

\[ P_+ (\lambda - \overline{\alpha} \varphi(\lambda)) = \lambda - \overline{\alpha} \tilde{\varphi}(\lambda). \] (32)

If the inequality

\[ \|\beta(\lambda^k)\|^2 \leq \Gamma^2 \tilde{\varphi}(\lambda^k)^T \varphi(\lambda^k) \] (33)

holds, then we call the iterate \( \lambda^k \) strictly proportional. The test (33) is used to decide which component of the projected gradient \( \nu(\lambda^k) \) will be reduced in the next step.

The **proportioning step** is defined by

\[ \lambda^{k+1} = \lambda^k - \alpha_{cg} \beta(\lambda^k). \]

The steplength \( \alpha_{cg} \) is chosen to minimize \( f(\lambda^k - \alpha \beta(\lambda^k)) \) with respect to \( \alpha \), i.e.,

\[ \alpha_{cg} = \frac{\beta(\lambda^k)^T g(\lambda^k)}{\beta(\lambda^k)^T H \beta(\lambda^k)}. \]

The purpose of the proportioning step is to remove indexes from the active set.

The **conjugate gradient step** is defined by

\[ \lambda^{k+1} = \lambda^k - \alpha_{cg} p^k \] (34)
where $p^k$ is the conjugate gradient direction [2] which is constructed recurrently. The recurrence starts (or restarts) with $p^s = \varphi(\lambda^s)$ whenever $\lambda^s$ is generated by the expansion step or the proportioning step. If $p^k$ is known, then $p^{k+1}$ is given by the formulae [2]

$$p^{k+1} = \varphi(\lambda^k) - \gamma p^k, \quad \gamma = \frac{\varphi(\lambda^k)^T H p^k}{(p^k)^T H p^k}.$$  \hfill (35)

The conjugate gradient steps are used to carry out the minimization in the face $\mathcal{W}_J = \{\lambda : \lambda_i = -\vec{\lambda}_i \text{ for } i \in J\}$ given by $J = A(\lambda^s)$ efficiently. The algorithm that we use may now be described as follows.

**Algorithm 6.2.** Modified proportioning with reduced gradient projections (MPRGP).

Let $\lambda^0$ be an $n$-vector such that $\lambda_i \geq -\vec{\lambda}_i$ for $i \in \mathcal{I}$, $\vec{\pi} \in (0,\|H\|^{-1}]$, and $\Gamma > 0$ be given. For $k \geq 0$ and $\lambda^k$ known, choose $\lambda^{k+1}$ by the following rules:

*Step 1.* If $\nu(\lambda^k) = 0$, set $\lambda^{k+1} = \lambda^k$.

*Step 2.* If $\lambda^k$ is strictly proportional and $\nu(\lambda^k) \neq 0$, try to generate $\lambda^{k+1}$ by the conjugate gradient step. If $\lambda^{k+1} \geq -\vec{\lambda}_i$ for $i \in \mathcal{I}$, then accept it, else generate $\lambda^{k+1}$ by the expansion step.

*Step 3.* If $\lambda^k$ is not strictly proportional, define $\lambda^{k+1}$ by proportioning.

More details about the implementation of Algorithm 6.2 may be found in [22]. This algorithm was proved to converge for any set of initial parameters that satisfy the prescribed inequalities. The rate of the R-linear convergence of $\|g^p(\lambda^k)\|$ in terms of the condition number of the Hessian of $\Theta$ was given in [9]. In the next section, we show that, if Step 1 of Algorithm 6.1 is implemented by Algorithm 6.2, then Algorithm 6.1 is optimal.

### 7 Optimality

For any set of indices $\mathcal{I}$ and for any $t \in \mathcal{I}$, consider the following problem:

$$\text{minimize } \Theta_t(\lambda) \quad \text{s.t. } C_t \lambda_t = 0 \text{ and } \lambda_t \geq \ell_t, \quad \hfill (36)$$

with $\Theta_t(x) = \frac{1}{2} x^T A_t x - b_t^T x$. Here, $A_t \in \mathbb{R}^{nt \times nt}$ is a symmetric positive definite matrix, $b_t, \ell_t \in \mathbb{R}^{nt}$, and $C_t \in \mathbb{R}^{m_t \times nt}$. Let $\lambda_{\min}(A_t)$ and $\lambda_{\max}(A_t)$ be the smallest and the largest eigenvalues of $A_t$, respectively, and let $\ell_t \leq 0$. Our optimality result reads as follows.

**Theorem 7.1** Let $a_{\max} > a_{\min} > 0$ and let $c_{\max} > 0$ be given constants. Assume that the class of problems (36) satisfies

$$a_{\min} \leq \lambda_{\min}(A_t) \leq \lambda_{\max}(A_t) \leq a_{\max} \quad \text{and} \quad \|C_t\| \leq c_{\max}. \quad \hfill (37)$$

Let $\{\lambda^k\}, \{\mu^k\}$ and $\{\rho_{t,k}\}$ be generated by Algorithm 6.1 (SMALBE) for (36) with $\|b_t\| \geq \eta_t > 0, \beta > 1, M > 0, \rho_{t,0} = \rho_0 > 0$, and $\rho_0^s = 0$. Let $s \geq 0$ denote the smallest integer such that $\beta^s \rho_0 \geq M^2 / a_{\min}$. Implement Step 1 of Algorithm 6.1 by Algorithm 6.2 (MPRGP) with parameters $\Gamma > 0$ and $\vec{\pi} \in (0,(a_{\max} + \beta^s \rho_0)^{-1}]$ to generate the iterates
\( \lambda_t^{k,0}, \lambda_t^{k,1}, \ldots, \lambda_t^{k,l} = \lambda_t^k \) for the solution of (36) starting from \( \lambda_t^{k,0} = \lambda_t^{k-1} \) with \( \lambda_t^{-1} = 0 \), where \( l = l_{t,k} \) is the first index satisfying
\[
\|g^P(\lambda_t^{k,l}, \mu_t^k, \rho_{t,k})\| \leq M\|C_t \lambda_t^{k,l}\|
\]
(38)
or
\[
\|g^P(\lambda_t^{k,l}, \mu_t^k, \rho_{t,k})\| \leq \epsilon\|b_t\| \min\{1, M^{-1}\}.
\]
(39)
Then for any problem (36), Algorithm 6.1 generates an approximate solution \( \lambda_t^{kq} \) which satisfies
\[
M^{-1}\|g^P(\lambda_t^{kq}, \mu_t^{kq}, \rho_{t,k})\| \leq \|C_t \lambda_t^{kq}\| \leq \epsilon\|b_t\|
\]
(40)
at the cost of \( O(1) \) matrix-vector multiplications by the Hessian of the augmented Lagrangian \( L_t \) for (36) and \( \rho_{t,k} \leq \rho_s \).

**Proof:** See [9].

We apply Theorem 7.1, to the following problem which is equivalent to (9):
\[
\min \frac{1}{2} \lambda^T (PFP + CQ)\lambda - \lambda^T Pd \quad \text{s.t.} \quad G\lambda = 0 \quad \text{and} \quad \lambda_t \geq -\bar{\lambda}_I.
\]
(41)
The optimality results may be applied to the solution of (41) by observing that the Lagrangian for (41) with \( \rho > C \) may be written in the form
\[
\frac{1}{2} \lambda^T (PFP + CQ)\lambda - \lambda^T Pd + \lambda^T G\mu + \frac{\rho - C}{2} \lambda^T Q\lambda
\]
which may be interpreted as Lagrangian for (41) with the penalty parameter equal to \( \rho - C \).

**Theorem 7.2.** Let \( C_1 > 0 \), let \( C \) denote a constant that satisfies (18), and let \( \rho_0 > 0 \), \( \beta > 0 \), \( M > 0 \). Let \( \{\lambda_{t,H,h}^I\}, \{\mu_{t,H,h}^I\}, \{\rho_{H,h,k}\} \) denote the iterates generated by Algorithm 6.1 with \( \|d_{H,h}\| \geq \eta_{H,h} > 0 \), \( \rho_{H,h,0} = \rho_0 \), \( \mu_{H,h}^0 = 0 \) for the solution \( \lambda_{H,h} \) of the problem (41) arising from the regular discretization of (1) with the decomposition and discretization parameters \( H \) and \( h \), respectively. Let \( s \geq 0 \) denote the smallest integer such that \( \beta^s \rho_0 \geq M^2/C \) and implement Step 1 of Algorithm 6.1 by Algorithm 6.2 in the same way as in Theorem 7.1 with \( \Gamma > 0 \), \( \sigma \in (0, (CC_1^2 + \beta^s \rho_0)^{-1}] \). For any problem (41) discretized so that \( H/h \leq C_1 \), the Algorithm 6.1 generates an approximate solution \( \lambda_{H,h} \) which satisfies
\[
M^{-1}\|g^P(\lambda_{H,h}, \mu_{H,h}, \rho_{H,h})\| \leq \|G_{H,h} \lambda_{H,h}\| \leq \epsilon\|b_{H,h}\|
\]
(42)
at a cost of \( O(1) \) matrix-vector multiplication by the Hessian of the augmented Lagrangian \( L_{H,h} \) for (41).

**Proof:** Using Theorem 5.1, it is easy to see that
\[
\lambda_{\min}(PFP + CQ) \geq C \quad \text{and} \quad \|PFP + CQ\| \leq CH^2/h^2 \leq CC_1^2
\]
for any \( H \) and \( h \) such that \( H/h \leq C_1 \). Since \( \|G\| = 1 \), we can apply Theorem 7.1 and the desired result follows. \( \square \)
8 Numerical experiments

In this section we report some results for the numerical solution of the model semi-coercive contact problem to illustrate the performance of our FETI–DP algorithm, mainly its numerical scalability. In our experiments, we used the function $f$ specified by

$$f(x_1, x_2) = \begin{cases} 
-5 & \text{for } (x_1, x_2) \in (0, 1) \times [0.75, 1) \\
0 & \text{for } (x_1, x_2) \in (0, 1) \times [0, 0.75) \cup (1, 2) \times [0.25, 1) \\
-1 & \text{for } (x_1, x_2) \in (1, 2) \times [0, 0.25)
\end{cases}.$$  

Each domain $\Omega^i$ was partitioned into identical squares with side $H = 1/2, 1/4, 1/8$. These squares were then discretized by a regular grid with the stepsize $h$. For each partition, the number of nodes on each edge, $H/h$, was taken to be 4, 8, and 16. The meshes matched across the interface for every neighboring subdomains. The computations were performed using parameter values $\rho_0 = 100, \sigma = \rho^{-1}, \Gamma = 1$, and $\lambda_0 = 0$. The stopping criterion in the conjugate gradient iteration was $||\nu(\lambda^k)|| < 10^{-5}||d||$. All experiments were performed in MATLAB.

The solution of the model problem for $H = 1/4$ and $h = 1/4$ is presented in Figure 2. Selected results of the computations for varying values of $H$ and $H/h$ are given in Table 1. The primal dimension/dual dimension/number of corners are recorded in the upper row in each field of the table, while the number of the conjugate gradient iterations required for the convergence of the solution to the given precision is recorded in the lower row. Number of outer iterations was always 2. The dependence of the number of iterations on the number of subdomains is shown in Figure 3, where each line corresponds to a fixed value for $H/h$, i.e. to one row of Table 1. The key point is that the number of the conjugate gradient iterations for a fixed ratio $H/h$ varies very moderately with the increasing number of subdomains. This indicates that the unspecified constants in Theorem 7.2 are not very large and we can observe numerical scalability in practical computations.

Table 1
Convergence results for the FETI–DP algorithm

<table>
<thead>
<tr>
<th>$H$</th>
<th>1/2</th>
<th>1/4</th>
<th>1/8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H/h = 16$</td>
<td>2312/155/8</td>
<td>9248/791/36</td>
<td>36992/3503/140</td>
</tr>
<tr>
<td></td>
<td>61</td>
<td>51</td>
<td>53</td>
</tr>
<tr>
<td>$H/h = 8$</td>
<td>648/75/8</td>
<td>2592/375/36</td>
<td>10368/1647/140</td>
</tr>
<tr>
<td></td>
<td>38</td>
<td>36</td>
<td>46</td>
</tr>
<tr>
<td>$H/h = 4$</td>
<td>200/35/8</td>
<td>800/167/36</td>
<td>3200/719/140</td>
</tr>
<tr>
<td></td>
<td>29</td>
<td>28</td>
<td>35</td>
</tr>
</tbody>
</table>
Figure 2: Solution of the model problem

Figure 3: CG iterations vs. number of subdomains

References


[22] Dostál Z., and Schöberl J., Minimizing quadratic functions over non-negative cone with the rate of convergence and finite termination. *Computational Optimization and Applications* 30, 1, 2005; 23-44.


