Geosynthetic tubes – an inequality arising in 2D analysis

Josef Malík[1]

Institute of Geonics of the Academy of Sciences, Studetská 1768,
708 00 Ostrava–Poruba, Czech Republic

Abstract. Some problems connected with 2D modeling of geosynthetic tubes on rigid foundation are studied. Basic equations are derived and analyzed. The analysis of the equations in based on the implicit function theorem. Geosynthetic tubes are made of a special fabric and then filled with water or slurry. After being filled tubes take certain shapes and tensions are induced in geosynthetic. The modeling is based on the following hypotheses: the problem is two dimensional; the geosynthetic is flexible, inextensible and has negligible weight; the foundation is rigid; there is no friction between the foundation and the geosynthetic.

Keywords. rigid foundation, geosynthetic tube, implicit function theorem

1. Introduction

Geosynthetic tubes are comprised of thin sheets and pumped with water or slurry. The tubes are made of synthetic fabrics (geotextile). They have been used as dikes or breakwaters and to prevent beach erosion. They have got many other applications in geoengineering (see[2]).

The geosynthetic tubes are often very long and their cross sections are identical so they can be modelled as 2D objects. The geosynthetic tubes have been studied in many papers. The geosynthetic tubes on rigid foundation are studied, for instance, in [1, 3]. These results are generalized for tubes on elastic foundation [5]. The geosynthetic tubes in mutual contact are studied in [6]. Some problems connected with 3D modeling are solved in [7]. Similar techniques have been applied for solving some quite different problems. The floating liquid filled membranes are studied in [8, 9]. The shape of towed boom of logs is studied in [4].

The main purpose of this paper is to give the strict mathematical formulation and analysis of the geosynthetic tubes on the rigid foundation. An alternative formulation of the problem is presented. The derived equations are analyzed from the point of the existence and the algorithms for numerical solutions are proposed and tested on model problems.

2. Basic hypotheses and setting up problems

The geosynthetic tubes have diameters ranging from one to several meters and have theoretically infinite length. Let us consider that all cross sections are identical, so we can study the geosynthetic tubes as a two dimensional problem. The modeling is based on the following hypotheses:

[1]The research was supported by Grants FT-TA3/069 and 1ET400300415.

*E-mail address:*malik@ugn.cas.cz.

(1) The geosynthetic is inextensible and flexible and its weight can be neglected.
(2) The filling medium (water or slurry) behaves as an ideal liquid which generates the hydrostatic preassure in every point and act in the perpendicular direction to the geosynthetic.
(3) There is no friction between the foundation and the geosynthetic.

The geosynthetic tube is filled through the inlets on the top of the tube, which results in the proccess, where the certain part of the geosynthetic rises and the other part of the geosynthetic rests on the rigid foundation (see Fig.1.).

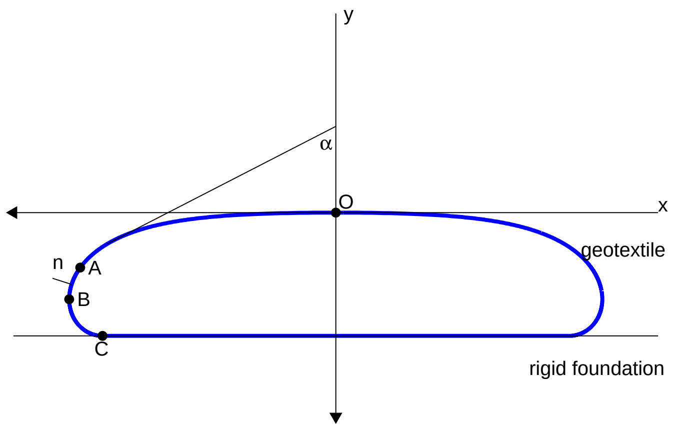

Fig.1 Cross section of a geosynthetic tube

Let us consider the coordinates in Fig.1 with the origin in the point O and with the axes x, y oriented in the way depicted in Fig.1. Let us use the notation

ρ the density of the water or slurry.

g the gravitational acceleration.

p the pressure of the water or slurry in the point O.

The pressure p can be interpreted as the pumping pressure of the water or slurry which is transported into the tube. Let us set up equilibrium conditions on the curve representing the shape of the cross section of the geosynthetic tube.

Let s be the parameter representing the length of the curve. The parameter is equal 0 in the point O and is oriented in the anticlockwise direction.

Let $n = (n_x, n_y)$ be the normal vector to the curve, $H(s)$ be the tension force in the geosynthetic in the point corresponding to the parameter s, and the functions $x(s)$, $y(s)$ describe the shape of the curve between the points O, C.

Now we are prepared to formulate the equilibrium conditions on the interval AB (see Fig.1), where the points A, B correspond to the parameters s_1, s_2. The equilibrium conditions read

$$
(1) \qquad
\begin{aligned}
&H(s_2)\frac{dx(s_2)}{ds} - H(s_1)\frac{dx(s_1)}{ds} + \int_{s_1}^{s_2} n_x(s)\,(g\rho y(s) + p)\,ds = 0,\\
&H(s_2)\frac{dy(s_2)}{ds} - H(s_1)\frac{dy(s_1)}{ds} + \int_{s_1}^{s_2} n_y(s)\,(g\rho y(s) + p)\,ds = 0.
\end{aligned}
$$

Since the parameter s represents the length of the curve, the normal vector n corresponds to $\left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$. If we consider that the values s_1, s_2 are arbitrary, then the equations (1) can be rewritten into the differential equations

$$
(2) \qquad
\begin{aligned}
&\frac{d}{ds}\left(H\frac{dx}{ds}\right) + \frac{dy}{ds}\,(g\rho y + p) = 0,\\
&\frac{d}{ds}\left(H\frac{dy}{ds}\right) - \frac{dx}{ds}\,(g\rho y + p) = 0,
\end{aligned}
$$

which hold on the interval OC (see Fig. 1.). Since s represents length, we have the identity

$$
(3) \qquad \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1
$$

Let us consider (3), then the equations

$$
(4) \qquad
\begin{aligned}
\frac{d}{ds}H &= \frac{d}{ds}\left(H\left(\frac{dx}{ds}\right)^2 + H\left(\frac{dy}{ds}\right)^2\right)\\
&= \frac{d}{ds}\left(H\frac{dx}{ds}\right)\frac{dx}{ds} + \frac{d}{ds}\left(H\frac{dy}{ds}\right)\frac{dy}{ds} + H\left(\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2}\right)
\end{aligned}
$$

hold. Since the equations (2) hold, then from (3), (4) it follows the identity

$$
\frac{dH}{ds} = 0,
$$

which means that H is constant on the whole curve OC.

From now the parameter H in the equations (2) is constant and these equations represent the equilibrium of forces on the curve OC. Moreover, the equilibrium of forces in the points O, C results in the equations

$$
(5) \qquad \frac{dy}{ds}(s_O) = \frac{dy}{ds}(s_C) = 0,
$$

where s_O, s_C represent the values of the parameter s in O, C. It is obvious as well that the curve describing the shape of the cross section is symmetric with respect to the axis x (see Fig. 1).

In the rest of this paper the parameters p, H will be positive. Let us formulate some useful relations. Let s_C be the value of the parameter s in the point C, then the perimeter L is

$$L = 2s_C + 2x(s_C). \tag{6}$$

Since the value of the parameter s in O vanishes, the area V of the cross section reads

$$V = 2\int_0^{s_C} x\frac{dy}{ds}ds, \tag{7}$$

where s_C is the value of the parameter s in the point C.

In some practical problems it is useful to know the hight h of the geosynthetic tube. The high can be express by the formula

$$h = y(s_C), \tag{8}$$

where s_C corresponds with the point C.

Let us set up some problems which are of practical use. In the following three problems the value of the perimeter is prescribed.

Problem 1. Let the length of the perimeter $L > 0$ and the pumping pressure $p > 0$ be given. Find the values of parameters $H > 0$, $s_C > 0$, so that the solutions $x(s)$, $y(s)$ to the equations (2) on the interval $(0, s_C)$ satisfy the equations (5), (6) together with the following equations

$$x(0) = y(0) = 0. \tag{9}$$

Moreover, the inequality

$$\frac{dy}{ds} > 0 \tag{10}$$

holds on the interval $(0, s_C)$.

Let us mention that the relations (3) and (5) yield the equations

$$\frac{dx(0)}{ds} = 1, \quad \frac{dx(s_C)}{ds} = -1.$$

Problem 2. Let the length of the perimeter $L > 0$ and the high of the tube $h > 0$ be given. Find the values of parameters $H > 0$, $p > 0$, $s_C > 0$, so that the solutions $x(s)$, $y(x)$ to the equations (2) on the interval $(0, s_C)$ satisfy the conditions (5), (6), (8), (9), (10).

Problem 3. Let the length of the perimeter $L > 0$ and the area of the cross section $V > 0$ be given. Find the values of parameters $H > 0$, $p > 0$, $s_C > 0$, so that the solutions $x(s)$, $y(x)$ to the equations (2) on the interval $(0, s_C)$ satisfy the conditions (5), (6), (7), (9), (10).

Let us formulate another problem which can be of practical use. We have a pump with the pumping pressure p. The tube, after being filled with the pump, shout achieve the prescribed high h. We look for the value of perimeter so that the conditions above are fullfilled.

Problem 4. Let the pumping pressure $p > 0$ and the high of the tube $h > 0$ be given. Find the values of parameters $H > 0$, $s_C > 0$, so that the solutions $x(s)$, $y(x)$ to the equations (2) on the interval $(0, s_C)$ satisfy the conditions (5), (8), (9), (10). Then the value of L is given by (6).

The solutions to the problems formulated above can be of practical use for the designer who can set some parameter in an optimal way.

We will solve the problems in a new parameterization which was adopted, for instance, in [3, 4]. This parameterization makes easier the mathematical analysis of the problems.

Let α be the angle between the tangential vector $(\frac{dx}{ds}, \frac{dy}{ds})$ in the point (x, y) on the curve and the axis y (see Fig. 1), which yields the equations

$$\frac{dx}{ds} = -\sin\alpha, \quad \frac{dy}{ds} = \cos\alpha. \tag{11}$$

Lemma 2.1. *Let all assumption for Problem 1 – 3 hold, then*

$$\frac{d\alpha}{ds} > 0$$

on the interval $(0, s_C)$.

Proof. Let α be the angle between the two vectors u, v, then the following formula holds

$$\cos\alpha = \frac{< u, v >}{\| u \| \| v \|},$$

where $< ., . >$, $\| . \|$ are the scalare product and norm in R^2. Let $u = (\frac{dx}{ds}, \frac{dy}{ds})$ and $v = (0, 1)$, then the last formula can be revritten into

$$\cos\alpha = \frac{dy}{ds}.$$

Let us derivate the last formula with respect to s and apply the second equation in (2), then we have the equation

$$-\sin\alpha \frac{d\alpha}{ds} = \frac{dx}{ds}\left(\frac{g\rho y + p}{H}\right).$$

From (11) it follows

$$\frac{d\alpha}{ds} = \frac{g\rho y + p}{H}. \tag{12}$$

If we consider that p, H are positive numbers and $y(s) > 0$ on the interval $(0, s_C)$, which follows from (10), then we have the desired result. □

Lemma 2.1 guarantees the regularity of the parameterization with respect to α. This parameterization defines the bijection from the interval $(0, s_C)$ to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Applying the equations (11), we can rewrite the first equation in (2) into

$$-H\cos\alpha \frac{d\alpha}{ds} + \cos\alpha(g\rho y + p) = 0. \tag{13}$$

From the second equation in (11) it follows the equations

$$\frac{d\alpha}{ds} = \frac{d\alpha}{dy}\frac{dy}{ds} = \frac{d\alpha}{dy}\cos\alpha.$$

These equation together with the equation (13) yield

$$\cos\alpha\frac{d\alpha}{dy} = \frac{g\rho y}{H} + \frac{p}{H},$$

which results in the relation

$$[\sin\theta]_{-\frac{\pi}{2}}^{\alpha} = \left[\frac{g\rho z^2}{M} + \frac{pz}{H}\right]_0^{y(\alpha)}, \tag{14}$$

where $y(\alpha)$ is equal to $y(s)$ in the parameterization described above. After simple operations (14) yields the equation

$$y(\alpha) = \frac{-p + (p^2 + 2g\rho H(1+\sin\alpha))^{\frac{1}{2}}}{g\rho} \tag{15}$$

The last equation is the solution to the quadratic equation (14), where we have chosen the solution satisfying the condition $y(-\frac{\pi}{2}) = 0$ which corresponds to the condition in (5). Moreover, the equation (15) can be rewritten into

$$y(\alpha) = \int_{\frac{-\pi}{2}}^{\alpha} \frac{H\cos\theta d\theta}{(p^2 + 2g\rho H(1+\sin\theta))^{\frac{1}{2}}}. \tag{16}$$

This formula will be useful in the analysis in the next section.

Let us consider (11) and (12), then we have the equations

$$\begin{aligned}\frac{d\alpha}{ds} &= \frac{d\alpha}{dx}\frac{dx}{ds} = -\frac{d\alpha}{dx}\sin\alpha = \frac{g\rho y}{H} + \frac{p}{H},\\ \frac{d\alpha}{ds} &= \frac{d\alpha}{dy}\frac{dy}{ds} = \frac{d\alpha}{dx}\cos\alpha = \frac{g\rho y}{H} + \frac{p}{H}.\end{aligned}$$

From the last equations and (15) it follows the formulas

$$\begin{aligned}\frac{dx}{d\alpha} &= -\frac{H\sin\alpha}{(p^2 + 2g\rho H(1+\sin\alpha))^{\frac{1}{2}}},\\ \frac{dy}{d\alpha} &= \frac{H\cos\alpha}{(p^2 + 2g\rho H(1+\sin\alpha))^{\frac{1}{2}}},\\ \frac{ds}{d\alpha} &= \frac{H}{(p^2 + 2g\rho H(1+\sin\alpha))^{\frac{1}{2}}}.\end{aligned} \tag{17}$$

The formulas (17) and the equations $x(-\frac{\pi}{2}) = s(-\frac{\pi}{2}) = 0$ yield

$$(18)\qquad \begin{aligned} x(\alpha) &= -\int_{-\frac{\pi}{2}}^{\alpha} \frac{H\sin\theta d\theta}{(p^2 + 2g\rho H(1+\sin\theta))^{\frac{1}{2}}}, \\ s(\alpha) &= \int_{-\frac{\pi}{2}}^{\alpha} \frac{H d\theta}{(p^2 + 2g\rho H(1+\sin\theta))^{\frac{1}{2}}} \end{aligned}$$

which together with (15) and (16) describe the geometry of the cross section with respect to the parameter α.

From the definition of the length of the perimeter (6), the equation $\alpha(s_C) = \frac{\pi}{2}$, and (18) it follows

$$(19)\qquad L = 2\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{H(1-\sin\theta)d\theta}{(p^2 + 2g\rho H(1+\sin\theta))^{\frac{1}{2}}}.$$

From the definition of the area of the cross section (7), the equation $\alpha(s_C) = \frac{\pi}{2}$, (15), (17), and (18) it follows

$$(20)\qquad \begin{aligned} V = +2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\frac{dy}{d\alpha}d\alpha &= \\ -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{H\cos\alpha d\alpha}{(p^2 + 2g\rho H(1+\sin\alpha))^{\frac{1}{2}}} \int_{-\frac{\pi}{2}}^{\alpha} \frac{H\sin\theta d\theta}{(p^2 + 2g\rho H(1+\sin\theta))^{\frac{1}{2}}} &= \\ -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{H\sin\theta\,(p^2 + 4g\rho H)^{\frac{1}{2}} d\theta}{(p^2 + 2g\rho H(1+\sin\theta))^{\frac{1}{2}} g\rho}&. \end{aligned}$$

From the definition (8), the equation $\alpha(s_C) = \frac{\pi}{2}$, (15), and (16) it follows the formula for the high of the tube

$$(21)\qquad h = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{H\cos\theta d\theta}{(p^2 + 2g\rho H(1+\sin\theta))^{\frac{1}{2}}} = \frac{(p^2 + 4g\rho H)^{\frac{1}{2}} - p}{g\rho}.$$

The relations derived in this section will be applied in the next section for the analysis of Problem 1 – 3.

3. Analysis of solvability

First of all we analyze Problem 1, where the length of the perimether $\bar{L}$ together with the pumping pressure $\bar{p}$ are prescribed. In the sequel all prescribed paramether will be denoted by the bar. Moreover, all expressions in (19), (20), (21) are considered as the functions $L(p, H)$, $V(p, H)$, $h(p, H)$ with the variables p, H.

Lemma 3.1. *Let $L(p, H)$ be defined by (19), then for all $p > 0$, $H > 0$ the inequality*

$$\frac{\partial L(p, H)}{\partial H} > 0$$

holds. Moreover, for any fixed $p > 0$ the expression L as a function of H is bijective from $(0, \infty)$ to $(0, \infty)$.

Proof. Let us consider the following inequalities

$$\begin{aligned} \frac{1 - \sin\theta}{(p^2 + 2g\rho H(1 + \sin\theta))^{\frac{1}{2}}} &> 0, \\ \frac{g\rho H(1 + \sin\theta)}{p^2 + 2g\rho H(1 + \sin\theta)} &< \frac{1}{2}, \end{aligned} \tag{22}$$

which hold on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then from the second inequality in (22) it follows the inequality

$$1 - \frac{g\rho H(1 + \sin\theta)}{p^2 + 2g\rho H(1 + \sin\theta)} > \frac{1}{2}, \tag{23}$$

which hold on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ as well. The inequalities (22) and (23) yield the relations

$$\begin{aligned} &\frac{\partial}{\partial H}\left(\frac{2H(1 - \sin\theta)}{(p^2 + 2g\rho H(1 + \sin\theta))^{\frac{1}{2}}}\right) = \\ &\frac{2(1 - \sin\theta)}{(p^2 + 2g\rho H(1 + \sin\theta))^{\frac{1}{2}}}\left(1 - \frac{g\rho H(1 + \sin\theta)}{p^2 + 2g\rho H(1 + \sin\theta)}\right) > 0 \end{aligned} \tag{24}$$

which holds on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then from (19) and (24) it follows the inequality

$$\frac{\partial L(p, H)}{\partial H} > 0, \tag{25}$$

which is the first assertion of this lemma.

If $p > 0$, then the limits

$$\begin{aligned} \lim_{H \to 0} \frac{2H(1 - \sin\theta)}{(p^2 + 2g\rho H(1 + \sin\theta))^{\frac{1}{2}}} &= 0, \\ \lim_{H \to \infty} \frac{2H(1 - \sin\theta)}{(p^2 + 2g\rho H(1 + \sin\theta))^{\frac{1}{2}}} &= \infty \end{aligned} \tag{26}$$

hold for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. From (24) it follows that the convergence is monotone for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Due to the monotone convergence theorem and (26) we have the limits

$$\lim_{H\to 0} L(p,H) = 0, \qquad \lim_{H\to\infty} L(p,H) = \infty.$$

These limits together with (25) mean that L as a function of H is bijective from $(0,\infty)$ to $(0,\infty)$ for any p, which is the second assertion of this lemma. □

Lemma 3.2. *Let $L(p,H)$ be defined by (19), then for all $p > 0$, $H > 0$ the inequality*

$$\frac{\partial L(p,H)}{\partial p} < 0$$

holds. Moreover, for any fixed $H > 0$ the expression L as a function of p is bijective from $(0,\infty)$ to $(0,\infty)$.

Proof. Let us consider the relations

$$\frac{\partial}{\partial p}\left(\frac{2H(1-\sin\theta)}{(p^2+2g\rho H(1+\sin\theta))^{\frac{1}{2}}}\right) = -\frac{2pH(1-\sin\theta)}{(p^2+2g\rho H(1+\sin\theta))^{\frac{3}{2}}} < 0 \tag{27}$$

which hold for all $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$. Then from (19) it follows

$$\frac{\partial L(p,H)}{\partial p} < 0,$$

which proves the first assertion of this lemma.

Now let H be fixed. Let us consider the substition $\varphi = \theta + \frac{\pi}{2}$, where $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$, then there exist an interval $(0,a)$ and a constant $C_1 > 0$ such that for any $\varphi \in (0,a)$ the inequality

$$p^2 + 2g\rho H\left(1+\sin\left(\varphi - \frac{\pi}{2}\right)\right) \le p^2 + C_1\varphi^2$$

holds. From the last inequality it follows the existence of $C_2 > 0$ such that the relations

$$L(p,H) \ge \int_0^a \frac{C_2\, d\varphi}{(p^2+C_1\varphi^2)^{\frac{1}{2}}} = \frac{C_2}{\sqrt{C_1}}\left(\ln\left(C_1 a + \sqrt{C_1 a^2 + p^2}\right) - \ln(p)\right)$$

hold (see any textbook of calculus). The last relations yield the limit

$$\lim_{p\to 0} L(p,H) = \infty. \tag{28}$$

On the other hand the following limit

$$\lim_{p\to\infty} \frac{2H(1-\sin\theta)}{(p^2+2g\rho H(1+\sin\theta))^{\frac{1}{2}}} = 0 \tag{29}$$

holds for all $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$. From (27) it follows that the convergence is monotone, which together with (29) yields

$$\lim_{p\to\infty} L(p,H) = 0. \tag{30}$$

The limits (28), (30) yield the second assertion of this lemma. □

Lemma 3.3. *Let $h(p,H)$ be the function defined by (21). Then for any $p>0$, $H>0$ the inequalities*

$$\frac{\partial h(p,H)}{\partial p} < 0, \quad h(p,H) < 2\sqrt{\frac{H}{g\rho}}$$

hold.

Proof. Let us consider the relations

$$\frac{\partial}{\partial p}\left(\frac{H\cos\theta}{(p^2+2g\rho H(1+\sin\theta))^{\frac{1}{2}}}\right) = -\frac{pH\cos\theta}{(p^2+2g\rho H(1+\sin\theta))^{\frac{3}{2}}} < 0, \tag{31}$$

the from (21) it follows the inequality

$$\frac{\partial h(p,H)}{\partial p} < 0. \tag{32}$$

From the second equality in (21) it follows

$$h(0,H) = 2\sqrt{\frac{H}{g\rho}},$$

which together with the inequality (32) yields the inequality

$$h(p,H) < 2\sqrt{\frac{H}{g\rho}}.$$

The last inequality and (32) prove this lemma. □

Let $H(p)$ be the function from $(0,\infty)$ to $(0,\infty)$ defined by the equation

$$L(p,H(p)) = \bar{L}, \tag{33}$$

where $\bar{L}$ is the prescribed value of the perimeter. The existence and uniqueness of this function follows from Lemma 3.1 and the implicit function theorem.

Lemma 3.4. *Let $\bar{L}>0$ be prescribed and $H(p)$ be the function defined by (33). Then for any $\varepsilon>0$ there exists $p>0$ such that the inequality*

$$h(p,H(p)) < \varepsilon$$

holds.

Proof. Let us choose $\bar{H}$ such that the inequality

$$\varepsilon > 2\sqrt{\frac{\bar{H}}{g\rho}}$$

holds, then from Lemma 3.2 it follows the existence of $p>0$ such that the equation

$$L(p,\bar{H}) = \bar{L}$$

holds. Then Lemma 3.3 yields the inequality

$$h(p,\bar{H}) < \varepsilon.$$

From Lemma 3.1 and the implicit function theorem it follows

$$H(p) = \bar{H},$$

which proves this lemma. $\square$

Lemma 3.5. *Let $\bar{L} > 0$ be prescribed and $H(p)$ be defined by (33). Then for any $\varepsilon > 0$ there exists $p > 0$ such that the inequality*

$$h(p, H(p)) > \frac{\bar{L}}{\pi} - \varepsilon$$

holds.

Proof. Let $p \to \infty$ and $H \to \infty$ so that

$$\frac{2H}{p} \to \frac{\bar{L}}{\pi}, \tag{34}$$

then the following limits

$$\begin{aligned} \frac{2H(1-\sin\theta)}{(p^2 + 2g\rho H(1+\sin\theta))^{\frac{1}{2}}} &\to \frac{\bar{L}}{\pi}(1-\sin\theta), \\ \frac{2H\cos\theta}{(p^2 + 2g\rho H(1+\sin\theta))^{\frac{1}{2}}} &\to \frac{\bar{L}}{\pi}\cos\theta \end{aligned} \tag{35}$$

hold for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. From (19), (21), and the limits (35) it follows the limits

$$L(p, H) \to \bar{L}, \quad h(p, H) \to \frac{\bar{L}}{\pi}. \tag{36}$$

From (36) it follows that for any small $\varepsilon > 0$ there exist sufficiently large positive numbers p_1, p_2, H_0 such that the inequalities

$$\begin{aligned} L(p_1, H_0) > \bar{L}, \quad h(p_1, H_0) > \frac{\bar{L}}{\pi} - \varepsilon, \\ L(p_2, H_0) < \bar{L}, \quad h(p_2, H_0) > \frac{\bar{L}}{\pi} - \varepsilon \end{aligned} \tag{37}$$

hold. The continuity of the function $L(p, H)$ yields that there exists $p_0 \in (p_1, p_2)$ such that the equation

$$L(p_0, H_0) = \bar{L} \tag{38}$$

holds. Let us derivate (21) with respect to p, then after simple opeartions, we have

$$\frac{\partial h(p, H)}{\partial p} > 0.$$

The last inequality together with the inequalities (37) yield the inequality

$$h(p_0, H_0) > \frac{\bar{L}}{\pi} - \varepsilon. \tag{39}$$

From Lemma 3.1, the equality in (38), and the implicit function theorem it follows

$$H(p_0) = H_0,$$

which together with the inequality (39) proves this lemma. $\square$

Lemma 3.6. *Let $h(p, H)$ be defined by (21), then for all $p > 0$, $H > 0$ the inequality*

$$\frac{\partial h(p, H)}{\partial H} > 0$$

holds. Moreover, for any fixed $p > 0$ the expression h as a function of H is bijective from $(0, \infty)$ to $(0, \infty)$.

Proof. After derivating (21) with respect to H, we have the first assertion of this lemma. The following equality

$$\frac{\partial h(p, 0)}{\partial H} = 0$$

and the limit

$$\lim_{H \to \infty} h(p, H) = \infty,$$

which hold for any $p > 0$, yield together with the implicit function theorem this lemma. $\square$

If $p \to \infty$, $H \to \infty$ so that the limit (34) holds, then from the formulas (16), (18) it follows that the limiting shape of the cross section is described by the equations

$$\begin{aligned} x(\alpha) &= -\frac{\bar{L}}{\pi} \int_{\frac{-\pi}{2}}^{\alpha} \sin\theta d\theta = \frac{\bar{L}}{\pi} \cos\alpha, \\ y(\alpha) &= \frac{\bar{L}}{\pi} \int_{\frac{-\pi}{2}}^{\alpha} \cos\theta d\theta = \frac{\bar{L}}{\pi}(1 + \sin\alpha) \end{aligned} \tag{40}$$

which correspond to the circle with the perimeter $\bar{L}$.

The analysis above shows that the increase of pumping pressure results in the increase of the tension in the geotextile and for very large values the relation

$$H \approx \frac{\bar{L}p}{\pi}$$

holds.

Theorem 3.1. *Let the positive number $\bar{L}$, $\bar{p}$ be given. Then Problem 1 has exactly one solution.*

This theorem follows from Lemma 3.1 and the implicit function theorem.

Theorem 3.2. *Let the positive number $\bar{L}$, $\bar{h}$ be given and the following inequality*

$$\bar{h} < \frac{\bar{L}}{\pi}$$

hold. Then Problem 2 has a solution.

Proof. From Lemma 3.4 and Lemma 3.5 it follows that there exist the positive numbers p_1, p_2 such that the relations

$$\begin{aligned} L(p_1, H(p_1)) &= L(p_2, H(p_2)) = \bar{L}, \\ h(p_1, H(p_1)) &< \bar{h} < h(p_2, H(p_2)) \end{aligned} \tag{41}$$

hold.

Let us consider that the functions $L(p, H)$, $h(p, H)$ are continuous as well as the function $H(p)$ defined by the equation

$$L(p, H(p)) = L.$$

The function $H(p)$ is defined on $(0, \infty)$, which is the result of Lemma 3.1 and implicit function theorem. Then there exists $p_0 \in (p_1, p_2)$ such that the equation

$$h(p_0, H(p_0)) = \bar{h}$$

holds, which follows from (41). □

Theorem 3.3. *Let the positive numbers $\bar{L}$, $\bar{V}$ be given and the following inequality*

$$\bar{V} < \frac{\bar{L}^2}{4\pi}$$

hold. Then Problem 3 has a solution.

Proof. From Lemma 3.4 it follows that there exist $p_1 > 0$ such that the relations

$$L(p_1, H(p_1)) = \bar{L}, \quad h(p_1, H(p_1)) < \frac{2\varepsilon}{\bar{L}} \tag{42}$$

hold for sufficiently small $\varepsilon > 0$, where $H(.)$ is defined by (33). From the relations (42) it follows that the cross section is contained in the rectangle with the sites $\frac{2\varepsilon}{L}$, $\frac{L}{2}$, which yields the inequality

$$V(p_1, H(p_1)) < \varepsilon, \tag{43}$$

where the function $V(p, H)$ is defined by (20).

From Lemma 3.5 and the equations (40) it follows that there exists $p_2 > 0$ such that the relations

$$L(p_2, H(p_2)) = \bar{L}, \quad V(p_2, H(p_2)) < \frac{\bar{L}^2}{4\pi} - \varepsilon \tag{44}$$

hold, where $\varepsilon > 0$ is sufficiently small.

The relations (42), (43), (44), and the assumptions of this theorem yield the inequalities

$$V(p_1, H(p_1)) < \bar{V} < V(p_2, H(p_2)). \tag{45}$$

Then from the continuity of the functions $V(p, H)$, $H(p)$ and the inequalities (45) it follows the equations

$$L(p_0, H(p_0)) = \bar{L}, \quad V(p_0, H(p_0)) = \bar{V},$$

which proves the theorem. □

Theorem 3.4. *Let the positive number $\bar{h}$, $\bar{p}$ be given. Then Problem 4 has exactly one solution.*

This theorem follows from Lemma 3.6 and the implicit function theorem.

4. Algorithms for numerical solutions.

This section contains algorithms for solving the problems formulated above. The algorithms are based on the analyses worked out in Section 3.

Problem 1. For given $\bar{p}$ and $\bar{L}$ we look for H such that the equation

$$L(\bar{p}, H) = \bar{L}$$

is fullfilled. Due to Lemma 3.1 we can apply, for instance the Newton method.

Problem 2. For given $\bar{h}$ and $\bar{L}$ we look for p, H such that the equation

$$L(p, H) = \bar{L}, \quad h(p, H) = \bar{h}$$

are fullfilled. Moreover $\bar{h}$, $\bar{L}$ satisfy the inequality

$$\bar{h} < \frac{\bar{L}}{\pi}$$

which guarantees, due to Theorem 3.2, the existence of a solution to Problem 2. Let us select $\varepsilon > 0$, which represents the numerical accuracy, and suppose that we can solve the equation $L(\bar{p}, H) = \bar{L}$ with the respect to H for given $\bar{p}$, $\bar{L}$.

Algorithm for Problem 2:

(1) Choose $p > 0$, $\quad p_{\max} = p_{\min} = 0$.
(2) Solve $L(p, H) = \bar{L}$ with the respect to H.
(3) If $h(p, H) \geq \bar{h}$, then $p_{\max} = p$ else $p_{\min} = p$.
(4) If $p_{\max} = 0$, then go to (5) else go to (8).
(5) $p = 2p$.
(6) Solve $h(p, H) = \bar{L}$ with respect to H.
(7) If $h(p, H) \geq \bar{h}$ then $p_{\max} = p$ and go to (11) else go to (5).
(8) $p = p/2$.
(9) Solve $L(p, H) = \bar{L}$ with respect to H.
(10) If $h(p, H) \leq \bar{h}$, then $p_{\min} = p$ and go to (11) else go to (8).
(11) $p = (p_{\min} + p_{\max})/2$.
(12) Solve $L(p, H) = \bar{L}$ with respect to H.
(13) If $h(p, H) \geq \bar{h}$, then $p_{\max} = p$ else $p_{\min} = p$.
(14) If $\mid h(p, H) - \bar{h} \mid < \varepsilon$, then p, H are a solution to Problem 2 else go to (11).

The correctness and convergence of the algorithms follow from the analysis above.

Problem 3 For given $\bar{V}$, $\bar{L}$ we look for p, H such that the equations

$$V(p, H) = \bar{V}, \quad L(p, H) = \bar{L}$$

are fullfilled. Moreover, $\bar{L}$, $\bar{V}$ satisfy the inequality

$$\bar{V} < \frac{\bar{L}^2}{4\pi},$$

which guarantees, due to Theorem 3.3, the existence of a solution to Problem 3.

Let us select $\varepsilon > 0$, which represents the numerical accuracy, and suppose that we can solve the equations $L(\bar{p}, H) = \bar{L}$ with respect to H for any the given $\bar{p}$ and $\bar{L}$.

Algorithm for Problem 3:

This algorithm can be formulated in the same way as Algorithm for Problem 2. We only have to substitute the expressions

$$h(p, H) \geq \bar{h}, \quad h(p, H) \leq \bar{h}, \quad \mid h(p, H) - \bar{h} \mid < \varepsilon$$

for the expression

$$V(p, H) \geq \bar{V}, \quad V(p, H) \leq \bar{V}, \quad \mid V(p, H) - \bar{V} \mid < \varepsilon.$$

The algorithms were implemented in MATLAB. Now let us apply the MATLAB code for solving some model problems. Let us consider that we have a tube with the perimeter $10m$ filled with water ($\rho = 1000kg/m^3$) and $g = 10m/s^{-2}$. The graphs in Fig.2 describe the shapes of the tube for some values of the parameters p, H and $L = 10m$. The graphs in Fig.3-5 describe the functional dependences between h and p, H, V for $L = 10m$.

5. Conclusion

From the analysis above it follows that Problem 1 and 4 are uniquely solvable and Problem 2 and 3 are only solvable. The numerical experiment show that Problem 2 and 3 have unique solutions, but the author neither can prove uniqueness nor the opposite assertion.

From the graphs above it is clear that the dependence between the parameters p, H, L, h, V is nonlinear. The result show how to choose some parameters of the geotextile so that the tension H does not exceed the limits which can result in a destruction of the tube. Such information can contribute to the optimal design.

Acknowledgment

The author would like to thank Stanislav Sysala for stimulating discussions and comments.

REFERENCES

[1] K. Kazimierowicz, *Simple analysis of deformation of sand-sausages,* Proc. of 5th Conf. on Geotextile, Geomembranes, and related Products, Singapoure, 1994, pp. 775-778.

[2] R. M. Koerner, *Designing with Geosynthetics,* Prentice Hall, New York, 1994.

[3] D. Leshchinsky, O. Leshchinsky, H. I. Ling, P. A. Gilbert, *Geosynthetic tubes for confining pressurized slurry: some design aspects,* Journal of Geotechnical Engineering, 122 (1996), pp. 682-690.

[4] B. G. Newman, *Shape of towed boom of logs,* Proc. R. Soc. Lond., A.346 (1975), pp. 329-348.

[5] R. H. Plaut, S. Suherman, *Two-dimensional analysis of geosynthetic tubes,* Acta Mechanica, 129 (1998), pp. 207-218.

[6] R. H. Plaut, C. R. Klusman, *Two-dimensiona analysis of stacked geosynthetic tubes on deformable foundations,* Thin-Walled Structures, 34 (1999), pp. 179-194.

[7] P. A. Seay, R. H. Plaut, *Three-dimensional behavior of geosynthetic tubes,* Thin-Walled Structures, 32 (1998, pp. 263-274.

[8] R. A. Zhao, *A complete linear theory for two dimensional floating and liquid-filled membrane structures in waves,* Journal of Fluids and Structures, 9 (1995), pp. 937-956.

[9] R. A. Zhao, J. V. Aarsnes, *Numerical and experimental studies of a floating and liquid-filled membrane structure in waves,* Ocean Engineering, 25 (1998) pp. 753-765.