ON SOLVABILITY OF CONVEX NON-COERCIVE QUADRATIC PROGRAMMING PROBLEMS*

ZDENĚK DOSTÁL†

Abstract. Using a known result on minimization of convex functionals on polyhedral cones, the Frank–Wolfe theorem, and basic linear algebra, we give a simple proof that the general convex quadratic programming problem which satisfies a natural necessary condition has a solution.

Key words. Convex quadratic programming, existence of solution

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1. Introduction.

We are interested in sufficient conditions which guarantee the solvability of the quadratic programming problem to find

$$\min_{x \in \Omega} f(x) \tag{1}$$

with $\Omega = \{x : Bx \leq c\}$, $f(x) = \frac{1}{2}x^T Ax - x^T b$, $B \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n \times n}$ symmetric positive semidefinite. The letter asssumption on A implies that f is convex. Moreover, to avoid trivial cases, we assume that Ω is nonempty.

Let us recall that there is known a number number of conditions which guarantee solvability of (1). For example, if Ω is bounded, then the existence of a solution follows by the compactness argument [1], and if f is bounded from below on Ω, the there is a solution by the classical Frank–Wolfe theorem [4]. See also Eaves [3] or Blum and Oettli [2].

In many applications, the coercivity of f is one of the most useful assumptions which guarantee that there is a solution to (1). Let us recall that f is coercive with respect to Ω if $f(x) \to \infty$ for $x \to \infty$, $x \in \Omega$. For example, it is well known that the variational inequality which describes the equilibrium of a system of elastic bodies

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†Professor and Head of the Department of Applied Mathematics, FEECS VŠB–Technical University of Ostrava, 17. listopadu 15, CZ-70833 Ostrava, Czech Republic (zdenek.dostal@vsb.cz).

with some "floating" bodies in mutual contact has a solution if the energy functional is semicoercive, i.e. coercive due to the form of the linear term. See, e.g., Hlaváček et al. [5]. However, it is easy to see that a solution may exist in more general case. The point of this note is to give a necessary and sufficient condition which guarantees that a solution to (1) exists. We do not use any reference to coercivity.

Our main tools are some well-known facts about the structure of polyhedral sets like Ω, i.e. the sets described by linear inequalities, and some simple observations concerning the recession cone $\mathcal{C}$ of Ω which is defined by

$$\mathcal{C} = \{d \in \mathbb{R}^n : \ Bd \leq o\}.$$

We shall use the following nontrivial representation of the polyhedral sets like Ω.

PROPOSITION 1.1. *A set $\Omega \subseteq \mathbb{R}^n$ is polyhedral if and only if there is a nonempty set of n-vectors $\{x_1, \ldots, x_k\}$ and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that*

$$\Omega = \mathcal{C} + \mathrm{Conv}\{x_1, \ldots, x_k\},$$

where $\mathrm{Conv}\{x_1, \ldots, x_k\}$ denotes the convex hull of $x_1, \ldots, x_k$.

Proof. See, e.g., [1, Proposition B.17]. □

2. Main theorem. We shall go straight to the following formulation of the main result.

THEOREM 2.1. *Let the convex quadratic programming problem (1) be defined as above with $\Omega \neq \emptyset$. Then (1) has a solution if and only if*

$$d^T b \leq 0 \quad \mathit{for} \quad d \in \mathcal{C} \cap \mathrm{Ker} A, \tag{1}$$

where $\mathcal{C}$ is the recession cone of Ω.

Proof. Let $\overline{x}$ be a global solution of the minimization problem (1), and recall that

$$f(\overline{x} + \alpha d) - f(\overline{x}) = \alpha(A\overline{x} - b)^T d + \frac{\alpha^2}{2} d^T A d \tag{2}$$

for any $d \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. To see that (2) is satisfied, notice that if $d \in \mathcal{C} \cap \mathrm{Ker} A$, then (2) reduces to

$$f(\overline{x} + \alpha d) - f(\overline{x}) = -\alpha b^T d,$$

which is nonnegative for any $\alpha > 0$ if and only if $b^T d \leq 0$. Thus (1) is satisfied.

Let us now assume that (1) is satisfied and observe that if $c = o$, then Ω is a cone, so that a solution is known to exist even in infinite dimension (see Zeidler [6], pp. 553–556). To prove the general case, we shall use the latter fact to show that f is bounded from below on $\mathcal{C}$, so that a solution exists by the Frank–Wolfe theorem [4].

If c is arbitrary, then by Proposition 1.1 there are $x_1, \ldots, x_k \in \Omega$ such that

$$\Omega = \mathcal{C} + \mathrm{conv}\{x_1, \ldots, x_k\}, \quad \mathcal{C} = \{x : \ Bx \leq o\}.$$

Observing that $d \in \mathcal{C}$ if and only if $2d \in \mathcal{C}$, it follows that any $x \in \Omega$ can be written in the form

$$x = 2d + y, \quad d \in \mathcal{C}, \quad y \in \mathrm{Conv}\{x_1, \ldots, x_k\}.$$

Thus

$$\begin{aligned} f(x) = f(2d + y) &= d^T A d - 2b^T d + d^T A d + 2d^T A y + \frac{1}{2} y^T A y - b^T y \\ &\geq 2f(d) + (d^T A d + 2d^T A y) - b^T y. \end{aligned}$$

We have already seen that f is bounded from below on $\mathcal{C}$. Moreover, using the Euclidean norm, we get

$$-b^T y \geq -\|b\| \big(\|x_1\| + \ldots + \|x_k\| \big)$$

and

$$d^T Ad + 2d^T Ay \geq -(Ay)^T A^{\dagger} Ay = -y^T Ay \geq -\|A\|\big(\|x_1\| + \ldots + \|x_k\|\big)^2,$$

where $A^{\dagger}$ denotes the Moore–Penrose generalized inverse to A. Thus f is bounded from below on Ω and we can use the Frank–Wolfe theorem to finish the proof. □

3. Comments and conclusions. We gave a necessary and sufficient condition for solvability of convex quadratic programming problems. The proof uses a known result on minimization of the convex functional on a polyhedral cone, Frank–Wolfe theorem, and basic linear algebra. Our result is useful for analysis of solvability of convex quadratic programming problems arising from the discretization of variational inequalities that describe the equilibrium of a system of elastic bodies in mutual contact.

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