NUMERICAL MODELLING OF SEMI-COERCIVE BEAM PROBLEM WITH UNILATERAL ELASTIC SUBSOIL OF WINKLER'S TYPE

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Abstract. The presented work is continuation of the article [7], where the semi-coercive problem of a beam on a unilateral elastic subsoil and the corresponding problem approximation have been analysed. Numerical methods and the computational algorithms for the class of such problems are presented and their uniform converge properties are proved here. The methods are based on the minimisation of a total energy functional, where the descent directions of the functional are searched by solving the linear problems with a beam on bilateral elastic "springs". The influence of external loads on the convergence properties is also investigated. The effectiveness of the algorithms is illustrated on numerical examples.

Keywords: non-linear subsoil of Winkler's type, semi-coercive beam problem, approximation, iterative methods, convergence, projection, load stability

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1. Introduction

The semi-coercive problem of a beam on a unilateral elastic subsoil means to minimise a convex, differentiable and non-linear functional. The functional is coercive only if the additional assumptions on external loads are formulated. There are some methods how to numerically solve the class of such problems. The methods based on linear complementarity are presented in [4]. The methods for quadratic programming can also be used due to the dual formulations of the problems, see [6].

In this article, a total energy functional is minimised such that the descent directions of the functional are searched by solving the linear problems with a beam on bilateral elastic "springs". We obtain the so-called "descent direction method without projection" and prove its uniform convergence properties with respect to refinement of the partition. Since the problem is only semi-coercive it is also useful to investigate the influence of the load on convergence. Mainly for "unstable" cases

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of the load, the rate of convergence can be improved by adding of the so-called "projection" step. We obtain the "descent direction method with projection", which has the same convergence properties as the previous method.

In Section 2, the formulations of the problem, its approximation and the basic results of the article [7] are summarised. Moreover, two useful lemmas are added. In Section 3, the auxiliary linear problems with bilateral elastic "springs" are defined and their uniform properties are derived. In Section 4, the descent direction methods with and without projection are introduced and their uniform convergence properties are proved. In Section 5, the approximated problem and algorithms are rewritten to their algebraical forms and the reason of the "projection" step is explained. And in Section 6, the effectiveness of the algorithms is illustrated on numerical examples.

2. Overview to Semi-coercive Beam Problem on Unilateral Elastic Subsoil

2.1. **Notation.** We will use the Lebesgue spaces $L^p(\Omega)$, $p = 2, +\infty$, Sobolev spaces $H^k(\Omega) \equiv W^{k,2}(\Omega)$, $k = 0, 1, 2, 3, 4$, and the spaces of continuously differentiable functions $C^k(\overline{\Omega})$, where Ω is an open, bounded and non-empty interval in $\mathbb{R}^1$. The spaces are described in the book [1]. Their standard norms are denoted as $\|.\|_{p,\Omega}$, $\|.\|_{k,2,\Omega}$ and $\|.\|_{C^k(\overline{\Omega})}$, respectively. The i-th seminorms, $i = 0, 1, \ldots, k$, of the spaces $H^k(\Omega)$ are denoted as $|.|_{i,2,\Omega}$. The space of polynomials of the k-th degree is denoted as P_k.

Since we will mainly use the interval $\Omega := (0, l)$ in the remaining parts of the article, we will denote the norms and seminorms of the Sobolev spaces $H^k(\Omega)$, $k = 0, 1, 2, 3, 4$, without the symbol Ω for this concrete choice of the interval.

With respect to the well-known imbedding theorem of the Sobolev space $H^2(\Omega)$, see [1], we will assume that the functions $v \in H^2(\Omega)$ also belong to $C^1(\overline{\Omega})$ to define the values $v(x), v'(x)$, $x \in \overline{\Omega}$.

2.2. **Setting of the Problem.** We consider a beam of the length l with free ends which is situated in the interval $\Omega = (0, l)$, and assume that the beam is supported by a unilateral elastic subsoil in the interval $\Omega_s := (x_l, x_r)$, $0 \leq x_l < x_r \leq l$. Such a subsoil is active only if the beam deflects against it. Let E, I and q denote functions that represent, respectively, the Young's modulus of the beam material, the inertia moment of the cross-section of the beam and the stiffness coefficient of the subsoil. The aim is to find the deflection w^* of the axes of the beam caused by the beam load. The situation is depicted in Figure 1.

We will assume that the functions E, I, q belong to the Lebesgue space $L^\infty(\Omega)$ and there exist positive constants E_0, I_0 and q_0 such that

$$E(x) \geq E_0, \quad I(x) \geq I_0, \quad \text{a.e. in } \Omega, \quad \text{and} \quad q(x) \geq q_0 \quad \text{a.e. in } \Omega_s.$$

Then we can define the forms

$$a(v_1, v_2) \quad := \quad \int_\Omega EIv_1''v_2'' \, dx, \quad v_1, v_2 \in H^2(\Omega),$$

$$b(v_1, v_2) \quad := \quad \int_{\Omega_s} qv_1v_2 \, dx, \quad v_1, v_2 \in H^1(\Omega),$$
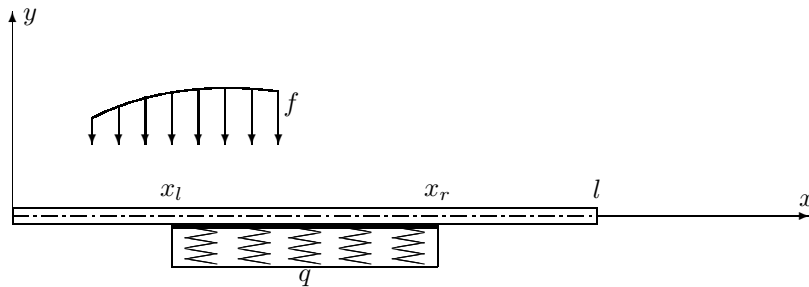
FIGURE 1. Scheme of the subsoiled beam with axes orientation.

to represent the work of the inner forces and the subsoil, respectively. The forms a, b are bilinear and bounded on the space $H^2(\Omega)$.

The space of all continuous and linear functionals defined on $H^2(\Omega)$ will be denoted V^* and its corresponding norm $\|.\|_*$. The work of the beam load will be represented by a functional $L \in V^*$.

The total potential energy functional for the problem has the form

$$(2.1) \qquad J(v) := \frac{1}{2}\big(a(v,v) + b(v^-, v^-)\big) - L(v), \quad v \in H^2(\Omega).$$

The functional J is Gâteaux differentiable and convex on the space $H^2(\Omega)$. Its Gâteaux's derivative at any point $w \in H^2(\Omega)$ and direction $v \in H^2(\Omega)$ has the form

$$(2.2) \qquad J'(w; v) = a(w, v) + b(w^-, v) - L(v).$$

The variational formulation of the problem can be written as the minimisation problem

$$(P) \qquad \text{find } w^* \in H^2(\Omega): \quad J(w^*) \le J(v) \quad \forall v \in H^2(\Omega),$$

or equivalently, with respect to (2.2), as the non-linear variational equation

$$(2.3) \qquad a(w^*, v) + b((w^*)^-, v) = L(v) \quad \forall v \in H^2(\Omega).$$

Notice that for sufficiently smooth data, problem means to solve the non-linear differential equation of the fourth order with the homogeneous Neumann boundary conditions.

2.3. **Solvability and Dependence on the Load.** Since the beam does not have fixed ends (it is only laid on the subsoil), the problem solvability depends on the beam load. The existence and uniqueness of the solution w^* of the problem (P) is ensured by the condition

$$(2.4) \qquad L(p) < 0 \quad \forall p \in P_1, \ \ p > 0 \ \text{ in } \ \Omega_s,$$

where the polynomials of the first degree represent the rigid beam motions for which the subsoil is not active. Notice that the functional J is coercive on $H^2(\Omega)$ if this condition holds.

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For other analyses, it will be usefull to rewrite equivalently the condition (2.4) in the following way:

$$(2.5) \qquad\qquad F < 0 \quad \text{and} \quad x_l < T < x_r,$$

where $F := L(1)$ is the load resultant and $T := L(x)/L(1)$ is the balance point of the load. The condition (2.5) means that the load resultant is situated in Ω_s and oriented against the subsoil, which causes that the beam deflection activates the subsoil on the set $M \subset \Omega_s$ with a positive one-dimensional Lebesgue measure, i.e. $w^* < 0$ in M. In addition, the balance point T lies in the convex closure of the set M.

To determine the dependence of the change of the problem (P) solution on the change of the load, we will consider the class $\mathcal{S}_{\delta,\xi,\eta}$ of the loads $L \in V^*$ such that $T \in [x_l + \delta, x_r - \delta]$, $F \leq -\xi < 0$ and $\|L\|_* \leq \eta$, with respect to positive parameters δ, ξ, η. If we will assume that $\mathcal{S}_{\delta,\xi,\eta}$ is non-empty then there exists a positive constant c which depends on the loads from $\mathcal{S}_{\delta,\xi,\eta}$ only through the parameters δ, ξ, η such that

$$(2.6) \qquad\qquad \|w_1^* - w_2^*\|_{2,2} \leq c\|L_1 - L_2\|_* \quad \forall L_1, L_2 \in \mathcal{S}_{\delta,\xi,\eta},$$

where $w_i^* = w_i^*(L_i)$ solves the problem (P) with respect to the load L_i, $i = 1, 2$.

The following lemma, which is also important for numerical modelling, describes dependence of the constant c from the estimate (2.6) on the parameters δ, ξ, η for the limit cases $\delta \to 0$ and $\xi \to 0$.

Lemma 2.1. *Let $\eta > 0$ and $0 < \delta_{max} < (x_r - x_l)/2$. Then there exists a positive constant ξ_{max} depending on η such that for any sequences $\{\delta_k\}_k$, $0 < \delta_k \leq \delta_{max}$, and $\{\xi_k\}_k$, $0 < \xi_k \leq \xi_{max}$, $k \geq 0$, it holds the following implication: if $\delta_k \to 0$ or $\xi_k \to 0$ then $c_k \to +\infty$, where $c_k = c_k(\delta_k, \xi_k, \eta)$ is the smallest constant which satisfies (2.6) for parameters δ_k, ξ_k, η.*

Proof. We will construct suitable sequences $\{L_{i,k}\}_k \subset V^*$, $i = 1, 2$, to prove the assertion. The corresponding load resultants, their balance points and solutions of the problems (P) will be respectively denoted $F_{i,k}$, $T_{i,k}$ and $w_{i,k}$, $i = 1, 2$. Subsequences of these sequences will be denoted in the same way. For the sake of brevity, some step of the proof will be done more briefly.

Case 1. Let $\eta > 0$ and $\delta_k \to 0$. Then there exists $\xi_{max} > 0$ such that $\|L_{i,k}\|_* \leq \eta$, $i = 1, 2$, where

$$L_{1,k}(v) := \xi_k v((x_l + x_r)/2), \quad L_{2,k}(v) := \xi_k v(x_l + \delta_k), \quad \xi_k \leq \xi_{max}, \quad k \geq 0.$$

We will assume that there exists $\xi_{min} > 0$ such that $\xi_k \geq \xi_{min}$, $k \geq 0$, in this first case. Then $F_{1,k} = F_{2,k} = \xi_k$, $T_{1,k} = (x_l + x_r)/2$ and $T_{2,k} = x_l + \delta_k$. Therefore $L_{i,k} \in \mathcal{S}_{\delta_k,\xi_{min},\eta}$, $i = 1, 2$. The sequence $\{w_{1,k}\}_k$ is bounded on $H^2(\Omega)$ by Theorem 3.2 in [7]. Suppose for a moment that some subsequence of $\{w_{2,k}\}_k$ is bounded on $H^2(\Omega)$. Then we can assume with loss of generality that there exists $w \in H^2(\Omega)$ such that $w_{2,k} \to w$ in $H^1(\Omega)$ by the Rellich theorem. The functions $w_{2,k}$ solve the equation

$$(2.7) \qquad\qquad a(w_{2,k}, v) + b(w_{2,k}^-, v) = L_{2,k}(v) \quad \forall v \in H^2(\Omega).$$

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The choice $v(x) = x - x_l \in P_1$ in (2.7) yields

$$b(w^-, v) = \lim_{k \to \infty} b(w_{2,k}^-, v) = \lim_{k \to \infty} L_{2,k}(v) = \lim_{k \to \infty} F_{2,k}(T_{2,k} - x_l) = 0.$$

Hence $w \geq 0$ in Ω_s. Then the choice $v(x) = 1 \in P_1$ in (2.7) yields contradiction:

$$0 = \lim_{k \to \infty} b(w_{2,k}^-, 1) = \lim_{k \to \infty} L_{2,k}(1) = \lim_{k \to \infty} F_{2,k} \leq -\xi_{min} < 0.$$

Therefore $\|w_{2,k}\|_{2,2} \to \infty$ and by (2.6),

$$c_k \geq \frac{\|w_{1,k} - w_{2,k}\|_{2,2}}{\|L_{1,k} - L_{2,k}\|_*} \to \infty.$$

Case 2. Let $\eta > 0$, $0 < \delta_{min} \leq \delta_k \leq \delta_{max} < (x_r - x_l)/2$ and $\xi_k \to 0$. Let us choose

$$L(v) \quad := \quad \eta_0 \left[v(x_l) - 2v \left(\frac{x_l + x_r}{2} \right) + v(x_r) \right],$$

$$L_{1,k}(v) \quad := \quad L(v) - \xi_k v \left(\frac{x_l + x_r}{2} \right),$$

$$L_{2,k}(v) \quad := \quad L_{1,k}(v) - \varepsilon_k v(x_l),$$

where $\varepsilon_k = \frac{\xi_k}{\delta_k}((x_l + x_r)/2 - (x_l + \delta_k)) > 0$ and $\eta_0 > 0$ is chosen such that $\|L_{i,k}\|_* \leq \eta$, $i = 1, 2$, for sufficiently large k. Then $L(1) = 0$, $L(x) = 0$, $F_{1,k} = -\xi_k \to 0$, $F_{2,k} = -\xi_k - \varepsilon_k$, $T_{1,k} = (x_l + x_r)/2$, $T_{2,k} = x_l + \delta_k$, $L_{i,k} \in \mathcal{S}_{\delta_{min}, \xi_k, \eta}$ and $L_{i,k} \to L$ in V^*, $i = 1, 2$.

By Theorem 3.2 in [7], the sequences $\{w_{1,k}\}_k$, $\{w_{1,k}\}_k$ are bounded on $H^2(\Omega)$. Therefore there exist subsequences $\{w_{i,k}\}_k$ and functions $w_i \in H^2(\Omega)$ such that $w_{i,k} \rightharpoonup w_i$ weakly in $H^2(\Omega)$ and $w_{i,k} \to w_i$ in $H^1(\Omega)$ (by the Rellich theorem), $i = 1, 2$. Since the functions $w_{i,k}$ solve the equations

$$a(w_{i,k}, v) + b(w_{i,k}^-, v) = L_{i,k}(v) \quad \forall v \in H^2(\Omega), \ i = 1, 2, \ k \geq 0,$$

the limit case $k \to \infty$ yields

$$a(w_i, v) + b(w_i^-, v) = L(v) \quad \forall v \in H^2(\Omega), \ i = 1, 2.$$

The choice $v = 1$ yields $b(w_i^-, 1) = 0$. Thus $w_1, w_2 \geq 0$ in Ω_s and consequently w_1, w_2 solve the following Neumann problem:

$$(2.8) \qquad\qquad a(w_i, v) = L(v) \quad \forall v \in H^2(\Omega), \ i = 1, 2.$$

Hence, there exists a polynomial $p \in P_1$ such that $w_1 - w_2 = p$. Notice that if a function $v \in H^2(\Omega)$ is convex and $v \notin P_1$ in Ω_s then $L(v) > 0$. From this result and the equation (2.8), it is possible to prove that $w_i'' > 0$ almost everywhere in Ω_s, $i = 1, 2$. It means that the functions w_1, w_2 are strictly convex in Ω_s and have just one minimum in $\overline{\Omega}_s$.

By Lemma 3.5 in [7], there exist sequences $\{x_{i,k}\}_k, \{y_{i,k}\}_k \subset \Omega_s$ and their limits x_i, y_i, $i = 1, 2$, such that

$$w_{i,k}(x_{i,k}) \leq 0, \quad w_{i,k}(y_{i,k}) \leq 0 \quad \text{and} \quad x_{i,k} \leq T_{i,k} \leq y_{i,k} \quad \forall k \geq 0, \ i = 1, 2.$$

Hence $w_i(x_i) = w_i(y_i) = 0$, since w_i are non-negative in Ω_s, $i = 1, 2$. Consequently,

$$x_1 = y_1 = (x_l + x_r)/2, \quad x_2 = y_2 = \lim_{k \to \infty} T_{2,k} < (x_l + x_r)/2,$$

since w_i are strictly convex in Ω_s, $i = 1, 2$. Thus $w_1((x_l + x_r)/2) = 0$ and $w_2(x_l + \delta) = 0 < w_1(x_l + \delta)$. Therefore $w_1 \neq w_2$ and consequently by (2.6),

$$c_k \geq \frac{\|w_{1,k} - w_{2,k}\|_{2,2}}{\|L_{1,k} - L_{2,k}\|_*} \to \infty.$$

This result holds for any subsequences $\{w_{i,k}\}_k$ with weak limits $w_i \in H^2(\Omega)$, $i = 1, 2$, which means that the whole sequence $\{c_k\}_k$ converges to ∞.

Case 3. Let $\eta > 0$, $\delta_k \to 0$, $\xi_k \to 0$ and $0 < \delta_{max} < (x_r - x_l)/2$. Since $\mathcal{S}_{\delta_{max}, \xi_k, \eta} \subset \mathcal{S}_{\delta_k, \xi_k, \eta}$ for sufficiently large k, $c_k(\delta_{max}, \xi_k, \eta) \leq c_k(\delta_k, \xi_k, \eta)$, which follows from the estimate (2.6). By Case 2, $c_k(\delta_{max}, \xi_k, \eta) \to \infty$. Hence, $c_k(\delta_k, \xi_k, \eta) \to \infty$. $\qquad\square$

Notice that the small change of the load causes the relatively large "rigid" displacement of the beam in Case 2 of the proof.

With respect to Lemma 2.1, the loads, for which the balance point T is closed to the end points of the subsoil or the size of the load resultant is small with respect to V^*-norm of the load, will be called *unstable*. Some unstable loads are illustrated in [8] on numerical examples.

2.4. Approximation of the Problem.

Let us define a partition τ_h,

$$0 = x_0 < x_1 < \ldots < x_N = l, \quad h := \max_{j=1,\ldots,N}(x_j - x_{j-1}), \quad h_{min} := \min_{j=1,\ldots,N}(x_j - x_{j-1})$$

of the interval $\overline{\Omega} = [0, l]$, with the nodal points x_j, $j = 0, 1, \ldots, N$, and with the parameters $h, h_{min} > 0$. With respect to a positive parameter θ, we will consider the system $\mathcal{T}_\theta$ of such partitions τ_h for which the inequality $\theta h \leq h_{min}$ holds.

For a partition $\tau_h \in \mathcal{T}_\theta$ with $N + 1$ nodal points, we will define the function space

$$V_h \subset H^2(\Omega), \quad V_h := \{v_h \in C^1(\overline{\Omega}) \mid v_h|_{(x_{j-1}, x_j)} \in P_3, \ j = 1, 2, \ldots, N\},$$

i.e. the space of continuously differentiable and piecewise cubic functions.

For the sake of simplicity, we will assume that the function q, which represents the stiffness coefficient of the subsoil, is piecewise constant in the interval Ω_s and that the considered partition $\tau_h \in \mathcal{T}_\theta$ take into account the points of discontinuity of q. Since the evaluation of the term $b(w_h^-, v_h)$, $w_h, v_h \in V_h$, cannot be computed exactly due to the non-linear term w_h^-, an approximation of the form b must be used. The form b will be approximated by a numerical quadrature on each subsoiled partition interval. Its approximation has the form

$$(2.9) \qquad b_h(v_1, v_2) := \sum_{i=1}^{m(h)} r_i v_1(z_i) v_2(z_i), \quad v_1, v_2 \in H^2(\Omega),$$

where z_i, $z_1 < z_2 < \ldots < z_{m(h)}$, are the points of the numerical quadratures and r_i contains the products of the stiffness coefficients and weights of the numerical quadrature. With respect to the assumption on $\tau_h \in \mathcal{T}_\theta$, it holds that there exists constants $c_1, c_2 > 0$ such that

$$(2.10) \qquad c_1 q_0 \theta h \leq r_i \leq c_2 \|q\|_\infty h, \quad i = 1, 2, \ldots, m(h).$$

From a mechanical point of view, the subsoil is substituted by insulated "springs". We will assume that the numerical quadrature is exact at least for polynomials of the first degree.

If we set

$$\mathcal{V}_M := \left\{ v \in H^2(\Omega) \mid \exists p \leq M, \ \exists y_1, y_2, \ldots, y_{2p} \in \overline{\Omega}_s : \right.$$

$$\left. \{x \in \overline{\Omega}_s \mid v^-(x) = 0\} = \bigcup_{i=1}^{p} [y_{2i-1}, y_{2i}] \right\}, M > 0,$$

then there exists positive constants c_1, c_2 and $c_3 = c_3(M)$, which are independent of the choice of τ_h, such that

$$(2.11) \qquad |b_h(u, v)| \leq c_1 \|q\|_{\infty, \Omega_s} \|u\|_{1,2} \|v\|_{1,2} \quad \forall u, v \in H^1(\Omega),$$

$$(2.12) \quad |b(v^-, u) - b_h(v^-, u)| \leq c_2 h \|v\|_{1,2} \|u\|_{1,2} \quad \forall u, v \in H^1(\Omega),$$

$$(2.13) \quad |b(v^-, u) - b_h(v^-, u)| \leq c_3 h^2 \|v\|_{2,2} \|u\|_{2,2} \quad \forall u \in H^2(\Omega), \ \forall v \in \mathcal{V}_M.$$

Now, we set the approximated problem. For the sake of simplicity, we will not consider a numerical quadrature of the forms a and L. The approximated problem corresponding to the partition $\tau_h \in \mathcal{T}_\theta$ has the form

$$(P_h) \qquad \begin{cases} \text{find } w_h^* \in V_h : \ J_h(w_h^*) \leq J_h(v_h) \quad \forall v_h \in V_h, \\ J_h(v_h) := \frac{1}{2} a(v_h, v_h) + \frac{1}{2} b_h(v_h^-, v_h^-) - L(v_h). \end{cases}$$

Since the functional J_h is convex and has the Gâteaux derivative on the space V_h, the problem (P_h) can be rewritten equivalently to the nonlinear variational equation

$$(2.14) \qquad a(w_h^*, v_h) + b_h((w_h^*)^-, v_h) = L(v_h) \quad \forall v_h \in V_h.$$

The existence of the problem (P_h) solution is ensured by the condition

$$(2.15) \qquad F < 0 \quad \text{and} \quad z_1 < T < z_{m(h)}.$$

This condition also ensures the uniqueness of the solution for sufficiently small h. Notice that if the condition (2.5) holds and the discretisation parameter h is sufficiently small, then the condition (2.15) also holds.

It holds that the set

$$(2.16) \qquad A_h^* := \{i \in \{1, \ldots, m(h)\} \mid w_h^*(z_i) < 0\},$$

which represents active "springs" is non-empty. In addition, the balance point T belongs to the convex closure of the points $\{z_i; \ i \in A_h^*\}$.

For the approximated problems (P_h), it holds the following estimates and convergence result:

$$\|w^* - w_h^*\|_{2,2} \leq c_1(M) h^2 \|w^*\|_{4,2} \qquad w^* \in H^4(\Omega) \cap \mathcal{V}_M, \ \forall \tau_h \in \mathcal{T}_\theta, \ h \leq h_0,$$

$$\|w^* - w_h^*\|_{2,2} \leq c_2 h \|w^*\|_{3,2} \qquad w^* \in H^3(\Omega), \ \forall \tau_h \in \mathcal{T}_\theta, \ h \leq h_0,$$

$$(2.17) \qquad \|w^* - w_h^*\|_{2,2} \to 0 \qquad \qquad w^* \in H^2(\Omega), \ h \to 0,$$

where w^* and w_h^* are respectively the solutions of the problems (P) and (P_h), and h_0 is a sufficiently small parameter. The first of these estimates is numerically illustrated in [8] for some numerical quadratures.

In the end of this section, we add one lemma, which describes when the functionals J_h are uniformly coercive on $H^2(\Omega)$. The lemma will be also useful for the following analysis.

Lemma 2.2. *Let $F < 0$, $x_l < T < x_r$, $0 < h_0 < \min\{T - x_l, x_r - T\}$, $c \in \mathbb{R}$ and $\theta > 0$. Then there exists a positive constant $\tilde{c}$ such that the following implication holds:*

$$J_h(u_h) \le c \quad \Longrightarrow \quad \|u_h\|_{2,2} \le \tilde{c} \qquad \forall \tau_h \in \mathcal{T}_\theta, \ h \le h_0, \ \forall u_h \in V_h.$$

Proof. Since the proof is similar to the first (existence) part of the proof of Theorem 3.1. in [7], some steps will be done more briefly.

Suppose that the lemma does not hold. Then, by the definition of J_h, there exist sequences $\{\tau_{h_k}\}_k$ and $\{u_k\}_k$, $u_k \in V_{h_k}$, $\|u_k\|_{2,2} \to +\infty$ such that

$$(2.18) \qquad 0 \le a(u_k, u_k) + b_{h_k}(u_k^-, u_k^-) \le 2L(u_k) + 2c.$$

If we divide (2.18) by $\|u_k\|_{2,2}^2$, we obtain

$$a(v_k, v_k) + b_{h_k}(v_k^-, v_k^-) \to 0, \quad v_k := u_k / \|u_k\|_{2,2}.$$

Hence, by the Rellich theorem and (2.11), there exist a subsequence $\{v_k\}_k$ (denoted in the same way) and a polynomial $p \in P_1$ such that $v_k \to p$ in $H^2(\Omega)$ and $b_{h_k}(p^-, p^-) \to 0$. By the assumption of h_0, (2.10) or eventually (2.12) for $h_k \to 0$, we obtain $p \ge 0$ in the neighbourhood of the point T.

If we divide (2.18) by $\|u_k\|_{2,2}$, then $0 \le L(p) = Fp(T)$. Therefore $p = 0$, since $F < 0$. However, it is in contradiction with $\|v_k\|_{2,2} = 1$. $\qquad \square$

3. Linear problems with bilateral elastic springs

In this section, we will define the family of linear problems with bilateral elastic "springs" and derive their uniform properties with respect to refinement of the partition. Such problems will be solved in each iteration of the algorithms, which will be presented below, in Section 4.

Let $\tau_h \in \mathcal{T}_\theta$ be a partition of $\overline{\Omega}$ and $A_h \subset \{1, \ldots, m(h)\}$ be a non-empty set of indices. Let us define the bilinear form

$$(3.1) \qquad b_h^{A_h}(v_1, v_2) := \sum_{i \in A_h} r_i v_1(z_i) v_2(z_i), \quad v_1, v_2 \in H^2(\Omega),$$

where the coefficients r_i and the spring points z_i have been described in the previous section. Let us define the functional

$$(3.2) \qquad J_h^{A_h}(v_h) := \frac{1}{2} a(v_h, v_h) + \frac{1}{2} b_h^{A_h}(v_h, v_h) - L(v_h).$$

The corresponding linear problem $(P_h^{A_h})$ with bilateral elastic springs has the form

$$(3.3) \qquad \text{find } w_h = w_h(A_h) \in V_h : \ J_h^{A_h}(w_h) \le J_h^{A_h}(v_h) \quad \forall v_h \in V_h,$$

or equivalently

(3.4)   find $w_h = w_h(A_h) \in V_h : \ a(w_h, v_h) + b_h^{A_h}(w_h, v_h) = L(v_h) \quad \forall v_h \in V_h$.

Lemma 3.1. *Let $\theta > 0$, $\tau_h \in \mathcal{T}_\theta$ and $card(A_h) \geq 2$. Then the problem $(P_h^{A_h})$ has a unique solution.*

If the condition (2.15) holds and $card(A_h) = 1$ then $(P_h^{A_h})$ has a solution if and only if $z_i = T$, where $i \in A_h$. In such a case, if $w_h(A_h)$ solves $(P_h^{A_h})$ then $w_h(A_h)+p$, where $p \in P_1$, $p(T) = 0$, also solves $(P_h^{A_h})$.

Proof. If $\tau_h \in \mathcal{T}_\theta$ and $card(A_h) \geq 2$ then there exists $c > 0$ such that the inequality

(3.5) $c\|v\|_{2,2}^2 \leq a(v, v) + b_h^{A_h}(v, v) \quad \forall v \in H^2(\Omega)$.

holds. The proof of the inequality (3.5) is quite similar to the proof of the Poincaré inequality, see [3] and also the proof of Lemma 3.2. Notice that if $b_h^{A_h}(1, 1) \to 0$ for $h \to 0$, then $c \to 0$.

The inequality (3.5) yields that the functional $J_h^{A_h}$ is coercive on V_h. Since J_h is also strictly convex and differentiable on V_h, the problem $(P_h^{A_h})$ has a unique solution by the well-known theorems of the variational calculus, see for example [2].

Suppose that $A_h = \{i\}$, $i \in \{1, 2, \ldots, m(h)\}$. Then the choices $v_h = 1$ and $v_h = x$ in the equation (3.4) and the definitions of T, F yield that $z_i = T$ and $w_h(z_i) = F/r_i$, if the problem $(P_h^{A_h})$ has a solution w_h. Let us define the auxiliary Neumann problem

(3.6)   find $\tilde{w}_h \in V_h : \ a(\tilde{w}_h, v_h) = L(v_h) - b_h^{A_h}(F/r_i, v_h) \quad \forall v_h \in V_h$.

Such a problem has a solution, since

$$L(p) - b_h^{A_h}(F/r_i, p) = 0 \quad \forall p \in P_1.$$

If $\tilde{w}_h$ is a solution of the problem (3.6) then the other solutions has the form $\tilde{w}_h + p$, $p \in P_1$. Therefore, we can assume that there exists a solution w_h of (3.6) such that $w_h(z_i) = F/r_i$. Now, it is easy to show that the functions $w_h + p$, where $p \in P_1$, $p(T) = 0$, also solves $(P_h^{A_h})$. $\qquad \square$

Corollary 3.1. *Let the condition (2.15) hold. Then the solution w_h^* of the problem (P_h) also solves the problem $(P_h^{A_h^*})$, where A_h^* is defined by (2.16).*

To show some uniform properties of the problems $(P_h^{A_h})$ with respect to $\tau_h \in \mathcal{T}_\theta$ and A_h, we introduce the notation

$$\mathcal{A} \ := \ \bigcup_h \{A_h \subset \{1, \ldots, m(h)\} \mid card(A_h) \geq 2\},$$

$$\mathcal{A}_\rho \ := \ \bigcup_h \{A_h \subset \{1, \ldots, m(h)\} \mid card(A_h) \geq \min\{m(h), \max\{2, \rho/h\}\}\}, \ \rho > 0.$$

Notice that the parameter ρ means the "relative" number of the spring points, since

$$\exists c_1, c_2 > 0 : \ c_1/h \leq m(h) \leq c_2/h \quad \forall \tau_h \in \mathcal{T}_\theta.$$

If $\{A_h\}_h \subset \mathcal{A}$ is such a sequence that $card(A_h)h \to 0$, or equivalently $b_h^{A_h}(1, 1) \to 0$ (see the estimate (2.10)), then $\{A_h\}_h \not\subset \mathcal{A}_\rho$ for any $\rho > 0$.

Lemma 3.2. *Let $\theta, \rho > 0$. Then there exist positive constants c_1, c_2 depending on $\theta, \rho > 0$ such that for any $\tau_h \in \mathcal{T}_\theta$ and any $A_h \in \mathcal{A}_\rho$ the estimate*

$$(3.7) \qquad c_1 \|v_h\|_{2,2}^2 \leq a(v_h, v_h) + b_h^{A_h}(v_h, v_h) \leq c_2 \|v_h\|_{2,2}^2 \quad \forall v_h \in V_h.$$

holds.

Proof. The second inequality in (3.7) follows from (2.11), since $b_h^{A_h}(v_h, v_h) \leq b_h(v_h, v_h)$. Suppose that the first inequality in (3.7) does not hold. Then there exist sequences $\{\tau_{h_k}\}_k$, $\{A_{h_k}\}_k$ and $\{v_{h_k}\}_k$ such that

$$a(u_k, u_k) + b_{h_k}^{A_{h_k}}(u_k, u_k) < \frac{1}{k}, \quad k \geq 1, \ u_k := \frac{v_{h_k}}{\|v_{h_k}\|_{2,2}}.$$

Hence, by the Rellich theorem and (2.11), we obtain

$$(3.8) \qquad \exists \{u_{k'}\}_{k'} \subset \{u_k\}_k : \ u_{k'} \to p \in P_1 \ \text{in} \ H^2(\Omega) \quad \text{and} \quad b_{h_{k'}}^{A_{h_{k'}}}(p, p) \to 0.$$

Since $\|u_{k'}\|_{2,2} = 1$, it holds that $p \neq 0$, i.e. there exists at most one point $x \in \mathbb{R}$ such that $p(x) = 0$. Therefore, for sufficiently small $\varepsilon > 0$, there exist $p_0 > 0$ and $0 < \tilde{\rho} \leq \rho$ such that

$$|p| \geq p_0 \ \text{in} \ \tilde{\Omega}_s \quad \text{and} \quad card(\tilde{A}_{h_{k'}}) \geq \tilde{\rho}/h_{k'},$$

where $\tilde{\Omega}_s := \overline{\Omega}_s \setminus (x - \varepsilon, x + \varepsilon)$ and $\tilde{A}_{h_{k'}} := \{i \in A_{h_{k'}} \mid z_i^{k'} \in \tilde{\Omega}_s\}$, $z_i^{k'}$ are the spring points of the partition $\tau_{h_{k'}}$. Then, by the estimate (2.10), there exists a positive constant c such that

$$b_{h_{k'}}^{A_{h_{k'}}}(p, p) \geq ch_{k'} p_0^2 \sum_{i \in \tilde{A}_{h_{k'}}} 1 \geq c\tilde{\rho} p_0^2 > 0.$$

However, it is in contradiction with (3.8). Therefore the estimate (3.7) holds. $\square$

Corollary 3.2. *Let $\theta, \rho > 0$. Then there exists a positive constant c depending on $\theta, \rho > 0$ such that for any $\tau_h \in \mathcal{T}_\theta$ and any $A_h \in \mathcal{A}_\rho$*

$$(3.9) \qquad \|w_h(A_h)\|_{2,2} \leq c\|L\|_*, \quad w_h(A_h) \ \text{solves} \ (P_h^{A_h}).$$

The proof immediately follows from the equation (3.4) and the estimate (3.7).

Let $\tau_h \in \mathcal{T}_\theta$ and $v \in H^2(\Omega)$. Then we can introduce the notation

$$(3.10) \qquad A_h(v) := \{i \in \{1, \ldots, m(h)\} \mid v(z_i) < 0\}.$$

Concretely, we will be interested in the relative cardinality of the set $A_h(w_h)$, where w_h solves the problem $(P_h^{A_h})$ for some $A_h \in \mathcal{A}$.

Lemma 3.3. *Let $v \in H^2(\Omega)$ and $v < 0$ in a non-empty open interval $(y_1, y_2) \subset \Omega_s$. Then there exists a positive constant ρ such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leq (y_2 - y_1)/2$, it holds $A_h(v) \in \mathcal{A}_\rho$.*

The proof clearly follows from the definition of the partitions $\tau_h \in \mathcal{T}_\theta$. Notice that the size of the parameter ρ depends on the length $y_2 - y_1$.

Lemma 3.4. *Let $F < 0$ and $\theta, \rho > 0$. Then there exist positive constants $\tilde{\rho}$ and h_0 such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_0$, and any $A_h \in \mathcal{A}_\rho$,*

$$(3.11) \qquad\qquad A_h \cap A_h(w_h) \in \mathcal{A}_{\tilde{\rho}},$$

where w_h solves the problem $(P_h^{A_h})$.

Proof. Suppose that (3.11) does not hold. Then there exist sequences $\{\tau_{h_k}\}_k$, $h_k \to 0$ and $\{A_k\}_k \subset \mathcal{A}_\rho$, $A_k \equiv A_{h_k}$, such that

$$(3.12) \qquad h_k card(A_k \cap A_k(w_k)) \to 0, \quad A_k(w_k) \equiv A_{h_k}(w_{h_k})$$

By Lemma 3.2, there exists $c_1 > 0$ such that $\|w_k\|_{2,2} \leq c_1$ for any $k \geq 0$. If we choose $v_h = 1$ in the equation (3.4) and denote the coefficients and spring points of the form b_{h_k} as r_i^k and z_i^k, then by the estimates (2.10) and (3.12), we obtain

$$
\begin{aligned}
F &= b_{h_k}^{A_k}(w_k, 1) \geq \sum_{i \in A_k \cap A_k(w_k)} r_i^k w_k(z_i^k) \geq \\
&\geq -c_2 h_k \|w_k\|_{C(\overline{\Omega})} card(A_k \cap A_k(w_k)) \to 0, \quad c_2 > 0.
\end{aligned}
$$

However, it is in contradiction with $F < 0$. Therefore (3.11) holds. $\qquad\square$

To show the other uniform properties of the problems $(P_h^{A_h})$, we will define the auxiliary problem $(P_{h,r}^{A_h})$ with the "rigid" beam:

$$(3.13) \qquad\qquad \text{find } p_h \in P_1 : \ J_h^{A_h}(p_h) \leq J_h^{A_h}(p) \quad \forall p \in P_1,$$

or equivalently

$$(3.14) \qquad\qquad \text{find } p_h \in P_1 : \ b_h^{A_h}(p_h, p) = L(p) \quad \forall p \in P_1.$$

Notice that the problem $(P_{h,r}^{A_h})$ means to solve the linear system of two equations with two unknowns.

Lemma 3.5. *Let $\tau_h \in \mathcal{T}_\theta$ and $A_h \in \mathcal{A}$. Then $p_h(x) = t_1 x + t_2$, where*

$$(3.15) \qquad t_1 = \frac{F}{det} \sum_{i \in A_h} r_i(T - z_i) \quad \text{and} \quad t_2 = \frac{-F}{det} \sum_{i \in A_h} r_i z_i(T - z_i),$$

with

$$(3.16) \qquad det = \sum_{i,j \in A_h, \ i<j} r_i r_j (z_i - z_j)^2 > 0, \ \ F = L(1), \ \ T = L(x)/F.$$

Proof. The relations (3.15) can be easily derived if we choose $p = 1$ and $p = x$ in the equation (3.14). $\qquad\square$

Lemma 3.6. *Let $F < 0$ and $\theta > 0$. Let $\{\tau_{h_k}\}_k \subset \mathcal{T}_\theta$ and $\{A_k\}_k \subset \mathcal{A}$, $A_k \equiv A_{h_k}$ be such sequences that*

$$(3.17) \qquad\qquad h_k \to 0 \quad \text{and} \quad h_k card(A_k) \to 0.$$

Then there exists a positive constant c, which is independent of the choice of the above sequences with the property (3.17), such that

$$(3.18) \qquad p_k(T) \to -\infty, \quad \|p_k\|_{2,2} \to +\infty \quad \text{and} \quad \|p_k\|_{2,2} \le c\frac{-p_k(T)}{h_k card(A_k)},$$

where $\{p_k\}_k$ is the corresponding sequence of the solutions of the problems $(P_{h_k,r}^{A_k})$.

Proof. Since the polynomial space P_1 has a finite dimension and since

$$p(x) = p(T) + (x - T)p',$$

there exist $c_1, c_2 > 0$ such that

$$(3.19) \qquad c_1\|p\|_{2,2} \le \max\{|p(T)|, |p'|\} \le c_2\|p\|_{2,2} \quad \forall p \in P_1.$$

Let us denote $n_k := card(A_k) \ge 2$. The coefficients and spring points of the form $b_{h_k}^{A_k}$ will be denoted r_i^k and z_i^k, $i = 1, \ldots, n_k$, $z_1^k < \ldots < z_{n_k}^k$. The determinant (3.16) will be denoted det_k for the problem $(P_{h_k,r}^{A_k})$. Let

$$(3.20) \quad d_i^k := z_{i+1}^k - z_i^k, \;\; i = 1, \ldots, n_k - 1, \;\; \text{i.e.} \;\; z_i^k = z_1^k + \sum_{j<i} d_j^k, \;\; i = 2, \ldots, n_k.$$

Since $\tau_{h_k} \in \mathcal{T}_\theta$, there exists $c_1 > 0$ such that

$$(3.21) \qquad d_i^k \ge c_1 h_k, \quad \forall k \ge 0, \;\; i = 1, \ldots, n_k.$$

We will also use the notation

$$(3.22) \qquad \sigma_0^k := \sum_{i=1}^{n_k} r_i^k, \quad \sigma_1^k := \sum_{i=1}^{n_k} r_i^k \sum_{j<i} d_j^k \quad \text{and} \quad \sigma_2^k := \sum_{i=1}^{n_k} r_i^k \left(\sum_{j<i} d_j^k\right)^2,$$

where $d_0^k := 0$. Then

$$
\begin{aligned}
\sum_{i=1}^{n_k} r_i^k (T - z_i^k)^2 &= \sigma_0^k (T - z_1^k)^2 - 2\sigma_1^k (T - z_1^k) + \sigma_2^k \\[2mm]
&\ge \frac{1}{\sigma_0^k}(\sigma_0^k \sigma_2^k - (\sigma_1^k)^2) \\[2mm]
&= \frac{1}{\sigma_0^k} \sum_{i_1,i_2=1}^{n_k} r_{i_1}^k r_{i_2}^k \left(\sum_{j_1<i_1} d_{j_1}^k\right) \left(\sum_{j_1<i_1} d_{j_1}^k - \sum_{j_2<i_2} d_{j_2}^k\right) \\[2mm]
&= \frac{1}{\sigma_0^k} \sum_{i_1,i_2;\; i_1<i_2} r_{i_1}^k r_{i_2}^k \left(\sum_{j_1<i_1} d_{j_1}^k - \sum_{j_2<i_2} d_{j_2}^k\right)^2 \\[2mm]
(3.23) \qquad &= \frac{1}{\sigma_0^k} \sum_{i_1,i_2;\; i_1<i_2} r_{i_1}^k r_{i_2}^k \left(\sum_{i_1 \le j < i_2} d_j^k\right)^2 = \frac{1}{\sigma_0^k} det_k.
\end{aligned}
$$

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Hence, by Lemma 3.5, the assumption (3.17) and the estimate (2.10), we obtain

$$p_k(T) = \frac{F}{det_k} \sum_{i=1}^{n_k} r_i^k (T - z_i^k)^2 \leq F/\sigma_0^k \leq cF/(h_k card(A_k)) \rightarrow -\infty, \ \ c > 0,$$

which implies $\|p_k\|_{2,2} \rightarrow +\infty$. The estimates (3.23), (3.21) and (2.10) also yield

$$
\begin{aligned}
\frac{\sum_{i=1}^{n_k} r_i^k (T - z_i^k)^2}{\sum_{i=1}^{n_k} r_i^k} &\geq c_2 \frac{h_k^2}{n_k^2} \sum_{i_1, i_2; \ i_1 < i_2} \left(\sum_{i_1 \leq j < i_2} 1 \right)^2 \\
&= \frac{1}{12} c_2 h_k^2 (n_k^2 - 1), \quad c_2 > 0.
\end{aligned}
$$

Hence, by the Cauchy-Schwarz inequality, Lemma 3.5, and the assumption (3.17), we obtain

$$
\begin{aligned}
\frac{|p_k'|}{-p_k(T)} &= \frac{|\sum_{i=1}^{n_k} r_i^k (T - z_i^k)|}{\sum_{i=1}^{n_k} r_i^k (T - z_i^k)^2} \leq \left(\frac{\sum_{i=1}^{n_k} r_i^k (T - z_i^k)^2}{\sum_{i=1}^{n_k} r_i^k} \right)^{-1/2} \\
&\leq c_3 (h_k^2 (n_k^2 - 1))^{-1/2} \leq \frac{c_4}{h_k n_k}, \quad c_3 > 0, \ c_4 = \frac{2}{\sqrt{3}} c_3,
\end{aligned}
$$

which implies (3.18) due to the estimate (3.19). $\qquad\square$

Lemma 3.7. *Let $\theta > 0$. Then there exists a positive constant $c > 0$ such that the estimate*

$$(3.24) \qquad c\|v_h\|_{2,2}^2 \leq a(v_h, v_h) + \left(\frac{b_h^{A_h}(v_h, 1)}{b_h^{A_h}(1,1)} \right)^2 + \left(\frac{b_h^{A_h}(v_h, x)}{b_h^{A_h}(1,1)} \right)^2 \quad \forall v_h \in V_h$$

holds for any $\tau_h \in \mathcal{T}_\theta$ and $A_h \in \mathcal{A}$.

The proof of Lemma 3.7 is based on the generalised Poincaré inequality, see [3]. The denominators $b_h^{A_h}(1,1)$ in (3.24) keep the limit case $b_h^{A_h}(1,1) \rightarrow 0$ for $h \rightarrow 0$.

Corollary 3.3. *Let $\theta > 0$. Then there exists a positive constant $c > 0$ such that the estimates*

$$(3.25) \qquad \|w_h - p_h\|_{2,2} \leq c\|L\|_* \quad and \quad a(w_h, w_h) \leq c\|L\|_*^2$$

hold for any $\tau_h \in \mathcal{T}_\theta$ and $A_h \in \mathcal{A}$, where w_h, p_h respectively solve the problems $(P_h^{A_h})$, $(P_{h,r}^{A_h})$.

Proof. By Lemma 3.7 and the equations (3.4) and (3.14), we obtain

$$
\begin{aligned}
c\|w_h - p_h\|_{2,2}^2 &\leq a(w_h, w_h) \leq a(w_h, w_h) + b_h^{A_h}(w_h - p_h, w_h - p_h) \\
&= L(w_h - p_h) \leq \|L\|_* \|w_h - p_h\|_{2,2},
\end{aligned}
$$

which yields the first estimate in (3.25) and consequently the second one. $\qquad\square$

Corollary 3.4. *Let the assumptions of Lemma 3.6 be fulfilled. Then*

$$(3.26) \qquad \frac{\|w_k\|_{2,2}}{\|p_k\|_{2,2}} \rightarrow 1 \quad and \quad \frac{J_{h_k}^{A_k}(w_k)}{w_k(T)} \rightarrow -F/2, \quad k \rightarrow +\infty,$$

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where $\{w_k\}_k$, $\{p_k\}_k$ are respectively the corresponding sequences of the solutions of the problems $(P_{h_k}^{A_k})$ and $(P_{h_k,r}^{A_k})$.

Proof. By the estimate (3.25) and the limits (3.18), we obtain

$$\frac{\|w_k\|_{2,2}}{\|p_k\|_{2,2}} \leq \frac{\|p_k\|_{2,2} + \|w_k - p_k\|_{2,2}}{\|p_k\|_{2,2}} \to 1$$

$$\frac{\|w_k\|_{2,2}}{\|p_k\|_{2,2}} \geq \frac{\|p_k\|_{2,2} - \|w_k - p_k\|_{2,2}}{\|p_k\|_{2,2}} \to 1,$$

i.e. the first limit in (3.26) holds. Notice that due to (3.14),

$$\frac{J_{h_k}^{A_k}(p_k)}{p_k(T)} = \frac{-L(p_k)}{2p_k(T)} = -F/2,$$

which implies $J_{h_k}^{A_k}(p_k) \to -\infty$ by (3.18). In addition, due to (3.4) and (3.14),

$$\frac{J_{h_k}^{A_k}(w_k)}{J_{h_k}^{A_k}(p_k)} = \frac{J_{h_k}^{A_k}(p_k) - L(w_k - p_k)/2}{J_{h_k}^{A_k}(p_k)} \to 1$$

and by Lemma 3.6 and Corollary 3.3,

(3.27) $$\lim_{k\to\infty} \frac{w_k(T)}{p_k(T)} = 1 + \lim_{k\to\infty} \frac{w_k(T) - p_k(T)}{p_k(T)} = 1.$$

Therefore

$$\lim_{k\to+\infty} \frac{J_{h_k}^{A_k}(w_k)}{w_k(T)} = \lim_{k\to+\infty} \frac{J_{h_k}^{A_k}(p_k)}{p_k(T)} = -\frac{F}{2}.$$

$\square$

Corollaries 3.3 and 3.4 shows that the problems $(P_{h_k}^{A_k})$ and $(P_{h_k,r}^{A_k})$ have many common properties for the limit case $h_k card(A_k) \to 0$. This fact will be used to prove the following theorems and lemmas.

Theorem 3.1. *Let $F < 0$, $x_l < T < x_r$, and $\theta > 0$. Then there exist positive constants ρ and h_0 such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_0$, and any $A_h \in \mathcal{A}$,*

$$A_h(w_h) \in \mathcal{A}_\rho,$$

where w_h solves the problem $(P_h^{A_h})$.

Proof. Suppose that Theorem 3.1 does not hold. Then there exist sequences $\{\tau_{h_k}\}_k$, $h_k \to 0$, and $\{A_k\}_k \subset \mathcal{A}$, $A_k \equiv A_{h_k}$, such that

(3.28) $$h_k card(A_k(w_k)) \to 0, \quad A_k(w_k) \equiv A_{h_k}(w_{h_k}).$$

Let us denote $p_k := p_{h_k}$ as the solutions of the problems $(P_{h_k,r}^{A_k})$, $k \geq 0$.

Suppose that there exists $\rho_1 > 0$ and a subsequence $\{A_k\}_k$ (denoted by the same way) such that

$$A_k \in \mathcal{A}_{\rho_1}, \quad \forall k \geq 0.$$

Then, by Lemma 3.4, there exists $\rho_2 > 0$ such that $A_k(w_k) \in \mathcal{A}_{\rho_2}$ for sufficiently large k, which contradicts with (3.28).

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Suppose that there exists a subsequence $\{A_k\}_k$ such that

$$h_k \, card(A_k) \to 0.$$

By Lemma 3.6, $p_k(T) \to -\infty$. Therefore, $p_k \to -\infty$ in $[x_l, T]$ or in $[T, x_r]$, since $T \in \Omega_s = (x_l, x_r)$. Hence and by Corollary 3.3, there exists sufficiently small $\varepsilon > 0$ such that $w_k < 0$ in $[x_l, T-\varepsilon]$ or in $[T+\varepsilon, x_r]$ for sufficiently large k, which contradicts with (3.28) due to Lemma 3.3. $\qquad \square$

Lemma 3.8. *Let $F < 0$ and $x_l < T < x_r$. Then there exist positive constants ρ and h_0 such that $\{A_h(w_h^*)\}_{h \leq h_0} \subset \mathcal{A}_\rho$, where w_h^* solves the problem (P_h).*

In addition, if $\tau_h \in \mathcal{T}_\theta$, $A_h \in \mathcal{A}$ and $A_h(w_h) = A_h$, where w_h solves the problem $(P_h^{A_h})$, then w_h also solves the problem (P_h).

Proof. Let w_h^*, w^* respectively solve the problems (P_h) and (P). Since $w_h^* \to w^*$ in $H^2(\Omega)$ by (2.17) and since w^* is negative somewhere in Ω_s by Lemma 3.5 in [7], there exist $\rho, h_0 > 0$ such that $A_h(w_h^*) \in \mathcal{A}_\rho$ for $h \leq h_0$ by Lemma (3.3).

If $A_h(w_h) = A_h$ and w_h solves the problem $(P_h^{A_h})$ then

$$L(v) = a(w_h, v) + b_h^{A_h}(w_h, v) = a(w_h, v) + b_h(w_h^-, v) \quad \forall v \in H^2(\Omega).$$

Thus the function w_h also solves the problem (P_h). $\qquad \square$

By the following lemma, we estimate the difference between the solution w_h^* of the problem (P_h) and their approximations generated by the algorithms, which will be presented in Section 4, see the proof of Theorem 4.2.

Lemma 3.9. *Let $F < 0$, $x_l < T < x_r$, and $c, \theta > 0$. Then there exist positive constants $\tilde{c}$ and $h_0 > 0$ such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_0$, and any $u_h \in V_h$, $\|u_h\|_{2,2} \leq c$,*

$$(3.29) \qquad \tilde{c}\|w_h^* - u_h\|_{2,2}^2 \leq a(w_h^* - u_h, w_h^* - u_h) + b_h((w_h^*)^- - u_h^-, w_h^* - u_h),$$

where w_h^* solves the problem (P_h).

Proof. Since the proof is similar to a proof of Theorem 4.5 in [7], some steps will be done more briefly. By Lemma 3.8 and Corollary 3.2, there exist $c_1, c_2 > 0$ such that for any $\tau_h \in \mathcal{T}_\theta$ with sufficiently small h,

$$(3.30) \qquad\qquad \|w_h^*\|_{2,2} \leq c_1 \quad \text{and} \quad \|w_h^* - u_h\|_{2,2} \leq c_2.$$

Suppose that the lemma does not hold. Then there exist sequences $\{\tau_{h_k}\}_k$, $h_k \to 0$, $\{w_{h_k}^*\}_k$ and $\{u_{h_k}\}_k$ such that

$$(3.31) \qquad a(w_k - u_k, w_k - u_k) + b_h(w_k^- - u_k^-, w_k - u_k) \to 0,$$

where

$$(3.32) \qquad w_k := \frac{w_{h_k}^*}{\|w_{h_k}^* - u_{h_k}\|_{2,2}}, \quad u_k := \frac{u_{h_k}}{\|w_{h_k}^* - u_{h_k}\|_{2,2}}, \quad \|w_k - u_k\|_{2,2} = 1.$$

All subsequences of these sequences will be denoted by the same way. By the Rellich theorem, (3.31) and (3.32), there exist subsequences $\{w_k\}_k$ and $\{u_k\}_k$ and a polynomial $p \in P_1$, $p \neq 0$, such that $w_k - u_k \to p$ in $H^2(\Omega)$. By Lemma 3.8,

$$(3.33) \qquad\qquad \exists \rho_1 > 0: \quad A_{h_k}(w_{h_k}^*) \in \mathcal{A}_{\rho_1}.$$

Suppose that $\|w_{h_k}^* - u_{h_k}\|_{2,2} \to 0$. Then

$$(3.34) \qquad\qquad \exists \rho_2 > 0 : \quad A_{h_k}(w_{h_k}^*) \cap A_{h_k}(u_{h_k}) \in \mathcal{A}_{\rho_2}$$

for sufficiently large k by (3.33). Since

$$b_{h_k}(w_k^- - u_k^-, w_k - u_k) \geq b_{h_k}^{A_{h_k}(w_{h_k}^*) \cap A_{h_k}(u_{h_k})}(w_k - u_k, w_k - u_k),$$

(3.31), (3.34), (2.11) and (2.10) yield that $p = 0$, which is in contradiction with $p \neq 0$.

Therefore we can assume that the sequences $\{w_k\}_k$ and $\{u_k\}_k$ are bounded due to (3.30). It means that there exist their subsequences which converge to functions w and $u = w - p$ in $H^1(\Omega)$ by the Rellich theorem. Then, by (3.31) and (2.12),

$$(3.35) \qquad\qquad\qquad w^- - (w - p)^- = 0 \quad \text{in } \Omega_s.$$

Since $w_{h_k}^* \to w^*$ in $H^2(\Omega)$, w^* solves the problem (P), by (2.17), and since $w^* < 0$ somewhere in Ω_s, also $w < 0$ somewhere in Ω_s. Therefore, (3.35) yields that $p = 0$ which contradicts with $p \neq 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Descent Direction Methods with and without Projection

In this section, two methods are presented as a numerical realization of the problem (P_h). The methods are based on the minimisation of the total energy functional J_h, where the descent directions of the functional are searched by solving the linear problems of type $(P_h^{A_h})$ presented in the previous section. The difference between the methods is in the "projection step". The step is useful mainly for unstable loads as we see in Section 5.

Since the uniform convergence properties of the methods with respect to refinement of the partition are derived, the corresponding algorithms are firstly described in the functional form. Their algebraical form will be presented later, in Section 5. We will assume that the solvability conditions (2.5) hold.

4.1. **Descent Direction Method without Projection.** Let $\tau_h \in \mathcal{T}_\theta$ be a partition and z_i, $i \in \{1, 2, \ldots, m(h)\}$, be the corresponding set of springs.

Algorithm 1

> *Initialisation*
> > $w_{h,0} = 0,$
> > $A_{h,0} = \{1, 2, \ldots, m(h)\}.$
> *Iteration* $k = 0, 1, \ldots$
> > $s_{h,k} \in V_h, \; w_{h,k} + s_{h,k}$ solves $(P_h^{A_{h,k}}),$
> > $\alpha_{h,k} = arg \min_{0 \leq \alpha \leq 1} J_h(w_{h,k} + \alpha s_{h,k}),$
> > $w_{h,k+1} = w_{h,k} + \alpha_{h,k} s_{h,k},$
> > $A_{h,k+1} = A_h(w_{h,k+1}).$

In the remaining part of this subsection, we show that Algorithm 1 is well-defined, i.e. the problems $(P_h^{A_{h,k}})$ are uniquely solvable and that $w_{h,k} \to w^*$ in $H^2(\Omega)$ uniformly with respect to sufficiently small h.

Let $u_h \in V_h$, $A_h(u_h) \in \mathcal{A}$, $w_h \in V_h$ solves the problem $(P_h^{A_h(u_h)})$ and $s_h := w_h - u_h$. It will be usefull to introduce the notation $A_h^\alpha := A_h(u_h + \alpha s_h)$. Then $A_h^0 = A_h(u_h)$ and $A_h^1 = A_h(w_h)$. Notice that the equality

$$(u_h + \alpha s_h)(z_i) = \alpha w_h(z_i) + (1 - \alpha)u_h(z_i)$$

yields the inclusion

(4.1) $$A_h^0 \cap A_h^1 \subset A_h^0 \cap A_h^\alpha \quad \forall \alpha \in [0, 1]$$

and the implication

(4.2) $$A_h^1 \subset A_h^0 \implies A_h^\alpha \subset A_h^0 \quad \forall \alpha \in [0, 1].$$

Lemma 4.1. Let $u_h \in V_h$, $A_h^0 \equiv A_h(u_h) \in \mathcal{A}$, $w_h \in V_h$ solves the problem $(P_h^{A_h^0})$ and $s_h := w_h - u_h$. Let

$$\alpha_h := arg \min_{0 \leq \alpha \leq 1} J_h(u_h + \alpha s_h).$$

Then

(4.3) $$J_h'(u_h; s_h) = 2J_h^{A_h^0}(w_h) - 2J_h^{A_h^0}(u_h)$$

(4.4) $$= -a(s_h, s_h) - b_h^{A_h^0}(s_h, s_h) \leq 0,$$

where $J_h'(u_h; s_h) = 0$ if and only if u_h solves the problem (P_h), and

(4.5) $$\alpha_h \geq \frac{a(s_h, s_h) + b_h^{A_h^0}(s_h, s_h)}{a(s_h, s_h) + b_h^{A_h^0 \cup A_h^{\alpha_h}}(s_h, s_h)} > 0, \quad s_h \neq 0.$$

Proof. By Lemma 3.1, the problem $(P_h^{A_h^0})$ has a unique solution w_h. Then the choice $v_h = s_h$ in the variational equation (3.4) yields

$$\begin{aligned} J_h'(u_h; s_h) &= a(u_h, s_h) + b_h(u_h^-, s_h) - L(s_h) \\ &= a(u_h, s_h) + b_h^{A_h^0}(u_h, s_h) - L(s_h) \\ &= -a(s_h, s_h) - b_h^{A_h^0}(s_h, s_h) \leq 0. \end{aligned}$$

The choices $v_h = u_h$ and $v_h = w_h$ in the variational equation (3.4) yield the equality (4.3). By the inequality (3.5), $J_h'(u_h, s_h) = 0$ if and only if $s_h = 0$, i.e. if $u_h = w_h$. It means that in such a case, u_h solves the problem (P_h) by Lemma 3.8.

Let us denote $\varphi(\alpha) := J_h(u_h + \alpha s_h)$ and let $s_h \neq 0$. Since J_h is a convex and differentiable functional on V_h, there exists α_h, which minimises φ in $[0, 1]$. The inequality (4.4) yields $\alpha_h > 0$ and $\varphi'(\alpha_h) \leq 0$. If $\alpha_h = 1$, then the inequality (4.5) holds. Otherwise,

$$0 = \varphi'(\alpha_h) = a(u_h + \alpha_h s_h, s_h) + b_h((u_h + \alpha_h s_h)^-, s_h) - L(s_h)$$

(4.6) $$= J_h'(u_h; s_h) + \alpha_h \left[a(s_h, s_h) + b_h \left(\frac{(u_h + \alpha_h s_h)^- - u_h^-}{\alpha_h}, s_h \right) \right].$$

Notice that

$$b_h \left(\frac{(u_h + \alpha_h s_h)^- - u_h^-}{\alpha_h}, s_h \right) = b_h^{A_h^0 \cap A_h^{\alpha_h}}(s_h, s_h) - b_h^{A_h^0 \setminus A_h^{\alpha_h}}(u_h, s_h)/\alpha_h +$$

$$+ b_h^{A_h^{\alpha_h} \setminus A_h^0}(u_h + \alpha_h s_h, s_h)/\alpha_h =$$

$$= b_h^{A_h^0 \cup A_h^{\alpha_h}}(s_h, s_h) + b_h^{A_h^{\alpha_h} \setminus A_h^0}(u_h, s_h)/\alpha_h -$$

$$- b_h^{A_h^0 \setminus A_h^{\alpha_h}}(u_h + \alpha_h s_h, s_h)/\alpha_h.$$

If $i \in A_h^{\alpha_h} \setminus A_h^0$ then $u_h(z_i) \geq 0$ and $s_h(z_i) < 0$. If $i \in A_h^0 \setminus A_h^{\alpha_h}$ then $(u_h + \alpha_h s_h)(z_i) \geq 0$ and $s_h(z_i) > 0$. Therefore

$$b_h^{A_h^{\alpha_h} \setminus A_h^0}(u_h, s_h) \leq 0 \quad \text{and} \quad b_h^{A_h^0 \setminus A_h^{\alpha_h}}(u_h + \alpha_h s_h, s_h) \geq 0.$$

Hence,

$$b_h \left(\frac{(u_h + \alpha_h s_h)^- - u_h^-}{\alpha_h}, s_h \right) \leq b_h^{A_h^0 \cup A_h^{\alpha_h}}(s_h, s_h)$$

and (4.6) yields the estimate (4.5). $\qquad \square$

Notice that if $A_h^1 \subset A_h^0$, then the implication (4.2) and the estimate (4.5) yield $\alpha_h = 1$.

By the following lemma, we can estimate the relative cardinality of the sets $A_{h,k}$, which are generated by Algorithm 1, see the proof of Theorem 4.1.

Lemma 4.2. *Let c, θ be positive constants and the solvability condition (2.5) hold. Then there exist positive constants h_0, ρ such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_0$, and any $u_h \in V_h$, $\|u_h\|_{2,2} \leq c$, $A_h^0 \equiv A_h(u_h) \in \mathcal{A}_\rho$, it holds*

$$(4.7) \qquad A_h^{\alpha_h} \equiv A_h(u_h + \alpha_h s_h) \in \mathcal{A}_\rho,$$

where $\alpha_h = \arg\min_{0 \leq \alpha \leq 1} J_h(u_h + \alpha s_h)$, $s_h = w_h - u_h$ and $w_h \in V_h$ solves the problem $(P_h^{A_h^0})$.

Proof. Suppose that the lemma does not hold. Then there exist sequences $\{\tau_{h_k}\}_k$, $h_k \to 0$, $\{\rho_k\}_k$, $\rho_k \to 0$, $\{u_k\}_k$, $u_k \in V_{h_k}$, $\|u_k\|_{2,2} \leq c$, $A_k^0 \equiv A_{h_k}(u_k) \in \mathcal{A}_{\rho_k}$, such that

$$(4.8) \qquad A_k^{\alpha_k} \equiv A_{h_k}(u_k + \alpha_k s_k) \notin \mathcal{A}_{\rho_k} \quad \forall k \geq 0,$$

where $\{\alpha_k\}_k$, $\{s_k\}_k$ and $\{w_k\}_k$ are the corresponding sequences for the sequences $\{\tau_{h_k}\}_k$ and $\{u_k\}_k$. For the sake of simplicity, all subsequences of these sequences will be denoted in the same way. (4.8) implies that

$$(4.9) \qquad card(A_k^{\alpha_k}) < card(A_k^0), \quad \forall k \geq 0.$$

Suppose that there exists $\rho_1 > 0$ and a subsequence $\{A_k^0\}_k$ such that $A_k^0 \in \mathcal{A}_{\rho_1}$. Then, by Lemma 3.4, there exists $\rho_2 > 0$ such that $A_k^0 \cap A_k^1 \in \mathcal{A}_{\rho_2}$ for sufficiently large k. Hence and by (4.1), we obtain $A_k^{\alpha_k} \in \mathcal{A}_{\rho_2}$, which contradicts with (4.8). Therefore, we can assume that

$$(4.10) \qquad h_k card(A_k^0) \to 0, \quad k \to +\infty.$$

Corollary 3.4, (4.10) and the boundedness of u_k yield

$$(4.11) \qquad \|w_k\|_{2,2} \to \infty, \quad \|s_k\|_{2,2} \to \infty \quad \text{and} \quad \frac{\|s_k\|_{2,2}}{\|p_k\|_{2,2}} \to 1,$$

where $p_k \in P_1$ solves the problem $(P_{h_k,r}^{A_k^0})$ defined in Section 3. Consequently by Corollary 3.3, we obtain

$$(4.12) \qquad a(s_k, s_k)/\|s_k\|_{2,2}^2 \to 0.$$

Since $\|u_k\|_{2,2} \le c$, there exists $c_0 > 0$ such that $J_{h_k}(u_k) \le c_0$ for any $k \ge 0$ and since $J_{h_k}(u_k) \ge J_{h_k}(u_k + \alpha_k s_k)$,

$$(4.13) \qquad \exists c_1 > 0 : \quad \|u_k + \alpha_k s_k\|_{2,2} \le c_1 \quad \forall k \ge 0,$$

by Lemma 2.2. The boundedness of $\{u_k\}_k$, (4.13) and (4.11) yield

$$(4.14) \qquad \exists c_2 > 0 : \quad \|\alpha_k s_k\|_{2,2} \le c_2 \quad \forall k \ge 0 \quad \text{and} \quad \alpha_k \to 0.$$

Suppose that

$$(4.15) \qquad \exists c_3 > 0 : \quad \|\alpha_k s_k\|_{2,2} \ge c_3 \quad \forall k \ge 0.$$

Then by the Rellich theorem, (4.12), (4.14) and (4.15) there exist a subsequence $\{\alpha_k s_k\}_k$ and $p \in P_1$, $p \ne 0$, such that $\alpha_k s_k \to p$ and consequently $\alpha_k p_k \to p$ in $H^2(\Omega)$. Since the sequences $\{u_k\}_k$ and $\{u_k + \alpha_k s_k\}_k$ are bounded, there exist their subsequences with weak limits u and $u + p$ in $H^2(\Omega)$. We can also assume that $u_k \to u$ and $u_k + \alpha_k s_k \to u + p$ in $H^1(\Omega)$ by the Rellich theorem. The functions u and $u + p$ are non-negative in Ω_s with respect to the assumptions (4.8), (4.10) and Lemma 3.3.

Due to the assumption $F < 0$, it holds that $A_k^0 \cap A_k^1 \ne \emptyset$, see the proof of Lemma 3.4. Hence and by (4.1), we obtain $A_k^0 \cap A_k^{\alpha_k} \ne \emptyset$, i.e. there exists a sequence $\{i_k\}_k$ such that $i_k \in A_k^0 \cap A_k^{\alpha_k}$. Therefore there exist a subsequence $\{z_{i_k}^k\}_k$ and $z \in \overline{\Omega}_s$ such that $z_{i_k}^k \to z$. Non-negativity of u and $u + p$ yield

$$(4.16) \qquad u(z) = 0 \quad \text{and} \quad p(z) = 0$$

and consequently

$$(4.17) \qquad u'(z) \begin{cases} = 0 & z \ne x_l, x_r, \\ \ge 0 & z = x_l, \\ \le 0 & z = x_r, \end{cases} \quad \text{and} \quad u'(z) + p'(z) \begin{cases} = 0 & z \ne x_l, x_r, \\ \ge 0 & z = x_l, \\ \le 0 & z = x_r. \end{cases}$$

Since $p \ne 0$, there exists just one such a point z, with respect to (4.16). Moreover, by (4.17), $z = x_l$ or $z = x_r$. In the both cases, $p < 0$ in Ω_s, since $p_k(T) \to -\infty$ by Lemma 3.6.

Let $\varphi_k(\alpha) := J_{h_k}(u_k + \alpha s_k)$. Since $\alpha_k \to 0$, the definition of α_k yields,

$$0 = \varphi_k'(\alpha_k) = a(u_k + \alpha_k s_k, s_k) + b_{h_k}((u_k + \alpha_k s_k)^-, s_k) - L(s_k)$$

for sufficiently large k. If we multiply this equality by α_k then for $k \to \infty$, we obtain contradiction $0 = -L(p) = -Fp(T) < 0$ by (2.11) and non-negativity of $u + p$.

Suppose that

$$(4.18) \qquad \|\alpha_k s_k\|_{2,2} \to 0 \quad \text{for} \quad k \to \infty.$$

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Then by the estimates (4.5) and (4.3), we obtain

$$
\begin{aligned}
0 \;&\leq\; (1-\alpha_k)J'_{h_k}(u_k; s_k) + \alpha_k b_{h_k}^{A_k^{\alpha_k}\setminus A_k^0}(s_k, s_k) \\
&=\; 2(1-\alpha_k)\left(J_{h_k}^{A_k^0}(w_k) - J_{h_k}^{A_k^0}(u_k)\right) + \alpha_k b_{h_k}^{A_k^{\alpha_k}\setminus A_k^0}(s_k, s_k).
\end{aligned}
$$

If we divide this inequality by $-w_k(T)$, we obtain by Lemma 3.6, Corollary 3.4, (2.10), (3.27), (4.9), (4.10), (4.11) and (4.18),

$$
\begin{aligned}
0 \;&\leq\; F + \lim_{k\to\infty}\left\{ \|\alpha_k s_k\|_{2,2}\frac{\|p_k\|_{2,2}}{-p_k(T)}\frac{p_k(T)}{w_k(T)}\frac{\|s_k\|_{2,2}}{\|p_k\|_{2,2}} b_{h_k}^{A_k^{\alpha_k}\setminus A_k^0}\left(\frac{s_k}{\|s_k\|_{2,2}}, \frac{s_k}{\|s_k\|_{2,2}}\right)\right\} \\
&\leq\; F + c_4 \lim_{k\to\infty}\|\alpha_k s_k\|_{2,2}\frac{1}{h_k\,card(A_k^0)}\sum_{A_k^{\alpha_k}\setminus A_k^0} r_i^k \\
&\leq\; F + c_5 \lim_{k\to\infty}\|\alpha_k s_k\|_{2,2} = F < 0,
\end{aligned}
$$

which is contradiction. Therefore, (4.7) holds. $\qquad\square$

Theorem 4.1. *Let the condition (2.5) hold and $\theta > 0$. Then there exist positive constants ρ, c and h_1 such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_1$,*

$$(4.19) \qquad A_{h,k} \in \mathcal{A}_\rho \quad\text{and}\quad \|w_{h,k}\|_{2,2} \leq c \quad \forall k \geq 0,$$

where the sets $A_{h,k}$ and the functions $w_{h,k}$ are generated by Algorithm 1.

Proof. The theorem will be proved by a mathematical induction. By Lemma 2.2, there exist $c > 0$ and $h_0 > 0$ such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_0$, the implication

$$(4.20) \qquad J_h(u_h) \leq 0 \quad\Longrightarrow\quad \|u_h\|_{2,2} \leq c \qquad \forall u_h \in V_h$$

holds. Since $\|w_{h,0}\|_{2,2} = 0 \leq c$ and $A_{h,0} = \{1, \ldots, m(h)\}$, there exist $\rho > 0$ and $0 < h_1 \leq h_0$ (which depend only on θ and c) such that $A_{h,1} \in \mathcal{A}_\rho$ for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_1$, by Lemma 4.2. Suppose that

$$A_{h,i} \in \mathcal{A}_\rho \quad \forall \tau_h \in \mathcal{T}_\theta,\ h \leq h_1,\ i = 0, 1, \ldots, k.$$

Since

$$J_h(w_{h,k}) \leq \ldots \leq J_h(w_{h,1}) \leq J_h(w_{h,0}) \leq 0, \quad h \leq h_1,$$

also $\|w_{h,k}\|_{2,2} \leq c$ by the implication (4.20), which yields $A_{h,k+1} \in \mathcal{A}_\rho$ for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_1$, by Lemma 4.2. $\qquad\square$

Lemma 4.3. *Let the condition (2.5) hold and $\theta > 0$. Then there exist positive constants c and h_0 such that*

$$(4.21) \qquad \alpha_{h,k} \geq c \qquad \forall \tau_h \in \mathcal{T}_\theta,\ h \leq h_0,\ \forall k \geq 0,\ s_{h,k} \neq 0,$$

where the numbers $\alpha_{h,k}$ and the functions $s_{h,k}$ are generated by Algorithm 1.

Proof. Let $s_{h,k}$, $w_{h,k}$, $\alpha_{h,k}$, $A_{h,k}$, $k \geq 0$, be generated by Algorithm 1. By Theorem 4.1, there exist $\rho, h_0 > 0$ such that $A_{h,k} \in \mathcal{A}_\rho$, $h \leq h_0$, for any $k \geq 0$. Hence and by Lemma 3.2, there exist $c_1, c_2 > 0$ such that

$$
\begin{aligned}
a(v,v) + b_h^{A_{h,k}}(v,v) &\geq c_1\|v\|_{2,2}^2 \\
a(v,v) + b_h^{A_{h,k}\cup A_{h,k+1}}(v,v) &\leq c_2\|v\|_{2,2}^2
\end{aligned}
\qquad \forall v \in H^2(\Omega), \forall k \geq 0.
$$

Then the estimate (4.5) in Lemma 4.1 yields

$$\alpha_{h,k} \geq \frac{a(s_{h,k}, s_{h,k}) + b_h^{A_{h,k}}(s_{h,k}, s_{h,k})}{a(s_{h,k}, s_{h,k}) + b_h^{A_{h,k} \cup A_{h,k+1}}(s_{h,k}, s_{h,k})} \geq \frac{c_1}{c_2} > 0 \quad \forall k \geq 0, \ s_{h,k} \neq 0.$$

$\square$

Lemma 4.4. *Let the condition (2.5) hold and $\theta > 0$. Then there exist positive constants c and h_0 such that*

$$(4.22) \qquad J_h(w_{h,k+1}) \leq J_h(w_{h,k}) - c\|s_{h,k}\|_{2,2}^2 \qquad \forall \tau_h \in \mathcal{T}_\theta, \ h \leq h_0, \ \forall k \geq 0,$$

where the functions $s_{h,k}$, $w_{h,k}$ are generated by Algorithm 1.

Proof. Let $s_k \equiv s_{h,k}$, $w_k \equiv w_{h,k}$, $\alpha_k \equiv \alpha_{h,k}$, $A_k \equiv A_{h,k}$, $k \geq 0$, be generated by Algorithm 1. Let $\varphi_k(\alpha) := J_h(w_k + \alpha s_k)$. By the definition of α_k,

$$0 \geq \varphi_k'(\alpha_k) = a(w_{k+1}, s_k) + b_h(w_{k+1}^-, s_k) - L(s_k).$$

Hence and by the definition of A_k, A_{k+1} and w_{k+1},

$$\begin{aligned} J_h(w_{k+1}) &= J_h(w_k) + \alpha_k \varphi_k'(\alpha_k) - \frac{1}{2}\alpha_k^2 a(s_k, s_k) + \\ &\quad + \frac{1}{2}b_h(w_{k+1}^-, w_{k+1}) - \frac{1}{2}b_h(w_k^-, w_k) - \alpha_k b_h(w_{k+1}^-, s_k). \end{aligned}$$

Notice that

$$\begin{aligned} &\tfrac{1}{2}b_h(w_{k+1}^-, w_{k+1}) - \tfrac{1}{2}b_h(w_k^-, w_k) - \alpha_k b_h(w_{k+1}^-, s_k) = \\ &= \tfrac{1}{2}b_h^{A_{k+1}}(w_k + \alpha_k s_k, w_k + \alpha_k s_k) - \tfrac{1}{2}b_h^{A_k}(w_k, w_k) - \alpha_k b_h^{A_{k+1}}(w_k + \alpha_k s_k, s_k) \\ &= -\tfrac{1}{2}\alpha_k^2 b_h^{A_{k+1}}(s_k, s_k) + \tfrac{1}{2}b_h^{A_{k+1}}(w_k, w_k) - \tfrac{1}{2}b_h^{A_k}(w_k, w_k) = \\ &= -\tfrac{1}{2}\alpha_k^2 b_h^{A_{k+1} \cap A_k}(s_k, s_k) - \tfrac{1}{2}\alpha_k^2 b_h^{A_{k+1} \setminus A_k}(s_k, s_k) + \\ &\quad + \tfrac{1}{2}b_h^{A_{k+1} \setminus A_k}(w_k, w_k) - \tfrac{1}{2}b_h^{A_k \setminus A_{k+1}}(w_k, w_k) \leq \\ &\leq -\tfrac{1}{2}\alpha_k^2 b_h^{A_{k+1} \cap A_k}(s_k, s_k), \end{aligned}$$

since $-\alpha_k s_k(z_i) > w_k(z_i)$ and consequently $\alpha_k^2 s_k^2(z_i) > w_k^2(z_i)$ if $i \in A_{k+1} \setminus A_k$. Therefore

$$(4.23) \qquad J_h(w_{k+1}) \leq J_h(w_k) - \frac{1}{2}\alpha_k^2 \left(a(s_k, s_k) + b_h^{A_k \cap A_{k+1}}(s_k, s_k) \right).$$

By Theorem 4.1, there exist $\rho_1 > 0$ and $h_1 > 0$ such that $A_k \in \mathcal{A}_{\rho_1}$ for any $k \geq 0$ and any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_1$. Therefore, by Lemma 3.4, there exist $0 < \rho \leq \rho_1$ and $0 < h_0 \leq h_1$ such that $A_k \cap A_k(w_k + s_k) \in \mathcal{A}_\rho$ and consequently (see (4.1)) $A_k \cap A_{k+1} \in \mathcal{A}_\rho$ for any $k \geq 0$ and any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_0$. Then, by Lemma 3.2, there exists $c > 0$ such that

$$c\|s_k\|_{2,2}^2 \leq a(s_k, s_k) + b_h^{A_k \cap A_{k+1}}(s_k, s_k) \quad \forall \tau_h \in \mathcal{T}_\theta, \ h \leq h_0, \ \forall k \geq 0.$$

Hence, by (4.23) and Lemma 4.3, we obtain (4.22). $\square$

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Theorem 4.2. *Let the condition (2.5) hold and $\theta > 0$. Then there exists $h_0 > 0$ such that the sequence $\{w_{h,k}\}_k$ generated by Algorithm 1 converges uniformly (with respect to h) to the function w_h^* solving the problem (P_h) in $H^2(\Omega)$ for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_0$.*

In addition, for any fix $\tau_h \in \mathcal{T}_\theta$, $h \leq h_0$, there exists an iteration $k_0 = k_0(h) \geq 0$ such that $w_{h,k_0} + s_{h,k_0} = w_h^*$.

Proof. Let $s_k \equiv s_{h,k}$, $w_k \equiv w_{h,k}$, $\alpha_k \equiv \alpha_{h,k}$, $A_k \equiv A_{h,k}$, $k \geq 0$, be generated by Algorithm 1. By Lemma 4.4, there exist $c_1 > 0$ and $h_0 > 0$ such that

$$(4.24) \qquad J_h(w_h^*) \leq J_h(w_k) \leq -c_1 \sum_{i=0}^{k-1} \|s_i\|_{2,2}^2 \qquad \forall \tau_h \in \mathcal{T}_\theta,\ h \leq h_0,\ \forall k \geq 0.$$

By (2.17),

$$J_h(w_h^*) = -L(w_h^*)/2 \to -L(w^*)/2 = J(w^*), \quad h \to 0,$$

where w^* solves the problem (P). Hence and by (4.24), there exists $c_2 > 0$ such that

$$(4.25) \qquad \sum_{i=0}^{+\infty} \|s_i\|_{2,2}^2 \leq c_2 \qquad \forall \tau_h \in \mathcal{T}_\theta,\ h \leq h_0,$$

and consequently $\|s_k\|_{2,2} \to 0$ uniformly with respect to h for $k \to +\infty$. Since $w_k + s_k$ solves the problem $(P_h^{A_k})$, the variational equations (2.14) and (3.4) yield

$$a(w_h^* - w_k, w_h^* - w_k) + b_h((w_h^*)^- - w_k^-, w_h^* - w_k) = a(s_k, w_h^* - w_k) + b_h^{A_k}(s_k, w_h^* - w_k).$$

Hence, by Theorem 4.1, Lemma 3.9 and (2.11), there exists $c_3 > 0$ such that

$$\|w_h^* - w_k\|_{2,2} \leq c_3 \|s_k\|_{2,2} \to 0 \quad \forall \tau_h \in \mathcal{T}_\theta,\ h \leq h_0,\ \forall k \geq 0,$$

which implies the uniform convergence of the sequence $\{w_{h,k}\}_k$ to the function w_h^* solving the problem (P_h).

Since $w_k \to w_h^*$, also $A_k \to A_h^*$ and consequently $A_k(w_k + s_k) \to A_h^*$. Since $card(A_k) \leq m(h) < \infty$ for any fix $h \leq h_0$, there exists $k_0 \geq 0$, such that $A_{k_0} = A_{k_0}(w_{k_0} + s_{k_0})$. Then, by Lemma 3.8, $w_{k_0} + s_{k_0} = w_h^*$. $\qquad \square$

Remark 4.1. *The convergence result of Algorithm 1 holds for parameters $h \leq h_0$, for some h_0. With respect to analyses in [7], we can assume that the size of h_0 depends on the stability of the load, i.e. how much the balance point T is closed to the end points x_l, x_r of the subsoil and how much the size of the load resultant F is relatively closed to zero.*

Remark 4.2. *Numerical examples shows that Algorithm 1 converges for almost all initial choices of $A_{h,0}$. However, the initial choice $A_{h,0} = \{1, \ldots, m(h)\}$ ensures in the tested examples that $\alpha_{h,k} = 1$ for any $k \geq 0$ due to inclusions $A_{h,k+1} \subset A_{h,k}$. These inclusions are shown in [5] for a concrete choice of the load.*

Remark 4.3. *We can also substitute $\alpha_{h,k}$ by*

$$\tilde{\alpha}_{h,k} := \min_{\alpha \geq 0} J_h(w_{h,k} + \alpha s_{h,k}).$$

The corresponding algorithm will be denoted Algorithm 2 and it is shown on numerical examples that we can expect the same convergence properties as for Algorithm 1. However, it is necessary to generalise Lemma 4.2 to use Algorithm 2 correctly. The comparison of the algorithm will be illustrated on numerical examples in Section 6.

There are many numerical methods how to find the values $\alpha_{h,k}$ or $\tilde{\alpha}_{h,k}$ which do not depend on the parameter h. Here, the regula falsi method has been used.

Algorithms 1,2 can also be used for coercive beam problems with the same convergence result which can be proved without Lemma 4.2 and without the restricted assumption on the parameter h.

4.2. **Descent Direction Method with Projection.** First of all, we will define the class of the auxiliary problems, which are specified by a partition $\tau_h \in \mathcal{T}_\theta$ and by a function $v_h \in V_h$:

$$(P_h^{v_h}) \qquad \text{find } p_h = p_h(v_h) \in P_1 : \quad J_h(v_h + p_h) \leq J_h(v_h + p) \quad \forall p \in P_1,$$

or equivalently

$$(4.26) \qquad \text{find } p_h = p_h(v_h) \in P_1 : \quad b_h((v_h + p_h)^-, p) = L(p) \quad \forall p \in P_1.$$

The problem $(P_h^{v_h})$ means to solve the system of two non-linear equations with two unknowns. Similarly as for the problem (P_h), it is possible to prove that the condition (2.15) ensures the existence of the solution and the uniqueness of the solution holds for sufficiently small parameters h. Notice that if w_h^* solves the problem (P_h) then the problem $(P_h^{w_h^*})$ solves the zeroth polynomial.

Lemma 4.5. Let the solvability condition (2.5) hold and $c, \theta > 0$. Then there exist positive constants $\rho > 0$ and h_0 such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_0$, and any $v_h \in V_h$, $|v_h|_{2,2} \leq c$,

$$(4.27) \qquad A_h(v_h + p_h) \in \mathcal{A}_\rho,$$

where p_h solves $(P_h^{v_h})$.

Proof. We start with the well-known inequality

$$(4.28) \qquad \exists c_1 > 0 : \quad |v|_{2,2}^2 \geq c_1 \inf_{p \in P_1} \|v + p\|_{2,2}^2 \quad \forall v \in H^2(\Omega),$$

which can be proved by the Poincaré inequality. Notice that

$$v_h + p + p_h(v_h + p) = v_h + p_h(v_h) \quad \forall p \in P_1,$$

where $p_h(v_h + p)$ solves $(P_h^{v_h+p})$. Thus $A_h(v_h + p_h(v_h)) = A_h(v_h + p + p_h(v_h + p))$. Therefore, with respect to the assumption $|v_h|_{2,2} \leq c$ and the inequality (4.28), we can assume that $\|v_h\|_{2,2} \leq \tilde{c}$, $\tilde{c} > 0$, for any $v_h \in V_h$.

Suppose that Lemma 4.5 does not hold. Then there exist sequences $\{\tau_{h_k}\}_k$, $h_k \to 0$, and $\{v_k\}_k$, $v_k \equiv v_{h_k}$, $\|v_k\|_{2,2} \leq \tilde{c}$, such that

$$(4.29) \qquad h_k card(A_k) \to 0,$$

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where $A_k \equiv A_{h_k}(v_k + p_k)$, p_k solves $(P_{h_k}^{v_k})$. The choice $p = 1$ in the equation (4.26) and the estimate (2.10) yield

$$F = \sum_{i \in A_k} r_i^k(v_k + p_k)(z_i^k) \geq c_2 \min_{i \in A_k}(v_k + p_k)(z_i^k)h_k card(A_k), \quad c_2 > 0.$$

Hence, by (4.29) and the boundedness of $\{v_k\}$, we obtain that there exists a point $z \in [x_l, x_r]$ such that $p_k(z) \to -\infty$. If $z \in \Omega_s$, then the assumption (4.29) cannot hold with respect to Lemma 3.3. Therefore, $z = x_l$ or $z = x_r$.

Let us consider the first case. For the second one, we obtain similar contradiction. Then $p_k(x_l) \to -\infty$ and $p_k(z) \not\to -\infty$ for $z > x_l$. Hence $p_k(z) \to +\infty$ for $z > x_l$. It means that $z_i^k \to x_l$ for all $i \in A_k$, since the functions v_k are uniformly bounded. Therefore $z_i^k < T$ for all $i \in A_k$, where k is sufficiently large. If we choose $p = x$ in the equation (4.26), we obtain

$$T = \frac{\sum_{i \in A_k} r_i^k(v_k + p_k)(z_i^k)z_i^k}{\sum_{i \in A_k} r_i^k(v_k + p_k)(z_i^k)} \leq \max_{i \in A_k} z_i^k < T,$$

which is contradiction. $\qquad\square$

The descent direction method with projection is obtained from the previous method by adding of the "projection" step, where the problem of type $(P_h^{v_h})$ is solved in:

Algorithm 3

   Initialisation
     $w_{h,0} = p_h(0)$, $p_h(0)$ solves (P_h^0),
     $A_{h,0} = A_h(w_{h,0})$.
   Iteration $k = 0, 1, \ldots$
     $s_{h,k} \in V_h$, $w_{h,k} + s_{h,k}$ solves $(P_h^{A_{h,k}})$,
     $\alpha_{h,k} = arg \min_{0 \leq \alpha \leq 1} J_h(w_{h,k} + \alpha s_{h,k})$,
     $\tilde{w}_{h,k} = w_{h,k} + \alpha_{h,k}s_{h,k}$,
     $p_{h,k} = p_h(\tilde{w}_{h,k})$, $p_h(\tilde{w}_{h,k})$ solves $(P_h^{\tilde{w}_{h,k}})$,
     $w_{h,k+1} = \tilde{w}_{h,k} + p_{h,k}$,
     $A_{h,k+1} = A_h(w_{h,k+1})$.

Lemma 4.6. *Let the condition (2.5) hold and $\theta > 0$. Then there exist positive constants ρ, c_1, c_2 and h_0 such that for any $\tau_h \in \mathcal{T}_\theta$, $h \leq h_0$, and any $k \geq 0$,*

(4.30) $$A_{h,k} \in \mathcal{A}_\rho,$$

(4.31) $$\alpha_{h,k} \geq c_1,$$

(4.32) $$J_h(w_{h,k+1}) \leq J_h(w_{h,k}) - c_2\|s_{h,k}\|_{2,2}^2,$$

where $A_{h,k}$, $\alpha_{h,k}$, $s_{h,k}$ and $w_{h,k}$ are generated by Algorithm 3.

The proofs of (4.30)-(4.32) are quite similar to the proofs of (4.19),(4.21) and (4.22) for Algorithm 1. Only instead of Lemma 4.2, we use Lemma 4.5 and the inequality

$$J_h(w_{h,k+1}) \leq J_h(\tilde{w}_{h,k}),$$

which follows from the definition of the problem $(P_h^{\tilde{w}_{h,k}})$.

In the same way as for Algorithm 1, we obtain the following convergence result for Algorithm 3.

Theorem 4.3. *Let the condition (2.5) hold and $\theta > 0$. Then there exist $h_0 > 0$ such that the sequence $\{w_{h,k}\}_k$ generated by Algorithm 3 converges uniformly (with respect to h) to the function w_h^* solving the problem (P_h) for any $\tau_h \in \mathcal{T}_\theta$, $h \le h_0$.*

In addition, for any fix $\tau_h \in \mathcal{T}_\theta$, $h \le h_0$, there exists an iteration $k_0 = k_0(h) \ge 0$ such that $w_{h,k_0} + s_{h,k_0} = w_h^*$.

For an implementation of the "projection" step in Algorithm 3, i.e. for an implementation of the problem $(P_h^{v_h})$, we can use the small modification of Algorithm 1 with the same convergence results:

> *Initialisation*
> $\quad p_{h,0} \in P_1, \;\; b_h(v_h + p_{h,0}, p) = L(p) \quad \forall p \in P_1,$
> $\quad A_{h,0} = A_h(v_h + p_{h,0}).$
> *Iteration $k = 0, 1, \dots$*
> $\quad \tilde{p}_{h,k} \in P_1, \;\; b_h^{A_{h,k}}(v_h + p_{h,k} + \tilde{p}_{h,k}, p) = L(p) \quad \forall p \in P_1,$
> $\quad \alpha_{h,k} = arg \min_{0 \le \alpha \le 1} J_h(v_h + p_{h,k} + \alpha \tilde{p}_{h,k}),$
> $\quad p_{h,k+1} = p_{h,k} + \alpha_{h,k} \tilde{p}_{h,k},$
> $\quad A_{h,k+1} = A_h(v_h + p_{h,k+1}).$

Remark 4.4. *Due to the projection step, the functions $w_{h,k}$ generated by Algorithm 3 have some common properties with the unknown function w_h^* as we see in the end of the next section.*

Again, it is possible to substitute $\alpha_{h,k}$ by

$$\tilde{\alpha}_{h,k} = arg \min_{\alpha \ge 0} J_h(w_{h,k} + \alpha s_{h,k})$$

in Algorithm 3.

The projection step cannot be applied for coercive problems, since the polynomials of the first degree do not belong between tested functions for such problems.

5. Algebraic Formulation of the Problem

5.1. **Rewriting of the Approximated Problem.** Let $\tau_h \in \mathcal{T}_\theta$ be a partition with nodal points

$$0 = x_0 < x_1 < \dots < x_l = x_{j_l-1} < \dots < x_r = x_{j_r} < \dots < x_N = l$$

and let $z_1 < z_2 < \dots < z_m$ be the corresponding points, which are obtained from the chosen numerical quadrature.

The functions $v_h \in V_h$ will be standardly represented by the vector $v \in \mathbb{R}^n$, $n = 2N + 2$. The form a and the functional L will be represented by the stiffness matrix $K \in \mathbb{R}^{n \times n}$ and by the load vector $f \in \mathbb{R}^n$. Notice that the matrix K is symmetric and positive semi-definite.

Let the polynomials $p = 1$ and $p = x$ be represented by the vectors $p_1, p_x \in \mathbb{R}^n$. Then the matrix $R := (p_1, p_x) \in \mathbb{R}^{n \times 2}$ represents all polynomials from P_1 and creates the kernel of K, i.e. $KR = 0$.

The matrix, which transforms the function values and the values of the first derivatives at the nodal points x_j, $j = 0, 1, \ldots, N$, onto the points z_i, $i = 1, \ldots, m$, will be denoted by $B \in \mathbb{R}^{m \times n}$. Let $D \in \mathbb{R}^{m \times m}$ be a diagonal matrix containing the coefficients r_i, i.e. the products of the weights of the numerical quadrature and the stiffness coefficients of the subsoil.

The Euclidean scalar product and norm in $\mathbb{R}^k$, $k \geq 1$, will be denoted by $(.,.)_k$ and $\|.\|_k$.

For the sake of simplicity, the corresponding functional and the unknown vector in the algebraic formulation will be denoted in the same way as in the continuous problem (P). Then the algebraic formulation of the problem (P_h) has a form

$$(\mathbb{P}) \quad \begin{cases} \text{find } w^* \in \mathbb{R}^n : \ J(w^*) \leq J(w) \quad \forall w \in \mathbb{R}^n, \\ J(w) := \frac{1}{2}(Kw, w)_n + \frac{1}{2}\left(D(Bw)^-, (Bw)^-\right)_m - (f, w)_n, \end{cases}$$

where $u^- \in \mathbb{R}^m$ is the negative part of u, i.e.

$$(u^-)_i := \min\{0, u_i\}, \quad i = 1, 2, \ldots, m.$$

The problem $(\mathbb{P})$ can be rewritten equivalently as the non-linear system of equations:

$$(5.1) \qquad \text{find } w^* \in \mathbb{R}^n : \ Kw^* + B^T D(Bw^*)^- = f.$$

Let a set $A_h \subset \{1, 2, \ldots, m\}$ of indices be represented by the diagonal matrix $A \in \mathbb{R}^{m \times m}$ such that $A_{ii} = 1$ if $i \in A_h$, otherwise $A_{ii} = 0$. The algebraic representation of a set $A_h(v_h)$ will be denoted $A(v)$.

We also introduce the notation

$$(5.2) \qquad G := BR = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_m \end{pmatrix}^T, \quad e := R^T f = F \begin{pmatrix} 1 \\ T \end{pmatrix}.$$

Then the auxiliary problems $(P_h^{A_h})$ and $(P_h^{v_h})$ have the following algebraical forms:

$$(5.3) \qquad (\mathbb{P}^A) \quad \text{find } w = w(A) \in \mathbb{R}^n : \ (K + B^T DAB)w = f.$$

$$(5.4) \qquad (\mathbb{P}^v) \quad \text{find } c = c(v) \in \mathbb{R}^2 : \ G^T D(Bv + Gc)^- = e.$$

The corresponding algebraical formulations of Algorithms 1,3 are following:

Algorithm 1

Initialisation

$\quad w^{(0)} = 0,$

$\quad A_{(0)}, \ (A_{(0)})_{ii} = 1, \ i = \{1, \ldots, m\}.$

Iteration $k = 0, 1, \ldots$

$\quad s^{(k)}, \ w^{(k)} + s^{(k)}$ solves $(\mathbb{P}^{A_{(k)}}),$

$\quad \alpha_{(k)} = arg \min\limits_{0 \leq \alpha \leq 1} J(w^{(k)} + \alpha s^{(k)}),$

$\quad w^{(k+1)} = w^{(k)} + \alpha_{(k)} s^{(k)},$

$\quad A_{(k+1)} = A(w^{(k+1)}).$

Algorithm 3

Initialisation

$\quad w^{(0)} = Rc^{(0)}, \ c^{(0)}$ solves $(\mathbb{P}^0),$

$\quad A_{(0)} = A(w^{(0)}),$

Iteration $k = 0, 1, \ldots$

$\quad s^{(k)}, \ w^{(k)} + s^{(k)}$ solves $(\mathbb{P}^{A_{(k)}}),$

$\quad \alpha_{(k)} = arg \min\limits_{0 \leq \alpha \leq 1} J(w^{(k)} + \alpha s^{(k)}),$

$\quad \tilde{w}^{(k)} = w^{(k)} + \alpha_{(k)} s^{(k)},$

$\quad c^{(k)}, \ c^{(k)}$ solves $(\mathbb{P}^{\tilde{w}^{(k)}}),$

$$w^{(k+1)} = \tilde{w}^{(k)} + Rc^{(k)},$$
$$A_{(k+1)} = A(w^{(k+1)}).$$

5.2. **Analysis of the Projection Step.** To explain the reason of the "projection step", we will consider the set

(5.5) $$\Lambda := \{\lambda \in \mathbb{R}^m \mid \lambda \le 0,\ G^T D\lambda = e\}.$$

First of all, we derive some basic properties of the set Λ. Clearly, the set Λ is closed and convex on $\mathbb{R}^m$.

Lemma 5.1. *Let $F < 0$ and $z_1 < T < z_m$. Then the set Λ is non-empty and bounded on $\mathbb{R}^m$.*

Proof. The assumptions of the lemma ensure that there exists the solution w^* of the problem $(\mathbb{P})$. If we multiply the equation (5.1) by the vectors in the form $(Ra)^T$, $a \in \mathbb{R}^2$, we obtain that $(Bw^*)^- \in \Lambda$ by (5.2).

The boundedness follows from the definition of the set Λ and the estimate (2.10):

$$-F = -e_1 = -(G^T D\lambda)_1 = \sum_{i=1}^{m} r_i|\lambda_i| \ge c\|\lambda\|_m, \quad c > 0.$$

$\square$

Lemma 5.2. *Let $F < 0$, $z_1 < T < z_m$ and $\lambda \in \Lambda$. Let*

$$A(\lambda) := \{i \in \{1, 2, \ldots, m\} \mid \lambda_i < 0\}.$$

Then

(5.6) $$\min_{i \in A(\lambda)} z_i \le T \le \max_{i \in A(\lambda)} z_i.$$

Proof. The equation $G^T D\lambda = e$ yields that

$$T = \frac{\sum_{i \in A(\lambda)} r_i \lambda_i z_i}{\sum_{i \in A(\lambda)} r_i \lambda_i}.$$

Hence we obtain (5.6). $\square$

The following lemma says that the diameter of the set Λ is small for unstable loads.

Lemma 5.3. *Let $\{F_k\}_k$, $\{T_k\}_k$ be the sequences of the load resultants and their balance points such that $F_k < 0$, $z_1 < T_k < z_m$ for any $k \ge 0$. Let $\{\Lambda_k\}_k$ be the sequence of the corresponding sets defined by (5.5). If $T_k \to z_1$ or $T_k \to z_m$ or $F_k \to 0$ then $\mathrm{diam}(\Lambda_k) \to 0$.*

Proof. Let $T_k \to z_1$. Then by the definition of the set Λ_k, we obtain

$$0 = \sum_{i=1}^{m} r_i \lambda_i^k (z_i - T_k) = r_1 \lambda_1^k (z_1 - T_k) + \sum_{i=2}^{m} r_i \lambda_i^k (z_i - T_k) \quad \forall \lambda^k \in \Lambda_k,\ \forall k \ge 1.$$

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The first term on the right-hand side is non-negative and tends to zero for $k \to \infty$. The second term is non-positive for sufficiently large k and therefore $\lambda_i^k \to 0$, $i = 2, \ldots, m$. Since it also holds

$$(5.7) \qquad F_k = \sum_{i=1}^{m} r_i \lambda_i^k \quad \forall \lambda^k \in \Lambda_k, \ \forall k \geq 1,$$

we obtain

$$\lambda_1^k - \tilde{\lambda}_1^k = -\frac{1}{r_1} \sum_{i=2}^{m} r_i(\lambda_i^k - \tilde{\lambda}_i^k) \to 0, \quad \forall \lambda^k, \tilde{\lambda}^k \in \Lambda_k$$

which means that $diam(\Lambda_k) \to 0$.

Similarly, we can prove the assertion for the case $T_k \to z_m$. For the case $F_k \to 0$, the assertion also holds, since the equation (5.7) yields $\lambda^k \to 0$ for any $\lambda^k \in \Lambda_k$. $\quad\square$

Since λ is the closed, convex and non-empty set, we can define uniquely the projection P of the space $\mathbb{R}^m$ onto the set Λ with respect to the scalar product $(D.,.)_m$ in $\mathbb{R}^m$:

$$(5.8) \qquad (D(\eta - P(\eta)), \lambda - P(\eta))_m \leq 0 \quad \forall \lambda \in \Lambda.$$

Let $v \in \mathbb{R}^n$ and let $c = c(v) \in \mathbb{R}^2$ solve the problem $(\mathbb{P}^v)$. Then the vector $(Bv+Gc)^-$ belongs to Λ and

$$\begin{aligned}
\left(D(Bv - (Bv + Gc)^-), \lambda - (Bv + Gc)^- \right)_m &= \\
&= \left(D((Bv + Gc)^+ - Gc), \lambda - (Bv + Gc)^- \right)_m \\
&= \left(D(Bv + Gc)^+, \lambda \right)_m + \left(c, G^T D((Bv + Gc)^- - \lambda) \right)_2 \\
&= \left(D(Bv + Gc)^+, \lambda \right)_m \leq 0 \quad \forall \lambda \in \Lambda.
\end{aligned}$$

Therefore, by the definition (5.8) of the projection P,

$$P(Bv) = (Bv + Gc)^-.$$

It means that for the vectors $w^{(k)}$, $k \geq 0$, generated by the Algorithm 3, and for the solution w^*, we obtain $(Bw^{(k)})^-, (Bw^*)^- \in \Lambda$. Thus, these vectors have the common properties specified by the above lemmas. Mainly, for unstable loads, the vectors $(Bw^{(k)})^-$ are closed to the vector $(Bw^*)^-$, which means that the vectors $Bw^{(k)}$ have the similar set of the active "springs" as the vector Bw^*. Therefore we can expect better convergence properties for Algorithm 3 than for Algorithm 1 for such loads. It will be also demonstrated on numerical examples in the next section.

The set Λ is also important for the dual formulation of the problem, see [6], since the vectors $-\lambda$, where $\lambda \in \Lambda$, can represent admissible Lagrange multipliers.

6. NUMERICAL EXAMPLES

In this section, the convergence results of Algorithms 1-3 will be demonstrated on the numerical examples.

We will consider the beam of the length l=1m with the parameter $EI = 5 * 10^5 \ Nm^2$. The subsoil is situated in the interval (x_l, x_r), where $x_l = 0.1 \ m$ and $x_r = 0.9 \ m$, and its stiffness coefficient is $q = 5 * 10^8 \ Nm^{-2}$. At the end points $0, l$

of the beam, we will consider the point loads F_0 and F_l, which will be specified for the concrete examples. The interval $(0, l)$ will be divided into $10 * 2^j$, $j = 2, 3, \ldots, 8$, equidistant parts. The situation is depicted in Figure 2.
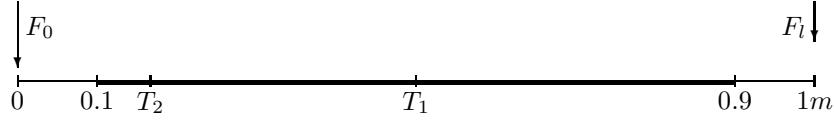

FIGURE 2. Scheme of the tested problem.

We use the following stopping criterion:

$$\frac{\|r^{(k)}\|_n}{\|f\|_n} \leq \varepsilon, \quad r^{(k)} := f - Kw^{(k)} - B^T D(Bw^{(k)})^-,$$

where $\varepsilon = 10^{-6}$ and $r^{(k)}$ is the k-th residuum of these algorithms. For an approximation of the bilinear form b, the reference numerical quadrature

$$\int_{-1}^{1} \phi(\xi) \, d\xi \approx \phi(-\sqrt{3}/3) + \phi(\sqrt{3}/3)$$

is used. The linear problems with bilateral elastic springs are solved by the Cholesky factorisation.

Example 1. Let $F_0 = -5000 \, N$ and $F_l = -5000 \, N$. Such a load fulfils the solvability condition (2.5) and is stable, since the balance point $T_1 = 0.5 \, m$ is situated in the centre of the subsoil interval. The dependence of the number of outer iterations on the refinement parameter j of the partition is shown in Table 1.

Notice that the number of outer iterations does not depend on j and are practically the same for all the algorithms. The number of iterations for the "projected" step in Algorithm 3 are about four. The approximated solution for $j = 8$, i.e. for 2560 elements, is depicted in Figure 3.

Example 2. Let $F_0 = -5000 \, N$ and $F_l = -1000 \, N$. Such a load fulfils the solvability condition (2.5) and is not too stable, since the balance point $T_2 = 0.1667 \, m$ is closed to the end point x_l of the subsoil. The dependence of the number of outer iterations on the refinement parameter j of the partition is shown in Table 1.

Notice that the number of outer iterations does not depend on j. The number of outer iterations for Algorithm 3 are smaller than for Algorithms 1,2, which is the expected result.

The approximated solution for $j = 8$ is depicted in Figure 3.

7. CONCLUSION

The descent direction methods with and without projection have been introduced and analysed. The methods can be generalised for the problems with more parts of the subsoil and also for two-dimensional models of thin elastic plates.

The methods have been illustrated on numerical examples. Other numerical examples, which confirm some theoretical results, can be found in [8].

Ex. 1	2	3	4	5	6	7	8	Ex. 2	2	3	4	5	6	7	8
ALG1	4	3	4	4	4	4	4	ALG1	6	6	7	8	7	8	8
ALG2	3	3	3	3	3	4	4	ALG2	5	5	6	6	6	6	6
ALG3	3	3	3	3	3	3	3	ALG3	2	2	2	2	2	2	2

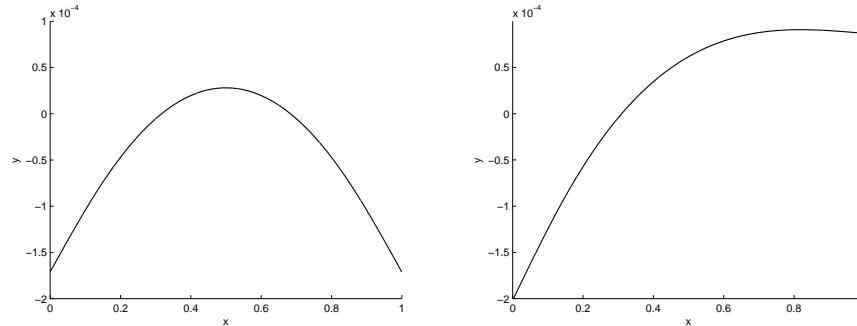TABLE 1. Numbers of outer iterations for Examples 1 and 2.

FIGURE 3. Approximated beam deflections w for Examples 1 and 2.

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