On homogenization of a quasilinear elliptic equation connected with heat conductivity

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Abstract. Homogenization of a quasilinear elliptic equation with periodic coefficients is studied. The problem describes heat conductivity in the magnetic cores of large transformers. Coefficients of the corresponding partial differential equation depend on temperature and are anisotropic due to the laminated structure of the material from which magnetic cores are made. The existence and uniqueness of solutions to the problem is proved. A homogenization procedure for the quasilinear elliptic equation is proposed and studied. The homogenized equation is a quasilinear elliptic equation whose coefficients only depend on temperature. The explicit formulas for laminated materials are given.

Keywords. quasilinear elliptic equation, homogenization, laminated material, Shauder fixed point theorem

1. Introduction

In this paper we deal with a nonlinear problem connected with heat conduction in the cores of large transformers. This problem was set up in [?] and leads to a quasilinear elliptic equation whose coefficient depend on temperature and are generally anisotropic due to the laminated structure of cores. This problem was analyzed in [?], [?] as well, where some questions connected with the reliability and finite element approximations of solutions were studied. In the paper [?] the authors proved that the problem is nor potential neither monotone and hence the usual technique guaranteeing the existence of a solution cannot be used.

In the same paper they proposed the existence proof based on the Galerkin approximation method and solved a numerical model problem connected with a quasilinear elliptic equation, whose coefficients were homogeneous and were obtained experimentally. We can say that they implicitly used some homogenous values for the material of the cores, since the original material was not homogenous because of its laminated structure.

In this paper we deal with the homogenization of a nonlinear problem connected with the same quasilinear elliptic equation with periodic coefficients and propose a method which makes possible to approximate this problem by a quasilinear elliptic equation with homogeneous coefficients.

Homogenization methods for some nonlinear variational problem were proposed and worked out in [?]. But these methods work for potential and convex functionals, which is not our case because our operator is not potential.

The formulas for homogenized coefficients are derived and the usual limit theorems connected with homogenization are proved. The explicit formulas for laminated materials are derived as well, which corresponds to the original problems of the heat conduction in the cores of transformers.

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We propose a new existence proof based on the Schauder theorem.

Let us remark that in the case of homogeneous and isotropic medium the well known
Kircho transformation (see [?]) can be applied and the problem can be transform into
a linear problem with the Laplace operator, but this method cannot be used for our
problem because of the anisotropy of coefficients.

2. Setting up the problem

In this section we formulate a quasilinear elliptic problem whose classical formulation
reads
\begin{equation}
-\frac{\partial}{\partial x_i} \left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) = f(x) \quad \text{in } \Omega,
\end{equation}
where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with a Lipschitz boundary. In the equation above
and in the rest of this paper summation over repeated indices is assumed.

The coefficients of the equation are bounded functions measurable in \( x \) for all \( u \).

The coefficients satisfy the symmetry assumptions
\begin{equation}
a_{ij}(x,u) = a_{ji}(x,u), \quad i, j = 1, \ldots n.
\end{equation}

For any \( \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n \) the inequality
\begin{equation}
\nu \eta_i \eta_i \leq a_{ij}(x,u) \eta_i \eta_j
\end{equation}
holds, where \( \nu \) is a positive constant independent of \( x \in \Omega \) and \( u \in R \).

For any \( u_1, u_2 \in R \) the inequalities
\begin{equation}
| a_{ij}(x,u_1) - a_{ij}(x,u_2) | \leq C_L | u_1 - u_2 |, \quad i, j = 1, \ldots n
\end{equation}
hold, where \( C_L \) is a positive constant independent of \( x \in \Omega \) and \( u \in R \).

The inequality (??) results in the continuity of the coefficients in \( u \) for all \( x \). The
last fact together with the measurability of coefficients in \( x \) yields the measurability of
\( a_{ij}(x,g(x)) \), \( i, j = 1, \ldots n \) for all measurable \( g(x) \), which is implicitly used in the whole
paper.

In this paper we analyze weak solutions to (??), which means that we solve the
variational equality
\begin{equation}
\int_{\Omega} a_{ij}(x,u) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = \int_{\Omega} f u \, dx,
\end{equation}
where \( f \in L^2(\Omega) \) and \( u, v \in H^1_0(\Omega) \) which is the Sobolev space of functions which belong
to \( L^2(\Omega) \) together with their first derivatives and whose traces vanish on \( \partial \Omega \). We say
that \( u \) is a solution to (??) if this equation holds for all \( v \in H^1_0(\Omega) \).

In the article [?], the existence and uniqueness of a solution to (??) was proved and the
proof of existence is based on the Galerkin method which makes possible to construct a
sequence of approximate solutions converging to a solution to (??).
3. Existence and uniqueness of a solution

In this section we deliver a new proof of the existence of a solution which is based on the Schauder theorem. Let us mentioned that the differential operator (2.1) is nor potential neither monotone (see [?]), so the usual methods guaranteeing the existence of a solution cannot be applied.

**Theorem 3.1.** Let \( a_{ij}(x, u), x \in \Omega, u \in \mathbb{R}, i, j = 1, \ldots, n \) be bounded, measurable in \( x \) for all \( u \), satisfy the assumptions (??) – (??), and \( f \in L^2(\Omega) \). Then there exists a unique solution to the variational equality (??)

**Proof.** Let us define the function \( S : L^2(\Omega) \to H^1_0(\Omega) \) as follows:

\[
S(u) = v,
\]

where \( v \in H^1_0(\Omega) \) is a solution to the linear variational equation

\[
\int_{\Omega} a_{ij}(x, u) \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_i} \, dx = \int_{\Omega} f w \, dx
\]

which holds for all \( w \in H^1_0(\Omega) \).

From (??) it follows that the solution \( v \) is unique and there exists a positive constant \( C \) independent of \( u \) such that the inequality

\[
\|S(u)\|_{H^1_0(\Omega)} \leq C \|f\|_{L^2(\Omega)}
\]

holds.

Let us prove the continuity of \( S \). Consider a sequence \( u_l \in L^2(\Omega) \) such that

\[
u_l \to u_0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad l \to \infty.\]

Let \( v_l, v_0 \) be the corresponding values of \( S \), which means that these functions are the solutions to the variational equations

\[
\int_{\Omega} a_{ij}(x, u_l) \frac{\partial v_l}{\partial x_j} \frac{\partial w}{\partial x_i} \, dx = \int_{\Omega} f w \, dx,
\]

\[
\int_{\Omega} a_{ij}(x, u_0) \frac{\partial v_0}{\partial x_j} \frac{\partial w}{\partial x_i} \, dx = \int_{\Omega} f w \, dx
\]

which hold for all \( w \in H^1_0(\Omega) \).

After simple operations, (??) yield the relation

\[
\int_{\Omega} a_{ij}(x, u_l) \frac{\partial (v_l - v_0)}{\partial x_j} \frac{\partial (v_l - v_0)}{\partial x_i} \, dx = \int_{\Omega} \left( a_{ij}(x, u_0) - a_{ij}(x, u_l) \right) \frac{\partial v_0}{\partial x_j} \frac{\partial (v_l - v_0)}{\partial x_i} \, dx.
\]

From (??) it follows the inequality

\[
\alpha \| v_l - v_0 \|_{H^1_0(\Omega)}^2 \leq \int_{\Omega} a_{ij}(x, u_l) \frac{\partial (v_l - v_0)}{\partial x_j} \frac{\partial (v_l - v_0)}{\partial x_i} \, dx,
\]

where \( \alpha \) is a positive constant independent of \( l \).
Summarizing (3.5) and (3.6), we have the inequality
\[ (a_{ij}(x, u_0) - a_{ij}(x, u_l)) \frac{\partial v_0}{\partial x_j} \to 0 \quad \text{in } L^2(\Omega) \quad \text{as } l \to \infty, \]

If we prove the limits
\[ (a_{ij}(x, u_0) - a_{ij}(x, u_l)) \frac{\partial v_0}{\partial x_j} \to 0 \quad \text{in } L^2(\Omega) \quad \text{as } l \to \infty, \]

then from (3.5) and the boundedness of the sequence \( v_l \) in \( H^1_0(\Omega) \) it follows the continuity of \( S \).

Let \( \epsilon \) be an arbitrary positive real number, \( \mu \) be the Lebesgue measure in \( \mathbb{R}^n \), then denote
\[ E_l = \{ x \in \Omega \mid |u_l(x) - u_0(x)| > \sqrt{\epsilon} \}. \]

From the convergence of \( v_l \) to \( v_0 \) in \( L^2(\Omega) \) it follows
\[ \mu(E_l) \to 0 \quad \text{as } l \to \infty. \]

Moreover, there exists \( \delta > 0 \) such that for any \( E \subset \Omega, \mu(E) < \delta \) the inequalities
\[ \int_E \left( \frac{\partial v_0}{\partial x_j} \right)^2 \, dx < \epsilon, \quad j = 1, \ldots, n \]
hold, which is a property of the Lebesgue measure (see [2]).

On considering (3.5) and the definition \( E_l \), the inequality
\[ \int_{\Omega \setminus E_l} \left( (a_{ij}(x, u_0) - a_{ij}(x, u_l)) \frac{\partial v_0}{\partial x_j} \right)^2 \, dx \leq \epsilon C \]
holds, where \( C \) is a positive constant independent of \( \epsilon \) and \( l \).

From (3.5), (3.6) it follows (3.7), which gives the continuity of \( S \).

The inequality (3.7) yields the boundedness of the range of \( S \) in \( H^1_0(\Omega) \). On considering that the embedding \( P : H^1_0(\Omega) \subset L^2(\Omega) \) is compact, we can see that the transformation \( T = P \circ S : L^2(\Omega) \rightarrow L^2(\Omega) \) is compact and continuous.

Applying the Schauder fixed point theorem to \( T \), we have a fixed point \( u \) of \( T \). From the definition of \( T \) it follows that \( u \) belongs to \( H^1_0(\Omega) \) and is a solution to the problem.

The uniqueness of the problem is based on the method described in [2] and we give the proof for the convenience of the reader.

Let there exist the two solutions \( u_1, u_2 \), then define the subsets of \( \Omega \)
\[ E = \{ x \mid u_2 - u_1 > 0 \}, \]
\[ E_\epsilon = \{ x \mid u_2 - u_1 > \epsilon \}, \]
where \( \epsilon > 0 \).

Let us define the function
\[ v_\epsilon(x) = \begin{cases} \epsilon & \text{if } x \in E_\epsilon \\ 0 & \text{if } x \in \Omega \setminus E \\ u_2 - u_1 & \text{if } x \in E \setminus E_\epsilon \end{cases}. \]
This function can be expressed as \( \gamma \circ (u_2 - u_1) \), where \( \gamma : R \to R \) is defined as follows:

\[
\gamma(t) = \begin{cases} 
\epsilon & \text{if } t > \epsilon \\
0 & \text{if } t < 0 \\
t & \text{if } t \in \langle 0, \epsilon \rangle.
\end{cases}
\]

From this it follows (see [?]) that \( v_\epsilon \in H^1_0(\Omega) \) and the equality

(3.11) \( \text{grad } v_\epsilon = 0 \quad \text{on } E_\epsilon \cup (\Omega \setminus E) \).

holds.

Let us consider the inequality

(3.12) \( \alpha \parallel \text{grad } v_\epsilon \parallel^2_{L^2(\Omega)} \leq \int_\Omega a_{ij}(x, u_1) \frac{\partial u_2}{\partial x_j} \frac{\partial v_\epsilon}{\partial x_i} dx \),

where \( \alpha \) is independent of \( u_1, v_\epsilon \), which is a result of (??)

If we employ the fact that \( u_1, u_2 \) are solutions to the problem, then we have the relation

(3.13) \( \int_\Omega (a_{ij}(x, u_1) - a_{ij}(x, u_2)) \frac{\partial u_2}{\partial x_j} \frac{\partial v_\epsilon}{\partial x_i} dx = \int_{E \setminus E_\epsilon} (a_{ij}(x, u_1) - a_{ij}(x, u_2)) \frac{\partial u_2}{\partial x_j} \frac{\partial v_\epsilon}{\partial x_i} dx \).

From (??) it follows

(3.14) \( \int_{E \setminus E_\epsilon} (a_{ij}(x, u_1) - a_{ij}(x, u_2)) \frac{\partial u_2}{\partial x_j} \frac{\partial v_\epsilon}{\partial x_i} dx \leq \epsilon C_L \parallel \text{grad } v_\epsilon \parallel_{L^2(\Omega)} \parallel \text{grad } u_2 \parallel_{L^2(E \setminus E_\epsilon)} \).

The assumption (??) yields

(3.15) \( \int_{E \setminus E_\epsilon} (a_{ij}(x, u_1) - a_{ij}(x, u_2)) \frac{\partial u_2}{\partial x_j} \frac{\partial v_\epsilon}{\partial x_i} dx \leq \epsilon C_L \parallel \text{grad } v_\epsilon \parallel_{L^2(\Omega)} \parallel \text{grad } u_2 \parallel_{L^2(E \setminus E_\epsilon)} \).

From (??), (??), (??) and (??) it follows the inequality

(3.16) \( \parallel \text{grad } v_\epsilon \parallel_{L^2(\Omega)} \leq \epsilon C \parallel \text{grad } u_2 \parallel_{L^2(E \setminus E_\epsilon)} \),

where \( C \) is a positive constant independent of \( \epsilon \).

On considering the definition \( v_\epsilon \), we have the estimate

(3.17) \( \mu(E_\epsilon) \leq \frac{1}{\epsilon^2} \int_\Omega |v_\epsilon|^2 dx \leq \frac{1}{\epsilon^2} C \parallel \text{grad } v_\epsilon \parallel^2_{L^2(\Omega)} \).

The second inequality in (??) follows from the Poincaré inequality in \( H^1_0(\Omega) \), where \( C \) is a constant independent of \( \epsilon \).

Employing (??), (??) and (??), we have the inequality

(3.18) \( \mu(E_\epsilon) \leq C \parallel \text{grad } u_2 \parallel^2_{L^2(E \setminus E_\epsilon)} \),

where \( C \) is a constant independent of \( \epsilon \). Moreover, the definitions of \( E, E_\epsilon \) yield the limit

(3.19) \( \mu(E \setminus E_\epsilon) \to 0 \quad \text{as } \epsilon \to 0 \).
This together with the last inequality yields

\[(3.18)\quad \mu(E_\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0.\]

On considering the definitions of $E$, $E_\epsilon$, the limit (3.18) yields $\mu(E) = 0$.

We can apply the same method for $u_1 - u_2$, which gives the whole proof. \qed

4. Homogenization – 1D case

We start with the simplest example of homogenization. Let us consider that $a(x, u)$

defined on $\mathbb{R} \times \mathbb{R}$ is measurable in $x$ for all $u$ and for all $x, u_1, u_2$ the inequalities

\[0 < \nu \leq a(x, u) \leq C_M,\]

\[|a(x, u_1) - a(x, u_2)| \leq C_L |u_1 - u_2|\]

hold. Moreover, $a(x, u)$ is periodic in $x$ with the period $l$.

Denote

\[a^\epsilon(x, u) = a\left(\frac{x}{\epsilon}, u\right).\]

Let $(c, d) \subset \mathbb{R}$ be a bounded interval and $f \in L^2(c, d)$, then we can study the variational equations

\[(4.2)\quad \int_c^d a^\epsilon(x, u) \frac{du}{dx} \frac{dv}{dx} dx = \int_c^d f(v) dx,
\]

where $u^\epsilon, v \in H^1_0(c, d)$ and $u^\epsilon$ is a solution to (4.2) if this equation is satisfied for all $v \in H^1_0(c, d)$.

If the function $v(x)$ is periodic with the period $l$, then we denote

\[\langle v \rangle = \frac{1}{l} \int_0^l v(x) dx.\]

If the function $w(x, u)$ defined on $\mathbb{R} \times \mathbb{R}$ is periodic in $x$ with the period $l$, then

\[\langle w \rangle(u) = \frac{1}{l} \int_0^l w(x, u) dx.\]

If we understand the variable $u$ in $w(x, u)$ as a parameter, then we can use the identity

\[\langle w \rangle(u) = \langle w(x, u) \rangle.\]

**Lemma 4.1.** Let the function $a(x, u)$ defined on $\mathbb{R} \times \mathbb{R}$ be bounded, measurable in $x$ for all $u$, periodic in $x$, and satisfy the assumption (4.1). Then the function $\langle a \rangle(u)$ defined on $\mathbb{R}$ satisfy the identity

\[|\langle a \rangle(u_1) - \langle a \rangle(u_2)| \leq C_L |u_1 - u_2|\]

for all $u_1, u_2$.

The proof of this lemma is a simple consequence of the definition of $\langle a \rangle(u)$. 

\[\text{6}\]
Lemma 4.2. Let $f(x)$ defined on $R$ is periodic and $f^\epsilon(x) = f(\frac{x}{\epsilon})$. Then for any bounded interval $(c,d) \subset R$ the limit

$$f^\epsilon \to f$$ in $L^2(c,d)$ as $\epsilon \to 0$

holds.

The proof of this lemma can be found in [?], [?].

Lemma 4.3. Let the function $a(x,u)$ defined on $R \times R$ be bounded, measurable in $x$ for all $u$, periodic in $x$, and satisfy the assumption (?). Let $u^\epsilon(x)$ be a sequence of continuous functions defined on the bounded interval $(c,d)$ such that the limit

$$u^\epsilon \to u^0$$ in $C((c,d))$ as $\epsilon \to 0$

holds. Then the following limit

$$a^\epsilon(x,u^\epsilon(x)) \to (a)(u^0(x))$$ in $L^2(c,d)$ as $\epsilon \to 0$

holds true.

Proof. We have to prove the limit

$$\int_c^d a^\epsilon(x,u^\epsilon(x))v(x)dx \to \int_c^d (a)(u^0(x))v(x)dx$$ as $\epsilon \to 0$

for arbitrary $v \in L^2(c,d)$.

On considering that $a^\epsilon(x,u^\epsilon(x))$ is a bounded sequence in $L^2(c,d)$, it is sufficient to prove

$$\int_e^g a^\epsilon(x,u^\epsilon(x))dx \to \int_e^g (a)(u^0(x))dx$$ as $\epsilon \to 0$

for any subinterval $(e,g) \subset (c,d)$. This follows from the fact that the linear span of characteristic functions defined on subintervals of $(c,d)$ is dense in $L^2(c,d)$.

Let $(e,g)$ be an arbitrary subinterval of $(c,d)$ and $\delta > 0$, then there exists a division of $(e,g)$

$$e = e_0 < e_1 < \ldots < e_k = g$$

and numbers $b_i$, $i = 0, \ldots, k - 1$ such that the inequalities

$$|b_i - u^\epsilon(x)| < \delta, \quad |b_i - u^0(x)| < \delta, \quad i = 0, \ldots, k - 1$$

hold for sufficiently small $\epsilon$ if $x \in (e_i, e_{i+1})$.

From (??) and (??) it follows the inequalities

$$\int_{e_i}^{e_{i+1}} | a^\epsilon(x,b_i) - a^\epsilon(x,u^\epsilon(x)) | \, dx \leq \delta C_L (e_{i+1} - e_i),$$

$$| (a)(b_i) - (a)(u^0(x)) | \leq \delta C_L, \quad i = 0, \ldots, k - 1$$

if $x \in (e_i, e_{i+1})$ and $\epsilon$ is sufficiently small.
Lemma 4.2 yields the inequalities

\[ \int_{e_i}^{e_{i+1}} (a^\varepsilon(x, b_i) - \langle a(x, b_i) \rangle) dx < \frac{\delta}{k}, \quad i = 0, \ldots, k - 1 \]

which holds for sufficiently small \( \varepsilon \).

Let us consider the inequalities

\[
\int_{e_i}^{e_{i+1}} \int_{c^1}^{2} \left| a^\varepsilon(x, b_i) - a^\varepsilon(x, u) \right| dx + \int_{e_i}^{e_{i+1}} \int_{c^1}^{2} \left| a^\varepsilon(x, b_i) - a^\varepsilon(x, b_i) \right| dx + \int_{e_i}^{e_{i+1}} \int_{c^1}^{2} \left| a^\varepsilon(x, b_i) - \langle a(x, b_i) \rangle \right| dx.
\]

These inequalities together with (??), (??) yield the inequality

\[ \int_{e_i}^{e_{i+1}} \left| \int_{c^1}^{2} a^\varepsilon(x, u^\varepsilon(x)) - \langle a(x, u^0(x)) \rangle dx \right| \leq \delta + 2\delta C_L (g - c), \]

which gives the limit (??).

\[ \square \]

**Theorem 4.1.** Let the function \( a(x, u) \) defined on \( R \times R \) be bounded, measurable in \( x \) for all \( u \), periodic in \( x \), and satisfy the assumption (??). Let \( u^\varepsilon \in H^1_0(c, d) \) be the sequence of solutions to (??). Then the limits

\[ u^\varepsilon(x) \to u^0(x) \quad \text{in} \quad H^1_0(c, d) \quad \text{as} \quad \varepsilon \to 0, \]

\[ \xi^\varepsilon(x) \to \xi^0(x) \quad \text{in} \quad C((c, d)) \quad \text{as} \quad \varepsilon \to 0 \]

hold, where \( u^0 \) is a solution to the variational equation

\[ \int_{c^1}^{2} a^0(u^0) \frac{du^0}{dx} dv = \int_{c^1}^{2} f v dx \]

which holds for all \( v \in H^1_0(c, d) \). The symbols above are defined by the identities

\[ a^0(u) = \left( \frac{1}{\alpha} \langle u \rangle \right)^{-1}, \]
\[ \xi^\epsilon(x) = a^\epsilon(x, u^\epsilon(x)) \frac{du^\epsilon(x)}{dx}, \]
\[ \xi^0(x) = a^0(u^0(x)) \frac{du^0(x)}{dx}. \]

\textbf{Proof.} From the inequalities \((??)\) it follows that the sequence \(u^\epsilon(x)\) and \(\xi^\epsilon(x)\) are bounded in \(H^1_0(c, d)\) and \(L^2(c, d)\). Thus there exist subsequences which weakly converge to \(u^* \in H^1_0(c, d)\) and \(\xi^* \in L^2(c, d)\). We denote these subsequences by \(u^\epsilon, \xi^\epsilon\). Since \((c,d)\) is a bounded interval in \(R\), the space \(H^1_0(c, d)\) is compactly embedded in \(C((c,d))\) and \(u^\epsilon\) converges to \(u^*\) in \(C((c,d))\) (see \([??]\)).

From the definition \(\xi^\epsilon\) it follows that the equations
\[ \int_c^d \xi^\epsilon \frac{dv}{dx} dx = \int_c^d f v dx \]
hold for all \(v \in H^1_0(c, d)\).

From \((??)\) it follows that \(\xi^\epsilon\) have the generalized derivatives which are equal to \(f \in L^2(c, d)\). The last fact yields the limit
\[ \xi^\epsilon \rightarrow \xi^* \quad \text{in} \quad C((c,d)) \quad \text{as} \quad \epsilon \rightarrow 0. \]

Let us consider the equations
\[ \frac{du^\epsilon(x)}{dx} = \frac{1}{a^\epsilon(x, u^\epsilon(x))} \xi^\epsilon, \]
then Lemma 4.3 and \((??)\) yield the limit
\[ \frac{1}{u^\epsilon(x, u^\epsilon(x))} \xi^\epsilon \rightarrow \left(\frac{1}{a}\right)(u^*(x)) \xi^* \quad \text{in} \quad L^2(c, d) \quad \text{as} \quad \epsilon \rightarrow 0. \]
The last limit together with \((??)\) yield the relation
\[ \frac{du^*(x)}{dx} = \left(\frac{1}{a}\right)(u^*(x)) \xi^* \]
which gives the equation
\[ \xi^* = a^0(u^*(x)) \frac{du^*}{dx}. \]

On considering \((??)\), the last equations yields that \(u^*\) is a solution to \((??)\). From lemma 4.1 and Theorem 3.1 it follows that this solution is unique and the whole sequence \(u^\epsilon\) converges to the solution of the problem \((??)\), which yields \(u^* = u^0\) and \(\xi^* = \xi^0\).

This theorem shows that some nonlinear nonhomogeneous material can be substituted by nonlinear homogeneous material in one–dimensional case. In the rest of this paper this result will be generalized for arbitrary dimension.
5. Some auxiliary results

In this section we are going to formulate some known results from the theory of elliptical partial differential equations and theory of homogenization for such type of equations.

Let \( a_{ij}(x) \) be bounded measurable functions defined on a bounded domain with a Lipschitz boundary and satisfy the assumptions
\[
\begin{align*}
\text{5.1} \\
a_{ij}(x) &= a_{ji}(x), \quad i, j = 1, \ldots, n, \\
\nu \xi_i \xi_j &\leq a_{ij}(x) \xi_i \xi_j,
\end{align*}
\]
where \( \nu \) is a positive constant and the inequality holds for all \( \xi = (\xi_1, \ldots, \xi_n) \) from \( \mathbb{R}^n \).

Moreover, the following estimates
\[
\text{5.2} \\
| a_{ij}(x) | \leq C_M, \quad i, j = 1, \ldots, n
\]
hold, where \( C \) is a positive constant.

Let us study the variational equations
\[
\text{5.3} \\
\int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx = \int_{\Omega} fv dx,
\]
where \( f \in L^2(\Omega) \) and \( u, v \in H^1_0(\Omega) \). We say that \( u \) is a solution to \( (5.3) \) if this variational equation is valid for all \( v \in H^1_0(\Omega) \).

The proof of the main homogenization result is based on the following theorem, the proof of which can be found, for instance, in [?], [?].

**Theorem 5.1.** Let \( a_{ij}(x), i, j = 1, \ldots, n \) satisfy (5.1, 5.2), \( \Omega \) be a bounded domain with a Lipschitz boundary and \( f \in L^p(\Omega) \), where \( p > \frac{2n}{n+2} \). Then the solution to the variational equation (5.3) is continuous on \( \Omega \) and the inequality
\[
\text{5.4} \\
| u(x) - u(y) | \leq C_N \| f \|_{L^p(\Omega)} | x - y |^\lambda
\]
holds for any \( x, y \in \Omega \), where \( C_N \) is a positive constant which depends only on \( \nu, C_M, \Omega, n \) and \( \lambda \) satisfies the inequalities \( 0 < \lambda < 1 \) and depends only on \( \nu, C_M, \Omega, n \) as well.

Now let us formulate some very well known results from the theory of homogenization for linear elliptic differential equations.

Let \( a_{ij}(x), i, j = 1, \ldots, n \) be defined on \( \mathbb{R} \) and be periodic with the period \( l = (l_1, \ldots, l_n) \). Moreover, let us define \( a'_{ij}(x) = a_{ij}(\frac{x}{l}) \).

Let \( f(x) \) and \( g(x, u) \) be defined on \( \mathbb{R}^n \) and \( \mathbb{R}^n \times \mathbb{R} \), and be periodic in \( x \) with the period \( l = (l_1, \ldots, l_n) \). Then we can define
\[
\langle f \rangle = \frac{1}{|Y|} \int_Y f(x) dx,
\]
\[
\langle g \rangle(u) = \frac{1}{|Y|} \int_Y g(x, u) dx,
\]
where \( Y = \langle 0, l_1 \rangle \times \ldots \times \langle 0, l_n \rangle \) and \( |Y| = l_1 \ldots l_n \).

If we understand the variable \( u \) in \( g(x, u) \) as a parameter, then we can use the identity
\[
\langle g \rangle(u) = \langle g(x, u) \rangle.
\]
The last formula will be applied in the next section in the proof of the main homoge-
nization theorem.

Let us denote $H^1_{\text{per}}(Y)$ the Sobolev space of periodic function with the period $l = (l_1, \ldots, l_n)$ whose first and second derivative belong to $L^2(Y)$. This space is equipped with the classical norm on $H^1(Y)$.

Let $a_{ij}(x)$, $i, j = 1, \ldots, n$ satisfy (5.5), (5.5), then we can study the variational equations

$$
\int_Y a_{ij}(x) \left( \frac{\partial \chi_k}{\partial x_j} + \delta_{jk} \right) \frac{\partial \psi}{\partial x_i} \, dx = 0, \quad k = 1, \ldots, n,
$$

where the functions $\chi_k \in H^1_{\text{per}}(Y)$ are solutions to (5.5) if these equations are satisfied for all $\psi \in H^1_{\text{per}}(Y)$. It is known that the solution are unique to within an additive constant.

Let us define the terms

$$
a_{ij}^0 = (a_{ij}) + \left\langle a_{ik} \frac{\partial \chi_i}{\partial x_k} \right\rangle, \quad i, j = 1, \ldots, n.
$$

The terms are the very well known formulas for the homogenized coefficients of the linear elliptic equation with rapidly oscillating coefficients.

Let $a_{ij}^z(x)$, $i, j = 1, \ldots, n$ be periodic functions in $x$ with the periods $l^z = (l_1^z, \ldots, l_n^z)$ for $z = 1, \ldots, m$, and $\Omega$ be a domain which is divided into subdomains $\Omega^z$, $z = 1, \ldots, m$ with Lipschitz boundaries.

Let us define, for any $\epsilon > 0$,

$$
a_{ij}^{\epsilon z}(x) = a_{ij}(x) \left( \frac{x}{\epsilon} \right) \quad \text{if } x \in \Omega^z,
\quad a_{ij}^{\epsilon 0}(x) = a_{ij}^0 \quad \text{if } x \in \Omega^z, \quad i, j = 1, \ldots, n,
$$

where $a_{ij}^{\epsilon z}$ are given by the formulas (5.5). In these formulas $a_{ij}(x)$ are substituted for $a_{ij}^z(x)$, the solutions $\chi_k(x)$, $k = 1, \ldots, n$, to the variational equation (5.5) for the corresponding periodic solutions to the variational equation with the coefficients $a_{ij}^z(x)$ on $Y^z = (0, l_1^z) \times \cdots \times (0, l_n^z)$, and the mean value $\langle \rangle$ integrates over $Y^z$, where $z = 1, \ldots, m$.

Now we can study the variational equations

$$
\int_\Omega a_{ij}^{\epsilon z}(x) \frac{\partial u^\epsilon}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = \int_\Omega f v \, dx,
\quad i, j = 1, \ldots, n
$$

where $u^\epsilon$, $u^0$ belong to $H^1_0(\Omega)$ and $f$ belongs to $L^2(\Omega)$. The functions $u^\epsilon$, $u^0$ are solutions to the equations (5.5) if these equations are satisfied for all $v \in H^1_0(\Omega)$.

**Theorem 5.2.** Let $u^\epsilon$, $u^0$ be solutions to (5.5), then the limits

$$
u^\epsilon \rightharpoonup u^0 \quad \text{in } H^1_0(\Omega) \quad \text{as } \epsilon \to 0,
\quad \xi^\epsilon_i \rightharpoonup \xi^0_i \quad \text{in } L^2(\Omega) \quad \text{as } \epsilon \to 0, \quad i = 1, \ldots, n
$$

hold, where

$$
\xi^\epsilon_i = a_{ij}^{\epsilon z} \frac{\partial u^\epsilon}{\partial x_j}, \quad \xi^0_i = a_{ij}^{\epsilon 0} \frac{\partial u^0}{\partial x_j}, \quad i = 1, \ldots, n.
$$

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The proof of this theorem is based on the G-convergence and can be found, for instance, in [7]. This theorem shows that solutions to the linear elliptic equation with rapidly oscillating coefficients converge to solution to the linear elliptic equation with constant coefficients on the subdomains $\Omega^z, z = 1 \ldots m$.

6. Homogenization – general case

In this section the formulas for the homogenized coefficients of the quasilinear elliptic equation are derived. In the sequel we consider that $a_{ij}(x, u)$ defined on $\mathbb{R}^n \times R$ be bounded, measurable in $x$ for all $u$, periodic in $x$ with the period $l = (l_1, \ldots, l_n)$, and satisfy the assumptions (8.1)-(8.5).

Let us study the following variational equations

$$
\int_Y a_{ij}(x, u) \left( \frac{\partial \chi_k(x, u)}{\partial x_j} + \delta_{jk} \right) \frac{\partial \psi}{\partial x_i} \, dx = 0, \quad k = 1, \ldots, n,
$$

where $u$ is a parameter from $R$. These equations play important role in our considerations.

To guarantee the uniqueness of (8.1), we study the equation on the space $W = \{ v \in H^1_{\text{per}}(Y) \mid \int_Y v \, dx = 0 \}$ which is equipped with the classical norm on $H^1(Y)$.

Solutions to (8.1) are the functions $\chi_k(x, u), k = 1, \ldots n$ defined on $Y \times R$ such that $\chi_k(x, u) \in W$ for all $u \in R$ and these functions satisfy the variational equations (8.1) for all $\psi \in W$.

**Lemma 6.1.** Let $a_{ij}(x, u)$ defined on $\mathbb{R}^n \times R$ be bounded, measurable in $x$ for all $u$, periodic in $x$ with the period $l = (l_1, \ldots, l_n)$, and satisfy the assumptions (8.1)-(8.5). Then there exists the unique solutions $\chi_k(x, u), k = 1, \ldots n$ to (8.1) such that the inequalities

$$
\| \chi_k(\cdot, u_1) - \chi_k(\cdot, u_2) \|_W \leq C | u_1 - u_2 |, \quad k = 1, \ldots n
$$

hold for any $u_1, u_2 \in R$, where $C$ is a constant independent of $u$.

**Proof.** First of all let us mention that the inequality

$$
\| v \|_{L^2(Y)} \leq C \| \text{grad } v \|_{L^2(Y)}
$$

holds for all $v \in W$, where $C$ is a constant independent of $v$. This estimate is a consequence of the Poincaré inequality (see [3]).

From (8.1) and (8.2) it follows the inequality

$$
\alpha \| v \|_W^2 \leq \int_Y a_{ij}(x, u) \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx,
$$

where $\alpha$ is a positive constant independent of $v$ and the parameter $u$. The last inequality guarantees the existence and uniqueness of the variational equation (8.1).
Let \( \chi_k(x,u_1), \chi_k(x,u_2) \) be solutions to (??), then after simple operation we have the relations
\[
\int_Y a_{ij}(x,u_1) \left( \frac{\partial \chi_k(x,u_1)}{\partial x_j} - \frac{\partial \chi_k(x,u_2)}{\partial x_j} \right) \left( \frac{\partial \chi_k(x,u_1)}{\partial x_i} - \frac{\partial \chi_k(x,u_2)}{\partial x_i} \right) dx = \\
(6.4) \int_Y (a_{ij}(x,u_2) - a_{ij}(x,u_1)) \left( \frac{\partial \chi_k(x,u_2)}{\partial x_j} + \delta_{kj} \right) \left( \frac{\partial \chi_k(x,u_1)}{\partial x_i} - \frac{\partial \chi_k(x,u_2)}{\partial x_i} \right) dx,
\]
\( k = 1, \ldots n \).

From the inequality (??) it follows the estimates
\[
\|\chi_k(x,u)\|_W < C, \quad k = 1, \ldots n,
\]
where \( C \) is a constant independent of \( u \). Moreover, (??) yields the inequalities
\[
(6.5) \quad \alpha \|\chi_k(.,u_1) - \chi_k(.,u_2)\|_W^2 \leq \int_Y a_{ij}(x,u_1) \left( \frac{\partial \chi_k(x,u_1)}{\partial x_j} - \frac{\partial \chi_k(x,u_2)}{\partial x_j} \right) \left( \frac{\partial \chi_k(x,u_1)}{\partial x_i} - \frac{\partial \chi_k(x,u_2)}{\partial x_i} \right) dx,
\]
\( k = 1, \ldots n \),

where \( \alpha \) is the same constant as in (??).

From the Hölder inequality it follows
\[
(6.6) \quad \leq \left( \sum_{i,j=1}^n \|a_{ij}(x,u_1) - a_{ij}(x,u_2)\|_{L^\infty(Y)} \right)^2 \cdot \\
\left( \|\chi_k(.,u_2)\|_W + \sqrt{Y} \right) \|\chi_k(.,u_1) - \chi_k(.,u_2)\|_W, \quad k = 1, \ldots n.
\]

From (??), (??), (??) it follows the inequalities
\[
(6.7) \quad \left( \sum_{i,j=1}^n \|a_{ij}(x,u_1) - a_{ij}(x,u_2)\|_{L^\infty(Y)} \right)^2 \cdot \\
\left( \|\chi_k(.,u_2)\|_W + \sqrt{Y} \right) \|\chi_k(.,u_1) - \chi_k(.,u_2)\|_W \leq C \|u_1 - u_2\| \|\chi_k(.,u_1) - \chi_k(.,u_2)\|_W, \quad k = 1, \ldots n,
\]
where \( C \) is independent of \( u_1, u_2 \).

Employing (??), (??), (??), (??), we have the assertion of this lemma.

Let us define
\[
(6.8) \quad a_{ij}^0(u) = \langle a_{ij} \rangle(u) + \left( a_{ik} \frac{\partial \chi_j}{\partial x_k} \right)(u), \quad i,j = 1, \ldots n,
\]
where \( \chi_j(x,u) \) are the solutions to (??).
Lemma 6.2. Let all assumptions of Lemma 6.1 be satisfied. The inequalities
\[ |a_{ij}^0(u_1) - a_{ij}^0(u_2)| < C|u_1 - u_2|, \quad i, j = 1, \ldots, n \]
hold for all \( u_1, u_2 \in R \), where \( C \) is a constant independent of \( u \).

Proof. From the definition of \( h_i(\cdot) \) it follows the inequality
\[ (6.10) \quad \left| \left<h_i(\cdot) - h_i(\cdot), \frac{\partial x_j(x, u_1)}{\partial x_k} - \frac{\partial x_j(x, u_2)}{\partial x_k}\right> \right| \leq C|u_1 - u_2|, \quad i, j = 1, \ldots, n, \]
where \( C \) is independent of \( u_1, u_2 \).

Let us analyze the second term in the formula for \( a_{ij}^0(u) \). Then we have the inequalities
\[ (6.11) \quad \left| \left<a_{ik}(x, u_1) - a_{ik}(x, u_2), \frac{\partial x_j(x, u_1)}{\partial x_k} - \frac{\partial x_j(x, u_2)}{\partial x_k}\right> \right| \leq C \sum_{k=1}^n \|a_{ik}(\cdot, u_1) - a_{ik}(\cdot, u_2)\|_{L^\infty(Y)} \|\chi_j(\cdot, u_1)\|_W, \]
where we can obtain after easy operations.

The following inequalities
\[ (6.12) \quad \left| \left<a_{ik}(x, u_1) - a_{ik}(x, u_2), \frac{\partial x_j(x, u_1)}{\partial x_k} - \frac{\partial x_j(x, u_2)}{\partial x_k}\right> \right| \leq C \sum_{k=1}^n \|a_{ik}(\cdot, u_2)\|_{L^\infty(Y)} \|\chi_j(\cdot, u_1) - \chi_j(\cdot, u_2)\|_W, \]
i, j = 1, \ldots, n,
are an easy consequence of the Hölder inequality, where \( C \) is independent of \( u \).

On considering (6.10) and Lemma 6.1, the inequalities (6.10) yield the estimates
\[ (6.13) \quad \left| \left<a_{ik} \frac{\partial x_j}{\partial x_k} \right> (u_1) - \left<a_{ik} \frac{\partial x_j}{\partial x_k} \right> (u_2) \right| \leq C|u_1 - u_2|, \quad i, j = 1, \ldots, n, \]
where \( C \) is independent of \( u \).

The estimates (6.11), (6.12) yield the assertion of this lemma. \( \square \)

Lemma 6.3. Let us assumption of Lemma 6.1 be satisfied. Then for any \( u \in R \) the relations
\[ a_{ij}^0(u) = a_{ji}^0(u), \quad i, j = 1, \ldots, n \]
hold true and the inequality
\[ \nu \eta_i \eta_j \leq a_{ij}^0(u) \eta_i \eta_j \]
is satisfied for all \( \eta = (\eta_1, \ldots, \eta_n) \in R^n \), where \( \nu \) is a positive constant independent of \( u \) and \( \eta \).
The proof of this lemma is completely parallel to the same result for the homogenized coefficients $a_{ij}$ corresponding to the linear elliptic equation. This proof can be found, for instance in [?], [?].

**Theorem 6.1.** Let $a_{ij}(x,u)$, $i,j = 1\ldots n$ defined on $\mathbb{R}^n \times \mathbb{R}$ be bounded, measurable in $x$ for all $u$, periodic in $x$, and satisfy the assumptions (?). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary and $f \in L^p(\Omega)$, where $p > \frac{n}{2}$ and $p \geq 2$. Let $u^\epsilon \in H^1_0(\Omega)$ be solutions to the quasilinear variational equation

$$
\int_\Omega a_{ij}(x,u^\epsilon) \frac{\partial u^\epsilon}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = \int_\Omega fv \, dx
$$

and $u^0 \in H^1_0(\Omega)$ be a solution of the quasilinear variational equations

$$
\int_\Omega a_{ij}^0(u^0) \frac{\partial u^0}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = \int_\Omega fv \, dx,
$$

where (?), (?) are fulfilled for all $v \in H^1_0(\Omega)$. Then the limits

$$
u^\epsilon \to u^0 \quad \text{in } H^1_0(\Omega) \quad \text{as } \epsilon \to 0,
$$

$$
\xi^\epsilon_i \to \xi^0_i \quad \text{in } L^2(\Omega) \quad \text{as } \epsilon \to 0, \quad i = 1,\ldots,n
$$

hold, where

$$
\xi^\epsilon_i(x) = a_{ij}(x,u^\epsilon) \frac{\partial u^\epsilon}{\partial x_j}, \quad \xi^0_i(x) = a_{ij}^0(u^0) \frac{\partial u^0}{\partial x_j}, \quad i = 1,\ldots,n.
$$

**Proof.** If $u^\epsilon$ are solution to the quasilinear variational equations (?), then these solutions are the solutions to the linear variational equations with the coefficients $a_{ij}^\epsilon(x,u^\epsilon(x))$, $i,j = 1,\ldots,n$ as well.

From Theorem 5.1 it follows that $u^\epsilon$ belong to $C(\Omega)$ and satisfy the assumption of the Arzelà–Ascoli theorem. Thus there exists a subsequence of $u^\epsilon$ which converges strongly in $C(\Omega)$ and weakly in $H^1_0(\Omega)$ to $u^\epsilon$. Let us denote this subsequence by $u^{\epsilon,\epsilon}$.

If $\delta$ is an arbitrary positive number, then the domain $\Omega$ can be divided into the subdomains $\Omega^z$, $z = 1,\ldots,m$ with Lipschitz boundaries and there exists a function $u^\delta$ constant on each subdomain $\Omega^z$, such that for sufficiently small $\epsilon$ the following inequality

$$
|u^{\epsilon}(x) - u^\delta(x)| \leq \delta
$$

holds on $\Omega$. The last inequality yields the inequality

$$
|u^\epsilon(x) - u^\delta(x)| \leq \delta.
$$

Let $u^{\delta,\epsilon}(x) \in H^1_0(\Omega)$ is the sequence of solutions to the linear variational equations

$$
\int_\Omega a_{ij}^{\delta,\epsilon}(x,u^{\delta,\epsilon}) \frac{\partial u^{\delta,\epsilon}}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = \int_\Omega fv \, dx
$$

which are fulfilled for all $v \in H^1_0(\Omega)$. 

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If we consider that \( u^\epsilon, u^{\delta, \epsilon} \) are solutions to (??), (??), then after simple operations we have the relation

\[
\begin{align*}
\int_\Omega a_{ij}^\epsilon(x, u^\epsilon) \left( \frac{\partial u^\epsilon}{\partial x_j} - \frac{\partial u^{\delta, \epsilon}}{\partial x_j} \right) \left( \frac{\partial u^\epsilon}{\partial x_i} - \frac{\partial u^{\delta, \epsilon}}{\partial x_i} \right) dx = \\
\int_\Omega \left( a_{ij}^\delta(x, u^\delta) - a_{ij}^\epsilon(x, u^\epsilon) \right) \left( \frac{\partial u^\epsilon}{\partial x_j} - \frac{\partial u^{\delta, \epsilon}}{\partial x_j} \right) \left( \frac{\partial u^\epsilon}{\partial x_i} - \frac{\partial u^{\delta, \epsilon}}{\partial x_i} \right) dx.
\end{align*}
\]

(6.19)

Employing the inequality (??), we have the estimate

\[
\alpha \| u^\epsilon - u^{\delta, \epsilon} \|_{H^1_0(\Omega)}^2 \leq \int_\Omega a_{ij}^\epsilon(x, u^\epsilon) \left( \frac{\partial u^\epsilon}{\partial x_j} - \frac{\partial u^{\delta, \epsilon}}{\partial x_j} \right) \left( \frac{\partial u^\epsilon}{\partial x_i} - \frac{\partial u^{\delta, \epsilon}}{\partial x_i} \right) dx,
\]

(6.20)

where \( \alpha \) is a positive constant independent of \( \epsilon \) and \( \delta \).

Moreover, from (??) it follows the estimate

\[
\| u^{\delta, \epsilon} \|_{H^1_0(\Omega)} < C,
\]

(6.21)

where \( C \) is independent of \( \delta, \epsilon \).

The Hölder inequality, (??), (??) yield the inequalities

\[
\begin{align*}
\left| \int_\Omega \left( a_{ij}^\epsilon(x, u^\delta) - a_{ij}^\epsilon(x, u^\epsilon) \right) \frac{\partial u^{\delta, \epsilon}}{\partial x_j} \left( \frac{\partial u^\epsilon}{\partial x_i} - \frac{\partial u^{\delta, \epsilon}}{\partial x_i} \right) dx \right| & \leq \\
\sum_{i,j=1}^n \| a_{ij}^\epsilon(x, u^\delta) - a_{ij}^\epsilon(x, u^\epsilon) \|_{L^\infty(\Omega)} \| u^{\delta, \epsilon} \|_{H^1_0(\Omega)} \| u^\epsilon - u^{\delta, \epsilon} \|_{H^1_0(\Omega)} & \leq \\
\delta C \| u^{\delta, \epsilon} \|_{H^1_0(\Omega)} \| u^\epsilon - u^{\delta, \epsilon} \|_{H^1_0(\Omega)},
\end{align*}
\]

where \( C \) is independent of \( \epsilon, \delta \). The last inequality in (??) holds for all sufficiently small \( \epsilon \).

The estimates (??), (??), (??) yield the inequality

\[
\| u^\epsilon - u^{\delta, \epsilon} \|_{H^1_0(\Omega)} \leq \delta C,
\]

(6.23)

where \( C \) is independent of \( \epsilon, \delta \), and this inequality holds for sufficiently small \( \epsilon \).

Let \( u^{\delta, 0} \in H^1_0(\Omega) \) be a solution to the linear variational equation

\[
\int_\Omega a_{ij}^{\delta, 0}(u^\delta) \frac{\partial u^{\delta, 0}}{\partial x_j} \frac{\partial v}{\partial x_i} dx = \int f v dx
\]

(6.24)

which holds for all \( v \in H^1_0(\Omega) \).

Since \( a_{ij}^{\delta, 0}(u^\delta) \) are constant on every \( \Omega^z, z = 1, \ldots, m \), so we can employ Theorem 5.2, which results in the limits

\[
\begin{align*}
u^{\delta, \epsilon} & \rightharpoonup u^{\delta, 0} \quad \text{in} \ H^1_0(\Omega) \quad \text{as} \ \epsilon \to 0, \\
\xi^{\delta, \epsilon} & \rightharpoonup \xi^{\delta, 0} \quad \text{in} \ L^2(\Omega) \quad \text{as} \ \epsilon \to 0, \quad i = 1, \ldots, n,
\end{align*}
\]

(6.25)

where

\[
\begin{align*}
u^{\delta, \epsilon} &= a_{ij}^\epsilon(x, u^\delta) \frac{\partial u^{\delta, \epsilon}}{\partial x_j}, \quad \xi^{\delta, 0} = a_{ij}^{\delta, 0}(u^\delta) \frac{\partial u^{\delta, 0}}{\partial x_j}, & i &= 1, \ldots, n.
\end{align*}
\]
Let \( w^0 \in H^1_0(\Omega) \) be a solution to the linear variational equation

\[
\int_\Omega a^0_{ij}(u^*) \frac{\partial u^0}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = \int_\Omega f v \, dx
\]

which holds for all \( v \in H^1_0(\Omega) \).

If we consider (6.26), then we have the relation

\[
\int_\Omega a^0_{ij}(u^*) \left( \frac{\partial u^0}{\partial x_j} - \frac{\partial u^0}{\partial x_i} \right) \frac{\partial v}{\partial x_i} \, dx =
\int_\Omega a^0_{ij}(u^*) \frac{\partial w^0}{\partial x_j} \left( \frac{\partial u^0}{\partial x_i} - \frac{\partial w^0}{\partial x_i} \right) \, dx.
\]

If we follow the same method we have applied in the proof of the estimate (6.27), then from (6.27) it follows the estimate

\[
\|\nu^{\delta,0} - w^0\|_{H^1_0(\Omega)} \leq \delta C,
\]

where \( C \) is independent of \( \delta \).

Let us consider the following equality

\[
(u^\epsilon, v) - (w^0, v) = (u^\epsilon - u^\delta, v) + (u^\delta - u^{\delta,0}, v) + (u^{\delta,0} - w^0, v),
\]

where \((\ldots)\) is the usual scalar product in \( H^1_0(\Omega) \) and \( v \) is arbitrary element from \( H^1_0(\Omega) \). Then (6.28), (6.27), (6.29) and (6.27) yield the estimate

\[
\int (u^\epsilon, v) - (w^0, v) \leq \delta C \|v\|_{H^1_0(\Omega)},
\]

where \( C \) is independent of \( \epsilon, \delta, v \) and the estimate holds for sufficiently small \( \epsilon \).

From the estimate (6.27), where \( \delta \) is arbitrary, it follows the limit

\[
u^\epsilon \to w^0 \quad \text{in} \quad H^1_0(\Omega) \quad \text{as} \quad \epsilon \to 0.
\]

On the other hand we know that

\[
(u^\epsilon) \to u^* \quad \text{in} \quad H^1_0(\Omega) \quad \text{as} \quad \epsilon \to 0,
\]

which yields \( u^* = w^0 \).

On considering (6.27), we can see that \( w^0 \) is a solution of the quasilinear variational equation

\[
\int_\Omega a^0_{ij}(w^0) \frac{\partial w^0}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = \int_\Omega f v \, dx
\]

which holds for all \( v \in H^1_0(\Omega) \).

From Lemma 6.2 and 6.3 it follows that \( w^0 \) is the unique solution to (6.27) and the whole sequence \( u^\epsilon \) converges to the solution of (6.27).

We have to prove the convergence of \( \xi^\epsilon_i \) to \( \xi^0_i, \; i = 1, \ldots, n \).
On considering the definition of $\xi_i^\epsilon$, $\xi_i^{\delta,\epsilon}$, $i = 1, \ldots, n$, we have the inequalities

$$\left\| \xi_i^\epsilon - \xi_i^{\delta,\epsilon} \right\|_{L^2(\Omega)} \leq \left( \sum_{j=1}^n \left| a_{ij}^\epsilon(x, u^\epsilon) \right| \right) \left\| u^\epsilon - u^{\delta,\epsilon} \right\|_{H^1_0(\Omega)}$$

(6.32)

Applying the Hölder inequality, we have the estimates

$$\left\| a_{ij}^\epsilon(x, u^\epsilon) \frac{\partial u^\epsilon}{\partial x_j} \right\|_{L^2(\Omega)} \leq \left( \sum_{j=1}^n \left| a_{ij}^\epsilon(x, u^\epsilon) \right| \right) \left\| u^\epsilon - u^{\delta,\epsilon} \right\|_{H^1_0(\Omega)},$$

(6.33)

where $C$ is independent of $\epsilon$, $\delta$, and the estimates hold for sufficiently small $\epsilon$.

From the first inequality in (??) and the estimate (??) it follows

$$\left\| a_{ij}^\epsilon(x, u^\epsilon) \left( \frac{\partial u^\epsilon}{\partial x_j} - \frac{\partial u^{\delta,\epsilon}}{\partial x_j} \right) \right\|_{L^2(\Omega)} \leq \delta C, \quad i = 1, \ldots, n,$$

where $C$ is independent of $\epsilon$, $\delta$, and the estimates hold for sufficiently small $\epsilon$.

Summarizing the inequalities (??)–(??), we have the estimates

$$\left\| \xi_i^\epsilon - \xi_i^{\delta,\epsilon} \right\|_{L^2(\Omega)} \leq \delta C, \quad i = 1, \ldots, n,$$

(6.36)

where $C$ is independent of $\epsilon$, $\delta$, and the estimates hold for sufficiently small $\epsilon$.

In a similar way we can prove the estimates

$$\left\| \xi_i^{\delta,0} - \xi_i^{0} \right\|_{L^2(\Omega)} \leq \delta C, \quad i = 1, \ldots, n,$$

(6.37)

where $C$ is independent of $\delta$.

Let us consider the relations

$$[\xi_i^\epsilon, v] - [\xi_i^\delta, v] = [\xi_i^\epsilon - \xi_i^{\delta,\epsilon}, v] + [\xi_i^{\delta,\epsilon} - \xi_i^{\delta,0}, v] + [\xi_i^{\delta,0} - \xi_i^{0}, v], \quad i = 1, \ldots, n,$$

(6.38)

where $[\cdot, \cdot]$ is the usual scalar product in $L^2(\Omega)$ and $v$ is arbitrary from $L^2(\Omega)$. 

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If we consider (??) and the estimates (??), (??), then from the relations (??) it follows the inequalities

\[(\xi_i^\varepsilon, v) - (\xi_i^0, v) \leq \delta C ||v||_{L^2(\Omega)}, \quad i = 1, \ldots n,\]

where \(C\) is independent of \(\varepsilon, \delta,\) and the inequality holds for sufficiently small \(\varepsilon.\) If we consider that \(\delta\) is arbitrary, then the last inequality yields the convergence of \(\xi_i^\varepsilon\) to \(\xi_i^0, \) \(i = 1, \ldots n,\) which finishes the proof. \(\square\)

The last theorem can be generalized for the domain divided into subdomains filled by different materials with periodic structure in a similar way as Theorem 5.2 does that for linear materials.

Let the domain \(\Omega\) be divided into the subdomains \(\Omega^z, \) \(z = 1, \ldots m.\) There exists \(a_{ij}^z(x, u)\), \(i, j = 1, \ldots n, z = 1, \ldots m,\) defined on \(R^n \times R,\) which are periodic in \(x\) with the periods \(l^z = (l_1^z, \ldots l_n^z)\) for \(z = 1, \ldots m.\) Then we define

\[\tilde{a}_{ij}^\varepsilon(x, u) = a_{ij}^z\left(\frac{x}{\varepsilon}, u\right) \quad \text{if} \quad x \in \Omega^z, \]

\[\tilde{a}_{ij}^0(u) = a_{ij}^0(u) \quad \text{if} \quad x \in \Omega^z,\]

where \(a_{ij}^z(x, u)\) are given by the formulas (??). In these formulas \(a_{ij}(x, u)\) are substituted for \(a_{ij}^z(x, u),\) the solutions \(\chi_k(x, u), k = 1, \ldots n,\) to the variational equations (??) for the corresponding periodic solutions to the variational equation with the coefficients \(a_{ij}^z(x, u)\) on \(Y^z = (0, l^z_1) \times \ldots \times (0, l^z_n) \times R,\) and the mean value \(\langle \cdot \rangle(u)\) integrates over \(Y^z,\) where \(z = 1, \ldots m.\)

**Theorem 6.2.** Let \(a_{ij}^z(x, u)\), \(i, j = 1 \ldots n\) defined on \(R^n \times R\) be bounded, measurable in \(x\) for all \(u,\) periodic in \(x\) with the periods \(l^z\) for \(z = 1, \ldots m,\) and satisfy the assumptions (??)-(??). Let \(\Omega\) be a bounded domain in \(R^n\) with a Lipschitz boundary which is divided into the subdomains \(\Omega^z, \) \(z = 1, \ldots m\) with Lipschitz boundaries, and \(f \in L^p(\Omega),\) where \(p > \frac{n}{2}\) and \(p \geq 2.\) Let \(u^\varepsilon \in H^1_0(\Omega)\) be solutions to the quasilinear variational equation

\[\int_{\Omega} \tilde{a}_{ij}^\varepsilon(x, u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial v}{\partial x_i} dx = \int_{\Omega} f v dx\]

and \(u^0 \in H^1_0(\Omega)\) be a solution of the quasilinear variational equations

\[\int_{\Omega} \tilde{a}_{ij}^0(u^0) \frac{\partial u^0}{\partial x_j} \frac{\partial v}{\partial x_i} dx = \int_{\Omega} f v dx,\]

where the coefficients \(\tilde{a}_{ij}^\varepsilon(x, u), \) \(\tilde{a}_{ij}^0(u)\) are defined above and the variational equations are fulfilled for all \(v \in H^1_0(\Omega).\) Then the limits

\[u^\varepsilon \rightarrow u^0 \quad \text{in} \quad H^1_0(\Omega) \quad \text{as} \quad \varepsilon \rightarrow 0, \]

\[\xi_i^\varepsilon \rightarrow \xi_i^0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \varepsilon \rightarrow 0, \quad i = 1, \ldots n\]

hold, where

\[\xi_i^\varepsilon(x) = \tilde{a}_{ij}^\varepsilon(x, u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j}, \quad \xi_i^0(x) = \tilde{a}_{ij}^0(u^0) \frac{\partial u^0}{\partial x_j}, \quad i = 1, \ldots n.\]
Proof. In the proof of Theorem 6.1 we employed version of Theorem 5.2, where all coefficients \( a_{ij}^z(x, u) \), \( z = 1, \ldots, m \) were periodic in \( x \) with the same period. The proof of this theorem is almost parallel to the proof of Theorem 6.1. It is enough to keep in mind the division of \( \Omega \) into \( \Omega^z \), \( z = 1, \ldots, m \). In the proof of Theorem 6.1 we employed divisions of \( \Omega \) to approximate the solution to the original problem. In the proof of this theorem all approximate divisions have to be subdivisions of the division mentioned the assumptions of this theorem. Then we can apply the general version of Theorem 5.2 and follow the ideas in the proof of Theorem 6.1.

Let us remark that for \( n = 1, 2, 3 \) it is enough to assume \( p = 2 \), so in real situations there are not any special requirements on the right hand side comparing to the linear homogenization theory.

Let us come back to the original problem connected with the heat conduction in a transformer core of laminated structure. In this case the coefficients of the variational equation \( (\mathcal{C}) \) explicitly depend on \( x_1 \) and are periodical in \( x_1 \). Thus the coefficients are the functions \( a_{ij}(x_1, u) \), \( i, j = 1, \ldots, n \) defined on \( \mathbb{R} \times \mathbb{R} \) and these functions are periodical in \( x_1 \) with the period \( l_1 \). Moreover, they are measurable in \( x_1 \) for all \( u \) and satisfy \((\mathcal{C})-(\mathcal{C})\).

**Theorem 6.3.** Let \( a_{ij}(x_1, u) \), \( i, j = 1, \ldots, n \) satisfy the assumptions mentioned above and all assumptions of Theorem 6.1 be fulfilled. Then the formulas

\[
a_{11}^0(u) = \left( \frac{1}{a_{11}} \right)(u),
\]

(6.39) \[
a_{ij}^0(u) = \left( \frac{a_{ij}}{a_{11}} \right)(u), \quad 2 \leq j \leq n,
\]

\[
a_{ij}^0(u) = \left( \frac{a_{ii}}{a_{11}} \right)(u) \left( \frac{a_{ij}}{a_{11}} \right)(u) + \left( \frac{a_{ij} - a_{ii}a_{ij}}{a_{11}} \right)(u), \quad 2 \leq i, j \leq n
\]

hold. In this case

\[
\langle g \rangle(u) = \frac{1}{l_1} \int_0^{l_1} g(s, u) ds,
\]

where \( g(x_1, u) \) is a periodic function in \( x_1 \) with the period \( l_1 \).

Proof. In the linear case, where the coefficients \( a_{ij}(x_1) \), \( i, j = 1, \ldots, n \) are independent of \( u \) and periodic in \( x_1 \) with the period \( l_1 \), the following formulas

\[
a_{11}^0 = \left( \frac{1}{a_{11}} \right),
\]

(6.40) \[
a_{ij}^0 = \left( \frac{a_{ij}}{a_{11}} \right), \quad 2 \leq j \leq n,
\]

\[
a_{ij}^0 = \left( \frac{a_{ii}}{a_{11}} \right) \left( \frac{a_{ij}}{a_{11}} \right) + \left( \frac{a_{ij} - a_{ii}a_{ij}}{a_{11}} \right), \quad 2 \leq i, j \leq n
\]

hold true. In this case

\[
\langle g \rangle = \frac{1}{l_1} \int_0^{l_1} g(s) ds,
\]
where \( g(x_1) \) is a periodic function in \( x_1 \) with the period \( l_1 \).

The formulas (??) were proved in [?] and their proof can be found in [?] as well. If we consider Theorem 6.1, then the formulas (??) are an easy consequence of the formulas (??).

\[ \square \]

REFERENCES


