Two partial results on preconditioning using hierarchical bilinear finite elements

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Abstract. *We present some new results on the preconditioning for the numerical solution of partial differential equations using hierarchical finite elements. The effectivity of the preconditioning is tied to the constant in the strengthened Cauchy-Buniakowski-Schwarz inequality for the coarse space and the complement of it in the finer level. We show the estimates of this constants for two special choices of bilinear finite element functions.*

1 Introduction

The purpose of this paper is to show some properties of one type of the multilevel preconditioning for the numerical solution of elliptic problems. We consider the following elliptic boundary value problem in a selfadjoint form

$$\begin{aligned} -\nabla(a\nabla u) = f \qquad & \text{in} \qquad \Omega, \\ u = 0 \qquad & \text{on} \qquad \Gamma, \end{aligned} \tag{1}$$

where Ω is a two-dimensional domain, Γ is the boundary of Ω and f is a given function in $L^2(\Omega)$. We will assume that a is a diagonal matrix, the diagonal elements of which are piecewise smooth and positive functions on $\overline{\Omega}$. The weak formulation of (1) consists in finding $u \in W_0^{1,2}(\Omega)$ such that

$$A(u,v) = (f,v),$$

for all $v \in W_0^{1,2}(\Omega)$, where

$$A(u,v) = \int_\Omega a\nabla u \nabla v \,\mathrm{d}x\mathrm{d}y.$$

Using the finite element method the discretized problem is to find an n-dimensional real vector $\mathbf{v}$ such that

$$\mathbf{A}_V\mathbf{v} = \mathbf{F}_V, \tag{2}$$

where

$$\begin{aligned} (\mathbf{A}_V)_{ij} &= A(v_i, v_j), \\ (\mathbf{F}_V)_i &= (f, v_i) \end{aligned}$$

where $\{v_i\}_{i=1}^n$ is a set of basis functions of the n-dimensional space $V \subset W_0^{1,2}(\Omega)$.

Throughout this paper we will consider the bilinear finite elements with rectangular supports; in section 3 we will deal with conforming bilinear elements, while in section 4 the non-conforming bilinear functions will be used.

2 Two-level setting and preconditioning

Let us consider two hierarchical finite element spaces U and V, $U \subseteq V$, corresponding to two regular grids of the sizes H and $h < H$, respectively. In this paper we will consider two examples in which $H = 2h$ and $H = 4h$, respectively. Let W be a space complementary to U in V,

$$V = U \oplus W.$$

Then we can reformulate (2); it is to find vectors $\mathbf{u} \in \mathbf{U}$ and $\mathbf{w} \in \mathbf{W}$ such that

$$\begin{pmatrix} \mathbf{A}_U & \mathbf{A}_{UW} \\ \mathbf{A}_{UW}^T & \mathbf{A}_W \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_U \\ \mathbf{F}_W \end{pmatrix},$$

where

$$(\mathbf{A}_U)_{ij} = A(u_i, u_j), \quad (\mathbf{A}_W)_{ij} = A(w_i, w_j), \quad (\mathbf{A}_{UW})_{ij} = A(u_i, w_j), \tag{3}$$

$$(\mathbf{F}_U)_i = (f, u_i) \quad \text{and} \quad (\mathbf{F}_W)_i = (f, w_i) \tag{4}$$

for all of the basis functions u_i and w_i of U and W, respectively.

Let us now compose a block diagonal matrix

$$\mathbf{M} = \begin{pmatrix} {\mathbf{A}_U}^{-1} & 0 \\ 0 & {\mathbf{A}_W}^{-1} \end{pmatrix}$$

and let us consider the equation

$$\mathbf{M} \begin{pmatrix} \mathbf{A}_U & \mathbf{A}_{UW} \\ \mathbf{A}_{UW}^T & \mathbf{A}_W \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{F}_U \\ \mathbf{F}_W \end{pmatrix}. \tag{5}$$

The matrix $\bar{\mathbf{A}}$ of the system (5),

$$\bar{\mathbf{A}} = \mathbf{M} \begin{pmatrix} \mathbf{A}_U & \mathbf{A}_{UW} \\ \mathbf{A}_{UW}^T & \mathbf{A}_W \end{pmatrix},$$

has the condition number

$$\kappa(\bar{\mathbf{A}}) \leq \frac{1+\gamma}{1-\gamma},$$

where the constant γ comes from the inequality

$$|A(u, w)| \leq \gamma \sqrt{A(u, u)} \sqrt{A(w, w)} \tag{6}$$

which should be valid for any $u \in U$ and $w \in W$ [1, 2]. Thus the constant γ depends on the splitting U and W of the basic space V and on the positive definite operator A defined by the equation (1). The inequality (6) is called the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality and the constant γ is the strengthened CBS constant. Let us stress that $\gamma < 1$ whenever $U \cap V = \{0\}$. Our aim is now to estimate the value γ for some finite element spaces. The universal estimate of the constant γ can be computed from the local properties of U and W on some element belonging to the coarser grid (macroelement) corresponding to the space U [1, 3]. The estimate does not depend on the mesh [1, 3]. The detailed description of the multilevel preconditioning techniques and the exploiting the strengthened CBS inequality can be found e.g. in [1, 2].

In the following parts we assume that the diagonal functions in a are constant on each macroelement. Studying the anisotropic case it is sufficient to consider the problem with the coefficient matrix a in the form

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Theorem 2.1. *Let a be a diagonal matrix piecewise constant and constant on macroelements. Then the CBS constant γ is less than or equal to γ_1 which is the upper estimate of the CBS constant for the case $a_{11} \equiv 1$ and $a_{22} \equiv 0$.*

Proof. Assuming symmetry of the functions and their supports, the quantity γ_1 as the CBS constant is valid also for the case that $a_{11} \equiv 0$, $a_{22} \equiv 1$. Then the proof follows directly from the estimating of the term

$$(a_{11} A_x(u,w) + a_{22} A_y(u,w))^2,$$

where

$$A_x(u,w) = \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,\mathrm{d}x\mathrm{d}y \quad \text{and} \quad A_y(u,w) = \int_\Omega \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,\mathrm{d}x\mathrm{d}y,$$

and from the Hölder inequality.

Now let us briefly recall some known results. The CBS constant γ for hierarchical finite element spaces of linear elements on triangles is not greater than $\sqrt{3/4}$ for general anisotropic case, and $\gamma \le \sqrt{1/2}$ for isotropic equation ($a_{11} = a_{22} \equiv 1$) and for equal mesh in x and y directions [3].

Linear 2D od 3D elements and general elasticity equation lead to $\gamma = \sqrt{\frac{m^2-1}{m^2}}$, where V is m-times finer than U [3].

3 Bilinear conforming finite elements

Let us consider a reference rectangle $\Omega_r = \langle -1, 1\rangle \times \langle -1, 1\rangle$ (macroelement) and four nodal basis functions $u_1, \ldots, u_4$ belonging to the coarse grid. Let $w_1, \ldots, w_5$ be the basis functions of W which are nonzero on Ω_r, see Figure 1 for the numbering. Let us denote

$$\tilde{U} = \mathrm{span}\{u_1, \ldots, u_4\}, \quad \tilde{W} = \mathrm{span}\{w_1, \ldots, w_5\}.$$

Then the upper estimate of the CBS constant γ can be obtained from

$$\gamma = \inf \left\{\gamma_0;\ |\mathbf{u}^T \mathbf{A}_{UW} \mathbf{w}| \le \gamma_0 \sqrt{\mathbf{u}^T \mathbf{A}_U \mathbf{u}} \sqrt{\mathbf{w}^T \mathbf{A}_W \mathbf{w}}\right\}$$

where the entries of $\mathbf{A}_U$, $\mathbf{A}_W$ and $\mathbf{A}_{UW}$ are given by (3) and (4) for the basis functions $u_1, \ldots, u_4$, $w_1, \ldots, w_5$ but the integrals are computed only over the domain Ω_r [1].

It was already introduced in [9] that the constant γ is not greater than $\sqrt{3/4}$ in the anisotropic case and it is not greater than $\sqrt{3/8}$ for isotropic coefficients and shape. Let us mention that for the letter case the estimate is even better than the corresponding upper bound for linear elements, which is $\sqrt{1/2}$.

In this paper we compute the estimate of γ for $H = 4h$. This means that we consider four nodal basis functions of $\tilde{U}$ and twenty one basis functions of $\tilde{W}$.

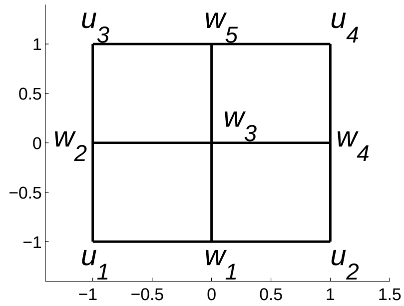

Figure 1: *Notation of the basis functions of $\tilde{U}$ and $\tilde{W}$ for $H = 2h$.*

Theorem 3.1. *Let us consider the problem* (1) *with diagonal matrix a constant on each macroelement and let us use two bilinear hierarchical finite element spaces with rectangular supports corresponding to meshes h and $H = 4h$, respectively. Then for the strengthened CBS constant γ we have*

$$\gamma \leq \sqrt{6/7}$$

and there exist functions $u \in U$ and $v \in V$ that the equality is achieved in (6) *for $\gamma = \sqrt{6/7}$.*

Proof. The proof is obtained from the computing the eigenvalues and eigenvectors of $\mathbf{A}_U$ and $\mathbf{A}_W$ and from the singular value decomposition of the matrix

$$\mathbf{U}^T \mathbf{A}_{UW} \mathbf{W},$$

where the columns of $\mathbf{U}$ and $\mathbf{W}$ are the normalized eigenvectors of $\mathbf{A}_U$ and $\mathbf{A}_W$, respectively, each of them divided either by the square root of the corresponding eigenvalue or by one if the eigenvalue is zero. All of the computations are carried out in exact arithmetic.

Let us note that the obtained estimate of γ is a little bit lower than the upper bound for the linear elements on triangles which is $\sqrt{15/16}$ for $H = 4h$. The reader can also compare this result with the estimates for other types of refinements [8].

4 Bilinear non-conforming finite elements

The bilinear non-conforming nodal [6] finite elements are also called Rannacher-Turek elements [7]. In this section we have

$$\tilde{V} = \text{span}\{v_1, \ldots, v_{12}\},$$

where $v_1, \ldots, v_{12}$ are bilinear nodal finite element functions defined on the reference rectangle $\Omega_r = \langle -1, 1\rangle \times \langle -1, 1\rangle$ and corresponding to the finer grid; see Figure 2 for the numbering of the functions.

While the conforming finite element spaces fulfill $U \subseteq V$ (or $\tilde{U} \subseteq \tilde{V}$ for the spaces of functions considered on the reference rectangle, respectively), this is not the case for the non-conforming elements. Then we have to find suitable subspaces U and W of V which lead to the sufficiently low constant γ not dependent on the mesh. Recall that for the linear finite elements (Crouzeix-Raviart elements) the resulting estimate is $\sqrt{3/4}$, see [4] and [5] for the choices of function subspaces. Trying to follow these types of splitting introduced for linear elements we were not successful. Also the analysis presented in [6] shows that the choice of the subspaces needs some more sophisticated approach than in the case of linear elements. Many "natural" constructions

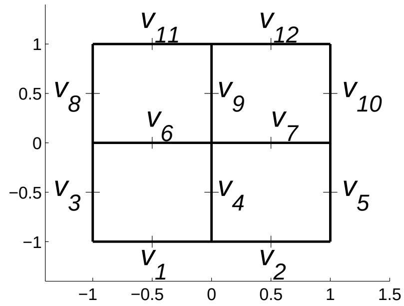

Figure 2: *Notation of the basis functions of $\tilde{V}$ for non-conforming elements.*

of U's and W's lead to $\gamma = 1$ for the anisotropic equation or for the anisotropic shape of the support. Two new possible approaches how to split the space V are presented in [6]. But any of them does not correspond to the "classical" method when the basis functions of U and W are the sparse linear combinations of the basis functions of V.

In this part we provide one partial result on this issue. Since it turns out that solving the anisotropic case is rather difficult and it will need some completely new approach, we introduce a simple scheme, which yields $\gamma = 1$ in the anisotropic case but $\gamma < 1$ in the isotropic case.

Let us choose the bases in the spaces $\tilde{U}$ and $\tilde{W}$ in the following way

$$\begin{aligned} u_1 &= 2v_1 + 2v_2 + v_6 + v_7, & u_2 &= 2v_3 + 2v_8 + v_4 + v_9, \\ u_3 &= 2v_5 + 2v_{10} + v_4 + v_9, & u_4 &= 2v_{11} + 2v_{12} + v_6 + v_7 \end{aligned} \tag{7}$$

and

$$\begin{aligned} w_1 &= v_1 + v_4, & w_2 &= v_2 + v_4, & w_3 &= v_3 + v_6, & w_4 &= v_5 + v_7, \\ w_5 &= v_6 + v_8, & w_6 &= v_7 + v_{10}, & w_7 &= v_9 + v_{11}, & w_8 &= v_9 + v_{12}, \end{aligned} \tag{8}$$

respectively.

Theorem 4.1. *Let us consider the problem* (1) *with the diagonal matrix a constant on each macroelement and let us use the bilinear non-conforming finite element functions with rectangular supports. Let us split this space into U and W the bases of which correspond to the relations* (7)-(8) *on macroelements. Then*

$$\gamma \leq \sqrt{19/115}$$

and there exist functions $u \in U$ and $v \in V$ that the equality is achieved in (6) *for $\gamma = \sqrt{19/115}$.*

Proof. The approach is identical to the proof of the Theorem 3.1.

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