The ThermoAcoustic Tomography Inverse Problem

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Principle of the TAT

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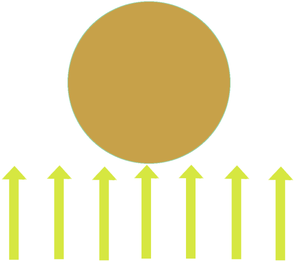
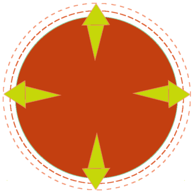
- A body is exposed to a radio frequency electromagnetic pulse,



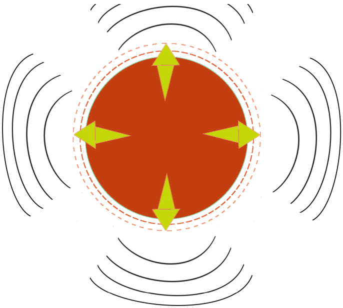
Principle of the TAT

- A body is exposed to a radio frequency electromagnetic pulse,
- Tissues heating causes expansion...



Principle of the TAT

- A body is exposed to a radio frequency electromagnetic pulse,
- Tissues heating causes expansion...
- ...which generates a pressure wave.



Biological observation

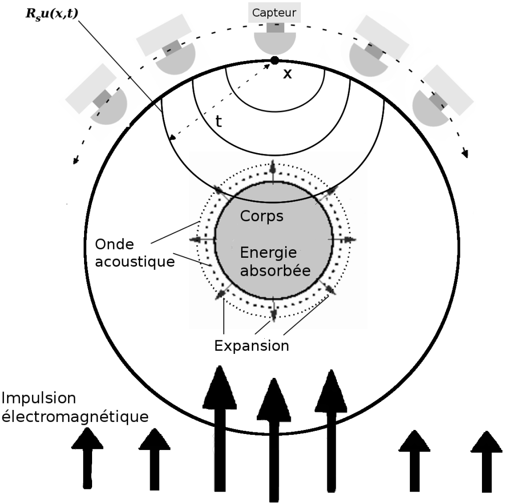
Cancerous tissues, by being more vascularised, absorb more electromagnetic energy.

$\Rightarrow$ The goal here is to reconstruct the absorptivity coefficients map, denoted by $\mu_{abs}(x)$, from the mesured acoustic wave.

Advantages of the TTA

- Non invasive;
- Combine the contrast skills of electromagnetic with high resolutions allowed by ultrasound waves;
- Simple and (farely) cheap equipment;
- Nevertheless, weak penetration capacity.

SET-UP FOR "POINT" DETECTORS



Back to problem
Back to model improvement

Plan

Fluids Mechanic equations

The linearized continuity equation

$$\frac{\partial \rho(x,\hat{t})}{\partial \hat{t}} = -\rho_0 \nabla \cdot v(x,\hat{t})$$

is derived from the principle of conservation of mass if the particle velocity $v(x,\hat{t})$ is small and the mass density $\rho_{tot}(x,\hat{t}) = \rho_0 + \rho(x,\hat{t})$ is weakly varying, i.e. $|\rho(x,\hat{t})| \ll \rho_0$.

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The linearized Euler equation

$$\rho_0 \frac{\partial v(x,\hat{t})}{\partial \hat{t}} = -\nabla p(x,\hat{t})$$

is derived from the principle of conservation of momentum for a non-viscous, non-turbulent flow in the absence of external forces with slowly varying pressure $p_{tot}(x,\hat{t}) = p_0 + p(x,\hat{t})$, i.e. $|p(x,\hat{t})| \ll p_0$, within the fluid.

Back to model improvement

Usual assumptions of the model

- The initial elecromagnetic pulse is considered to be a Dirac pulse,
- At time t_0, every part of the body receives the same amount of energy,
- The speed of the wave is assumed to be constant (homogeneous media),
- The wave is not subject to any attenuation.

Equation governing the acoustic pressure

Wave equation

$$
\left|
\begin{array}{l}
\dfrac{\partial^2 p(x,t)}{\partial t^2} - \Delta p(x,t) = 0, \\
p(x,0) = u(x) := \dfrac{\mu_{abs}(x)\beta(x)J(x){v_s}^2}{c_p(x)}, \\
\dfrac{\partial p(x,0)}{\partial t} = 0
\end{array}
\right.
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Plan

1. The ThermoAcoustic Tomography
   - Experimental set-up
   - Equations
   - Inverse problem for point detectors
   - Inverse problem for lineic detectors
2. Some inversion formulas
   - The filtered backprojection
   - Fourier-Bessel series expansion
   - Helmholtz equation
3. Variational approach
   - Fourier regularization
   - Regularization scheme
     - Regularizing data
     - Asymptotic behaviour of $\mathcal{P}_\beta$
4. Illustrations

Rewriting the solution

The classical way to solve the wave equation suggests the use of integral geometry tools :

Theorem

$$p(x,t) = \frac{\partial}{\partial t}\left[\frac{R_s u(x,t)}{4\pi t}\right]$$

where

$$R_s u(x,t) := \int_{\partial B_t(x)} u(y)\,\mathrm{d}\mathcal{S}(y), \qquad (x,t) \in \mathbb{R}^3 \times [0,\infty[.$$

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The inverse formula introduces some linear transform of the initial data u

Proposition

$$R_s u(x_{cent}, t) = 4\pi t \int_0^t p(x_{cent}, s)\,\mathrm{d}s$$

Inverse problem

The spherical Radon transform

$$R_s u(x,t) := \int_{\partial B_t(x)} u(y)\, \mathrm{d}\mathcal{S}(y), \qquad (x,t) \in \mathbb{R}^3 \times [0,\infty[,$$

with u supported in B.

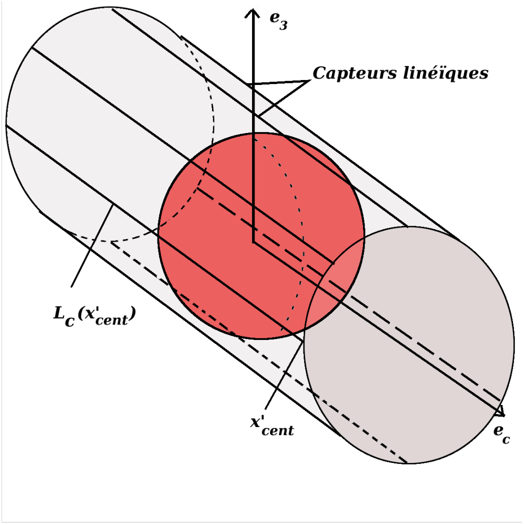
Problem

Can we reconstruct u, known to be with a compact support in B, from the knowledge of its integrals over spheres centered on the unit sphere, that is $R_s u$.

set-up

Plan

Set-up for lineic detectors



The 2-d wave equation $(e_c = e_1)$

Definition

$$\bar{u}(x') := \int_{\mathbb{R}} u(x_1, x')\,dx_1, \qquad x' \in \mathbb{R}^2, \tag{1}$$

$$\bar{p}(x', t) := \int_{\mathbb{R}} p(x_1, x', t)\,dx_1, \qquad (x', t) \in \mathbb{R}^2 \times [0, \infty[, \tag{2}$$

In this set-up, the mesured integrals appear to solve a 2-d wave equation :

Wave equation

$$\left|\begin{array}{l} \dfrac{\partial^2 \bar{p}(x', t)}{\partial t^2} - \Delta \bar{p}(x', t) = 0, \\ \bar{p}(x', 0) = \bar{u}(x'), \\ \dfrac{\partial \bar{p}(x', 0)}{\partial t} = 0 \end{array}\right.$$

Rewriting the solution

Theorem

$$\bar{p}(x'_{cent}, t) = \frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^t \frac{R_c(\bar{u})(x'_{cent}, s)}{\sqrt{t^2 - s^2}}\,\mathrm{d}s,$$

where R_c is the 2d equivalent of R_s.

Here again we have an inversion formula allowing to work with the circular Radon transform

Proposition

$$R_c(\bar{u})(x'_{cent}, t) = 4t\int_0^t \frac{\bar{p}(x'_{cent}, t')}{\sqrt{t^2 - t'^2}}\,\mathrm{d}t'$$

Inverse problem

A two step problem

- Reconstruct $\bar{u}$, supported in the unit disc, from the knowledge of $R_c\bar{u}$ on the unit circle (inversion of the circular Radon transform).
- Reconstruct u from its projections $\bar{u}$, i.e. inversion of the classical Radon transform.

Inverse problem

A two step problem

- Reconstruct $\bar{u}$, supported in the unit disc, from the knowledge of $R_c\bar{u}$ on the unit circle (inversion of the circular Radon transform).
- Reconstruct u from its projections $\bar{u}$, i.e. inversion of the classical Radon transform.

$\Rightarrow$ Even if the problem is originally designed in 3d, it makes sense to invert its 2d equivalent.

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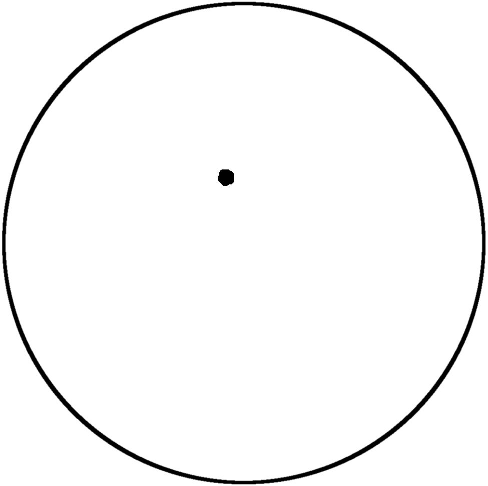
Idea of the backprojection



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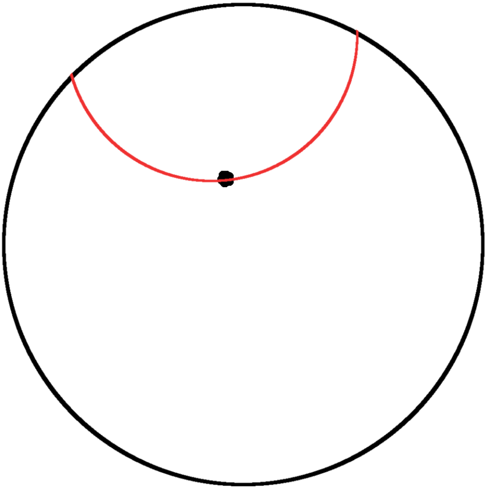

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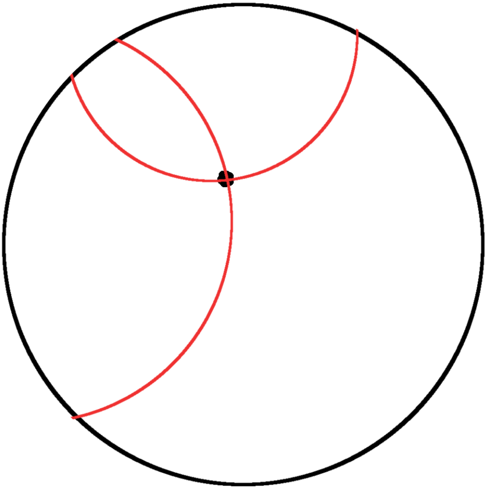

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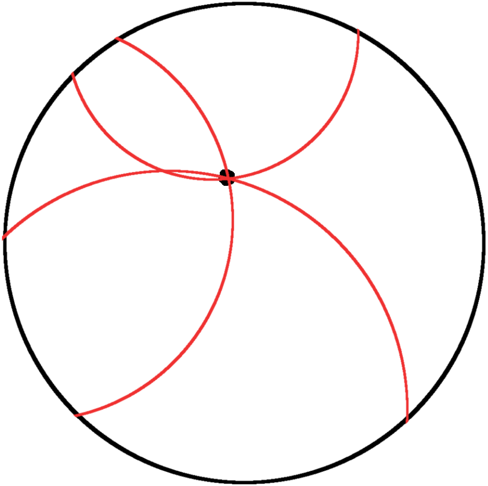

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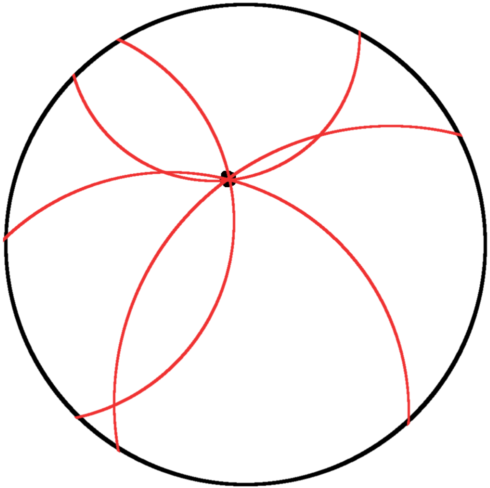

Some operators...

Definition

$$\begin{array}{rccc} \mathcal{N}: & C_0^\infty(\mathcal{B}) & \longrightarrow & \tilde{C}^\infty \\ & f(x) & \longmapsto & t^{n-2} R_s f(x,t) \end{array}$$

$$\begin{array}{rccc} \mathcal{D}: & \tilde{C}^\infty & \longrightarrow & \tilde{C}^\infty \\ & G(x,t) & \longmapsto & \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-3)/2} G(x,t) \end{array}.$$

Some operators...

Definition

$$\mathcal{N}: \quad C_0^\infty(\mathcal{B}) \longrightarrow \tilde{C}^\infty$$
$$f(x) \longmapsto t^{n-2} R_s f(x,t)$$

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$$G(x,t) \longmapsto \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-3)/2} G(x,t) \quad .$$

Proposition

$$\mathcal{N}^* G(x) = \frac{1}{\omega}\int_{\mathcal{S}} \frac{G(p,|p-x|)}{|p-x|}\, \mathrm{d}\mathcal{S}(p)$$

et

$$\mathcal{D}^* G(p,t) = (-1)^{(n-3)/2} t \mathcal{D}(G(p,t)/t).$$

Inversion formulas

Theorem [Patch, Finch, Rakesh, 2004]

Let n be odd and greater than 2, f in $\mathcal{C}_0^\infty(\mathcal{B})$, and assume that $R_s f$ is known on $\mathcal{S} \times \mathbb{R}_+$, then:

$$f(x) = -\frac{\pi}{2\Gamma(n/2)^2} (\mathcal{N}^* \mathcal{D}^* \partial_t^2 t \mathcal{D} \mathcal{N} f)(x), \qquad x \in \mathcal{B},$$

$$f(x) = -\frac{\pi}{2\Gamma(n/2)^2} (\mathcal{N}^* \mathcal{D}^* \partial_t t \partial_t \mathcal{D} \mathcal{N} f)(x), \qquad x \in \mathcal{B},$$

$$f(x) = -\frac{\pi}{2\Gamma(n/2)^2} \Delta_x (\mathcal{N}^* \mathcal{D}^* t \mathcal{D} \mathcal{N} f)(x), \qquad x \in \mathcal{B},$$

Comparison with the classical backprojection

- Spherical case :

$$f(x) = -\frac{\pi}{2\Gamma(n/2)^2}\Delta_x(\mathcal{N}^*\mathcal{D}^* t\mathcal{D}\mathcal{N}f)(x),$$

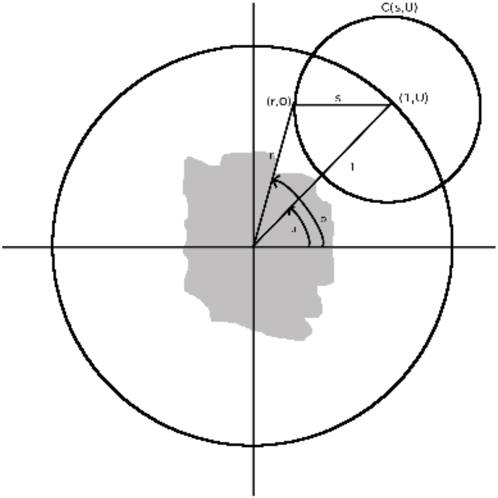
- Classical :

$$f(x) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}}\Delta_x^{(n-1)/2}(R^*Rf)(x).$$

Question : should we expect the same instability ?

Plan

Polar coordinates, illustration



Fourier-Bessel series expansion

- $$\int_{\mathcal{C}(\rho,\phi)} f(r,\theta)\,\mathrm{d}\mathcal{C} := g(\rho,\phi) = \sum_{n=-\infty}^{\infty} g_n(\rho)e^{in\phi},$$

  where

  $$g_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} g(\rho,\phi)e^{-in\phi}\,\mathrm{d}\phi;$$

- $$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta},$$

  where

  $$f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)e^{-in\theta}\,\mathrm{d}\theta.$$

Relation between coefficients

Notations

- J_i stands for the i^{th} Bessel function of first kind.
- $\mathcal{H}_n$ stands for the Hankel transform of n^{nt} kind, i.e. :

$$\mathcal{H}_n\{p(r)\}_z := \int_0^\infty p(r) J_n(rz) r \, dr.$$

Proposition

$$g_n(\rho) = 2\pi\rho\mathcal{H}_0\{J_n(z)\mathcal{H}_n\{f_n(r)\}_z\}_\rho, \forall n \in \mathcal{Z}, \forall \rho \in \mathbb{R}_+,$$

so that

$$f_n(r) = \mathcal{H}_n\left\{\frac{1}{J_n(z)}\mathcal{H}_0\left\{\frac{g_n(\rho)}{2\pi\rho}\right\}_z\right\}_r, \forall n \in \mathcal{Z}, \forall r \in \mathbb{R}_+.$$

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Plan

Solutions representation

Helmholtz equation

$$\begin{aligned} \Delta u_m(x) + \lambda_m^2 u_m(x) &= 0, \qquad x \in \mathcal{B} \\ u_m(x) &= 0, \qquad x \in \mathcal{S} \end{aligned} \tag{3}$$

and

$$\|u_m\|_{L^2} = 1$$

Representation

We denote by Φ_{λ_m} the Green functions, so :

$$u_m(x) = \int_{\mathcal{S}} \Phi_{\lambda_m}(|x - z|) \frac{\partial}{\partial n} u_m(z) \, \mathrm{d}\mathcal{S}(z), \quad x \in \mathcal{B}.$$

Series expansion of f [Kunyansky, 2007]

The eigen vectors $\{u_m(x)\}_0^\infty$ are an orthonormal basis of $L^2(\mathcal{B})$, so that f can be written :

$$f \overset{L^2}{=} \sum_0^\infty \alpha_m u_m,$$

where

$$\alpha_m = \int_{\mathcal{B}} u_m(x) f(x)\, \mathrm{d}x.$$

If g stands for the spherical Radon transform of f :

$$g(z,r) = \int_{\mathcal{S}} f(z+ry) r^{n-1}\, \mathrm{d}\mathcal{S}(y), \quad z \in \mathcal{S}, r \in \mathbb{R}_+.$$

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Computation of α_m

$$\begin{aligned}
\alpha_m &= \int_{\mathcal{B}} u_m(x) f(x)\,\mathrm{d}x \\
&= \int_{\mathcal{B}} \left(\int_{\mathcal{S}} \Phi_{\lambda_m}(|x-z|) \frac{\partial}{\partial n} u_m(z)\,\mathrm{d}\mathcal{S}(z) \right) f(x)\,\mathrm{d}x \\
&= \int_{\mathcal{S}} \left(\int_{\mathcal{B}} \Phi_{\lambda_m}(|x-z|) f(x)\,\mathrm{d}x \right) \frac{\partial}{\partial n} u_m(z)\,\mathrm{d}\mathcal{S}(z)
\end{aligned}$$

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&= \int_{\mathcal{S}} \left(\underbrace{\int_{\mathcal{B}} \Phi_{\lambda_m}(|x-z|) f(x)\,\mathrm{d}x}_{I(z,\lambda_m)} \right) \frac{\partial}{\partial n} u_m(z)\,\mathrm{d}\mathcal{S}(z) \qquad (4)
\end{aligned}$$

$I(z, \lambda_m)$

$$I(z,\lambda_m) \;=\; \int_{\mathcal{B}} \Phi_{\lambda_m}(|x-z|) f(x)\,\mathrm{d}x$$

$I(z, \lambda_m)$

$$
\begin{aligned}
I(z, \lambda_m) &= \int_{\mathcal{B}} \Phi_{\lambda_m}(|x - z|) f(x)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \Phi_{\lambda_m}(|x - z|) f(x)\, \mathrm{d}x,
\end{aligned}
$$

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&= \int_{\mathbb{R}^n} \Phi_{\lambda_m}(|x|) f(x + z)\,\mathrm{d}x \\
&= \int_{\mathbb{R}_+} \int_{\mathcal{S}} \Phi_{\lambda_m}(r) f(z + ry) r^{n-1}\,\mathrm{d}\mathcal{S}(y)\,\mathrm{d}r
\end{aligned}
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&= \int_{\mathbb{R}_+} g(z, r) \Phi_{\lambda_m}(r)\, dr.
\end{aligned}
$$

Comparison with the backprojection

Reconstruction of a function over a 2000000 , from 97000 mesurements :

Comparison with the backprojection

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- Helmholtz equation, $\mathcal{O}(n^3 \log(n))$:

Comparison with the backprojection

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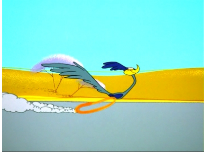

- Helmholtz equation, $\mathcal{O}(n^3 \log(n))$:
  $\longrightarrow$ 7 seconds,

Comparison with the backprojection

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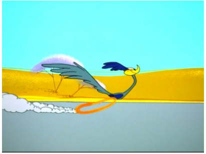

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- Exact inversion, $\mathcal{O}(n^5)$:

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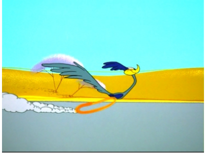

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- Exact inversion, $\mathcal{O}(n^5)$:
  $\longrightarrow$ 7 hours.

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Regularization scheme

Assuming we want to solve $Rf = g$, and that $g \approx Rf_0$ with $f_0 \in L^2(B)$:

- **Step 1:** Define the *object to be reconstructed* as $\phi_\beta * f_0$, where $(\phi_\beta)_{\beta>0}$ is an *approximation of unity*.
- **Step 2** Replace the original data g by *regularized data*: $\Phi_\beta g$.
- **Step 3** Finally, define the *reconstructed object* as the solution of the following optimization problem:

$$(\mathcal{P}_\beta) \quad \left| \begin{array}{rl} \text{Minimize} & \frac{1}{2}\|\Phi_\beta g - Rf\|^2_{L^2(S\times\mathbb{R}_+)} + \frac{\alpha}{2}\|(1-\hat{\phi}_\beta)\hat{f}\|^2_{L^2(\mathbb{R}^d)} \\ \text{s.t.} & f \in L^2(B), \end{array} \right.$$

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$\longrightarrow$ Here β is the relevant regularization parrameter.

Definition of a regularization operator

$$
\begin{aligned}
C_\beta : \quad L^2(\mathbb{R}^d) &\longrightarrow L^2(\mathbb{R}^d) \\
f &\longmapsto f * \phi_\beta
\end{aligned}
$$

Definition of a regularization operator

We investigate $\Phi_\beta \in L(L^2(S \times \mathbb{R}_+))$ such that:

$$\Phi_\beta R = R C_\beta.$$

If R is one-to-one, one gets easily:

$$\Phi_\beta = R C_\beta R^\dagger.$$

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If not, we can define Φ_β as some solution of...

Problem $\mathcal{Q}_\beta$

$$(\mathcal{Q}_\beta) \quad \left| \begin{array}{rl} \text{Minimize} & \frac{1}{2}\|RC_\beta - XR\|^2 \\ \text{s.t.} & X \in L(L^2(S \times \mathbb{R}_+)), \; X = 0 \text{ on } \operatorname{ran} R^\perp. \end{array} \right.$$

- This is a convex problem;
- R is compact $\Rightarrow$ level sets are not bounded.

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Nevertheless, we have

Proposition

If $RC_\beta R^\dagger$ is in $L(\mathcal{D}(R^\dagger), L^2(S \times \mathbb{R}_+))$, then $RC_\beta R^\dagger$ is the restriction of some bounded operator defined on $L^2(S \times \mathbb{R}_+)$ and solution of $(\mathcal{Q}_\beta)$. When R is injective, this solution is unique.

Problem $\mathcal{Q}_\beta$ in action

$$(\mathcal{Q}_\beta) \quad \left| \begin{array}{rl} \text{Minimize} & \frac{1}{2}\|RC_\beta - XR\|^2 \\ \text{s.t.} & X \in L(L^2(S \times \mathbb{R}_+)), \; X = 0 \text{ on } \operatorname{ran} R^\perp. \end{array} \right.$$

The computation of a solution could be achieve thanks to a *proximal point algorithm*.
$\longrightarrow$ This noise-free problem would be well posed.

$\looparrowright$ But $\mathcal{Q}_\beta$ is extremely huge !!

$\mathcal{Q}_\beta$ in finite dimension

$$F = \mathbb{R}^n \quad \text{and} \quad G = \mathbb{R}^m, \quad m < n \quad \text{in } \mathbb{N}$$

$$R \in \mathcal{M}_{m \times n} \quad \text{and} \quad \Phi_\beta \in \mathcal{M}_m$$

$\mathcal{Q}_\beta$ in finite dimension

$$(\mathcal{Q}_\beta) \quad \left| \begin{array}{rl} \text{Minimize} & \mathcal{J}(RC_\beta - XR) \\ \text{s.t.} & X \in \mathcal{M}_m, \;\; X = 0 \text{ on } \operatorname{ran} R^\perp. \end{array} \right.$$

Definition

The convex functional $\mathcal{J}$ is said to be $O(m) \times O(n)$-*invariant* iff

$$\forall\, (U_m, U_n) \in O(m) \times O(n),\;\; \mathcal{J}(U_m X U_n^t) = \mathcal{J}(X)$$

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Proposition

If $\mathcal{J}$ is $O(m) \times O(n)$-invariant, then $RC_\beta R^\dagger$ is solution of Problem $\mathcal{Q}_\beta$.

Change of framework

Remember...

Theorem (Maréchal *et al*)

I. Let $\alpha > 0$ and $\beta > 0$ fixed. Then Problem $(\mathcal{P}_\beta)$ is well posed.

II. Assume
- $\hat{\phi}(\xi) \neq 1, \forall \xi \in \mathbb{R}^d \setminus \{0\}$;
- $\exists K, s > 0$, $|1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} K\|\xi\|^s$;
- $g \in \mathcal{D}(T_W^\dagger)$ et $\tilde{g} = UT_W^\dagger g \ \in H^s(\mathbb{R}^d)$.

Than f_β converge in the strong sense to $T_W^\dagger g$, in $L^2(B)$, as $\beta \downarrow 0$.

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Φ_β solution of

$$\mathcal{Q}_\beta \quad \left| \begin{array}{rl} \text{Minimize} & \frac{1}{2}\|RC_\beta - XR\|^2_{L(H^s(B), L^2(S\times\mathbb{R}_+))} \\ \text{s.c.} & X \in L(G),\ \ X = 0 \text{ on } \operatorname{ran} R^\perp, \end{array} \right.$$

What happens if β tends to 0?

Lemma

When β goes down to zero, C_β converges to the identity and Φ_β converges the identity on ran R. In other words :

$$\|\Phi_\beta R - R\|_{L(H^s(B),L^2(S\times\mathbb{R}_+))} \underset{\beta\to 0}{\to} 0.$$

What happens if β tends to 0?

$$(\mathcal{P}_\beta) \quad \left| \begin{array}{rl} \text{Minimize} & \frac{1}{2}\|\Phi_\beta g - Rf\|^2_{L^2(S\times\mathbb{R}_+)} + \frac{\alpha}{2}\|(1-\hat{\phi}_\beta)\hat{f}\|^2_{L^2(\mathbb{R}^d)} \\ \text{s.c.} & f \in L^2(B), \end{array} \right.$$

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Taking β as a regularization parameter

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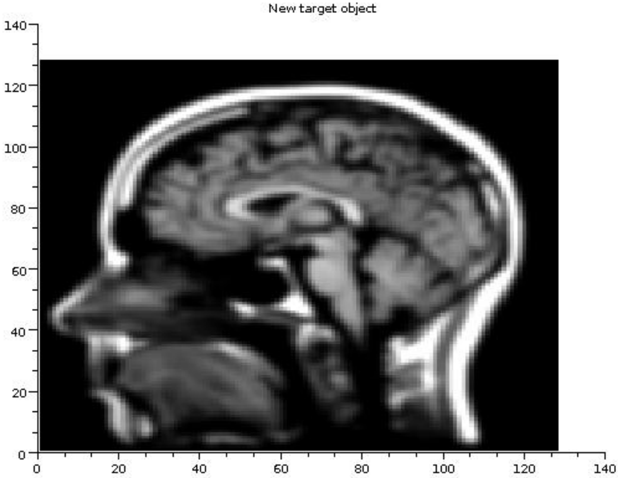

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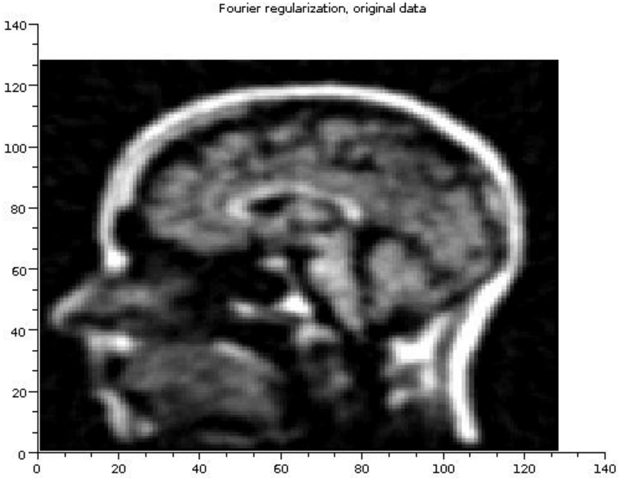

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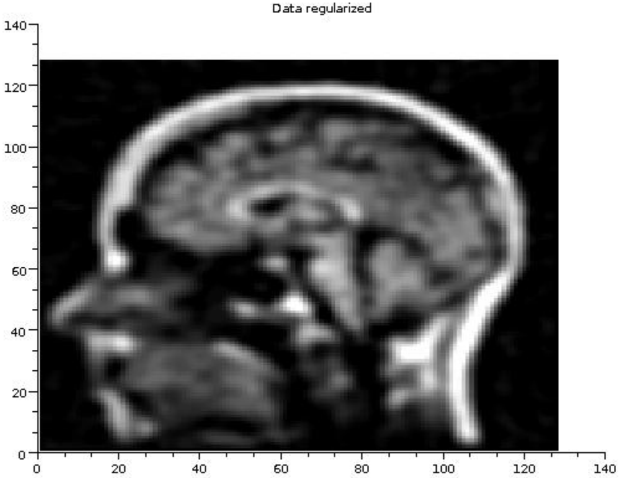

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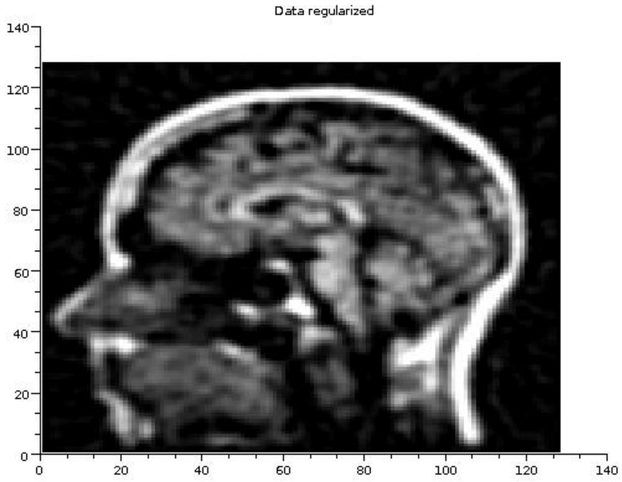

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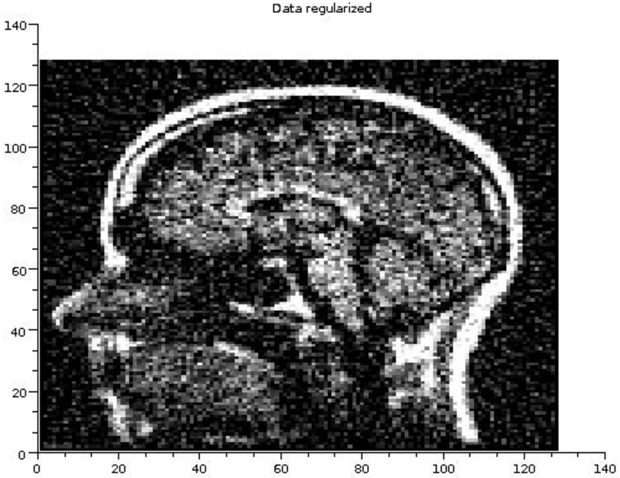

How do we compute $RCR^\dagger g$

We compute $R^\dagger g$ by means of a least square procedure :

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We compute $R^{\dagger}g$ by means of a least square procedure :

...and we apply RC to this nasty result.

Thank you