Recovering semialgebraic shapes from their moments with semidefinite programming

Didier Henrion[1,2]
Jean-Bernard Lasserre[1,3]

[1]LAAS-CNRS, Univ. Toulouse, FR
[2]Fac. Elec. Engr., Czech Tech. Univ. Prague, CZ
[3]Inst. Math. Toulouse, Univ. Toulouse, FR

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Outline

1. Shape from moments
2. SDP formulation
3. Examples

Moments of a set

Given a set $K \subset \mathbb{R}^n$, its moments are given by

$$y_\alpha = \int_K x^\alpha dx, \quad \forall \alpha \in \mathbb{N}^n$$

where dx is the Lebesgue (uniform) measure
and $x^\alpha = \prod_{k=1}^n x_k^{\alpha_k}$

Inverse problem: given sequence y, reconstruct set K

Chebyshev, Markov (1883), Hamburger, Stieltjes ($n = 1$)

Hadamard's well-posedness (1902): existence, uniqueness and continuous dependence on data of a solution

Tomography

Tomographic measurements of a body of constant density can be converted into moments, using the Radon transform

$$g_K(t,\theta) = \int_K \delta(t - x_1 \cos\theta - x_2 \sin\theta)dx$$

which is the line integral projection of K at the angle θ

It holds

$$\begin{array}{rcl} \int_{-T}^{T} g_K(t,\theta)t^k dt & = & \int_{-T}^{T}(x_1 \cos\theta + x_2 \sin\theta)^k dx \\ & = & \displaystyle\sum_{\alpha_1+\alpha_2=k} \left(\begin{array}{c} k \\ \alpha_1 \end{array}\right) \cos^{\alpha_1}(\theta) \sin^{\alpha_2}(\theta) y_\alpha \end{array}$$

Given line projections of K at $d+1$ distinct angles,
we can determine all the moments y_α of order up to d

Potential theory

Measurements of exterior gravitational field (or magnetic field, or thermal radiation) induced by a body of uniform mass can be converted into moments

Vector field $f(x) = \nabla\phi(x)$ corresponds to analytic function

$$f(z) = \frac{\partial\phi}{\partial x_1} + i\frac{\partial\phi}{\partial x_2}$$

where $z = x_1 + ix_2$, satisfying

$$f(z) = 2i\int_K \frac{dw}{z-w} = 2i\sum_{k=0}^{\infty} z^{-(k+1)} f_k$$

with harmonic moments

$$f_k = \int_K z^k dx$$

Uniqueness in the plane ($n = 2$)

A domain K is starlike with respect to a point P if it contains the chord PQ for every point $Q \in K$

Novikov (1938): a starlike domain is uniquely specified by its moments

Sakai (1978) constructed two distinct domains bounded by piecewise circular arcs with equal moments

Brodsky, Gabrielov, Strakhov (1985) constructed two distinct simply connected polygons with equal moments

Explicit counterexamples are scarce and complicated

Planar polygons

Davis (1977): a triangle is uniquely determined by its moments of up to order 3

Milanfar, Verghese, Karl, Willsky (1995): vertices of a simply connected n-gon are uniquely determined by moments of up to order $2n - 3$

Golub, Milanfar, Varah (1999): explicit numerical linear algebra algorithms on Hankel matrices

If the polygon is convex it is uniquely specified by its moments

Quadrature formulas

Connection with quadrature formulas: Motzkin, Schoenberg (1955) proved that given any function f analytic in a triangle T, the integral over T of the second derivative f'' is proportional to the second divided difference of f with respect to vertices of T

Davis (1977) generalized this to a polygon P with vertices b_k:

$$\int_P f''(x)dx = \sum_{k=1}^{n} a_k f(b_k)$$

where constants a_k depend on x_k but not on f

Quadrature formula: given a_k, b_k we want to find left-hand side

Inverse problem: $f(x) = x^\alpha$ so moments appear in left-hand side and unknowns a_k, b_k appear in right-hand side

Quadrature domains

Bounded planar domain K such that there exists an atomic measure μ supported in K such that

$$\int_K f(x_1 + ix_2)dx = \int_K f d\mu = \sum_{k=1}^{m} \sum_{j=0}^{n_k - 1} a_{kj} f^{(j)}(b_k)$$

where $b_k \in K$ are the quadrature nodes
and f is any analytic integrable function in K

A quadrature domain is described by a polynomial sublevel set

$$K = \{x \in \mathbb{R}^2 \,:\, g(x) > 0\}$$

where the degree of $g(x)$ in each variable separately is equal to the number of points $N = \sum_{k=1}^{m} n_k$ in the support of μ

Moments to quadrature domains

A quadrature domain of order N is uniquely specified by its moments up to order $2N$

For example, an ellipse of uniform mass is determined by 5 moments

A cardioid or a lemniscate is determined by 14 moments

Reconstructing quadrature domains and archipelagos

Tools from complex analysis were used by Golub, Gustafsson, He, Milanfar, Putinar, Varah (2000) to reconstruct exactly a quadrature domain K given its complex moments

$$\int_K z^{\alpha_1}\bar{z}^{\alpha_2}dxdy$$

and the Taylor expansion at infinity of the exponential transform

$$E(w,\bar{w}) = \exp\left(-\frac{1}{\pi}\int_K \frac{dxdy}{(z-w)(\bar{z}-\bar{w})}\right)$$

Roughly speaking, this is a reduction to the univariate case

Gustafsson, Putinar, Saff, Stylianopoulos (2008) extend these methods to reconstruct approximately what they call an archipelago, a finite union of mutually disjoint bounded Jordan domains

General semialgebraic sets

Quadrature domains are planar semialgebraic sets with a very particular structure

We would like to solve the shape reconstruction problem for a general basic semialgebraic set

$$K = \{x \in \mathbb{R}^n \,:\, g_k(x) \geq 0,\ k = 1, \ldots, m\}$$

where $g_k(x)$ are polynomials to be found, given moments of K

If the original K is not basic semialgebraic, we would like to approximate it with a basic semialgebraic set

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Moments and measures

In this work we use the algebraic moments

$$y_\alpha = \int_K x^\alpha d\mu(x), \quad \forall \alpha \in \mathbb{N}^n$$

If these relations are satisfied, we say that sequence y has representing measure μ supported on K

Inverse problem: given y, find μ and its support K

SDP conditions

Given a sequence y, define the moment matrix $M_d(y)$ of order d with entries indexed by multi-indices β (rows) and γ (columns)

$$[M_d(y)]_{\beta,\gamma} = y_{\beta+\gamma}, \quad |\beta| + |\gamma| \leq 2d$$

which are linear in y

Necessary condition: if y has a representing measure μ on $\mathbb{R}^n$ then $M_d(y) \succeq 0 \ \forall d$

Sufficient condition (Berg 1987): if $|y_\alpha| \leq 1 \ \forall \alpha$ and $M_d(y) \succeq 0$ $\forall d$, then y has a representing measure μ on $[-1, 1]^n$

SDP conditions

Given a sequence y and a polynomial $g(x) = \sum_\alpha g_\alpha x^\alpha$, define the localising matrix $M_d(gy)$ of order d with entries

$$[M_d(gy)]_{\beta,\gamma} = \sum_\alpha g_\alpha y_{\alpha+\beta+\gamma}, \quad |\alpha| + |\beta| + |\gamma| \leq 2d$$

Let $K = \{x \in \mathbb{R}^n : g_k(x) \geq 0, \forall k\}$ be compact basic semialgebraic with $\{x : g_k(x) \geq 0\}$ compact for some k

Necessary condition: if y has a representing measure μ with support in K, then $M_d(y) \succeq 0$, $M_d(g_k y) \succeq 0$ $\forall k$ $\forall d$

Sufficient condition (Putinar 1993): if $M_d(y) \succeq 0$, $M_d(g_k y) \succeq 0$ $\forall k$ $\forall d$ then y has a representing measure on K

SDP formulation

Define the finite-dimensional convex semialgebraic set

$$K_d = \{y \in \mathbb{R}^{r_d} \ : \ M_d(y) \succeq 0, \ M_d(g_k y) \succeq 0, \ k = 1, \ldots, m\}$$

in the r_d-dimensional space of moments up to degree $2d$

Theorem (Lasserre 2001): Sequence y has a representing measure μ supported in K if and only if $K_d \neq \emptyset \ \forall d$

Application in global polynomial optimization

Given a polynomial $g(x) = \sum_\alpha g_\alpha x^\alpha$ and a sequence y define $L(gy) = \sum_\alpha g_\alpha y_\alpha$ as a linear function of g

Consider the nonconvex polynomial optimization problem

$$\begin{array}{rcl} g^* & = & \min_x \;\; g_0(x) \\ & & \quad \text{s.t.} \;\; x \in K = \{x \in \mathbb{R}^n \,:\, g_k(x) \geq 0,\; k = 1,\ldots,m\} \end{array}$$

and the corresponding hierarchy of convex SDP relaxations

$$\begin{array}{rcl} g_d^* & = & \min_y \;\; L(g_0 y) \\ & & \quad \text{s.t.} \;\; y \in K_d = \{y \in \mathbb{R}^{r_d} \,:\, M_d(y) \succeq 0, \\ & & \qquad\qquad\qquad\qquad M_d(g_k y) \succeq 0,\; k = 1,\ldots,m\} \end{array}$$

Theorem (Lasserre 2001): $g_1^* \leq g_2^* \leq \cdots g_\infty^* = g^*$

Use measure supported on variety of globally optimal solutions

Shape reconstruction

For polynomial optimization, polynomials g_k are given and moments y are unknown

$$\begin{array}{rl} \min_y & L(g_0 y) \\ \text{s.t.} & y \in K_d = \{y \in \mathbb{R}^{r_d} \ : \ M_d(y) \succeq 0, \\ & \qquad M_d(g_k y) \succeq 0, \ k = 1, \ldots, m\} \end{array}$$

For shape reconstruction, moments y are given and polynomials g_k are unknown

$$\begin{array}{rl} \min_g & L(g_0 y) \\ \text{s.t.} & y \in K_d = \{y \in \mathbb{R}^{r_d} \ : \ M_d(y) \succeq 0, \\ & \qquad M_d(g_k y) \succeq 0, \ k = 1, \ldots, m\} \end{array}$$

Shape reconstruction

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$$\begin{array}{rl} \min_y & L(g_0 y) \\ \text{s.t.} & y \in K_d = \{y \in \mathbb{R}^{r_d} \;:\; M_d(y) \succeq 0, \\ & \qquad\qquad\qquad M_d(g_k y) \succeq 0,\; k = 1,\ldots,m\} \end{array}$$

For shape reconstruction, moments y are given and polynomials g_k are unknown

$$\begin{array}{rl} \min_g & L(g_0 y) \\ \text{s.t.} & M_d(g_k y) \succeq 0,\; k = 1,\ldots,m \end{array}$$

Reconstruction of a polynomial sublevel set

In the simplest case

$$K = \{x \in \mathbb{R}^n \,:\, f(x) = \sum_\alpha f_\alpha x^\alpha \geq 0\}$$

is a single polynomial sublevel set

K is the union of the closure of connected components of open set $\{x \,:\, f(x) > 0\}$ and polynomial f vanishes along ∂K

Assume K is compact

Our inverse problem: given y, find f

Our solution: hierarchy of SDP problems whose sequence of optimal solutions converges to f

Hierarchy of SDP problems

Given $x_0 \in \text{int}\, K$, we can enforce the normalization constraint

$$g(x_0) = \sum_\alpha g_\alpha x_0^\alpha = 1$$

Consider the hierarchy of SDP problems

$$\begin{array}{rrl} g_d^* = & \arg\min_g & L(gy) \\ & \text{s.t.} & M_d(gy) \succeq 0 \\ & & g(x_0) = 1 \end{array}$$

where the unknown are coefficients of a degree $2d$ polynomial g_d^*

Theorem: $K_d^* = \{x \,:\, g_d^*(x) \geq 0\} \subset K = \{x \,:\, f(x) \geq 0\}$

Theorem: $\lim_{d\to\infty} g_d^* = f$

Duality

The primal SDP problem reads

$$
\begin{array}{rrl}
g_d^* = & \min & \sum_\alpha g_\alpha y_\alpha \\
 & \text{s.t.} & \sum_\alpha g_\alpha M_d(x^\alpha y) \succeq 0 \\
 & & \sum_\alpha g_\alpha x_0^\alpha = 1
\end{array}
$$

and its dual SDP problem reads

$$
\begin{array}{rrl}
\sigma_d^* = & \max & \gamma \\
 & \text{s.t.} & \gamma x_0^\alpha + \int_K x^\alpha \sigma(x) d\mu = \int_K x^\alpha d\mu
\end{array}
$$

with unknown scalar γ and polynomial sum-of-squares $\sigma(x)$

Using conic complementarity, we can prove that

$$\lim_{d \to \infty} \int_K g_d^* \sigma_d^* d\mu = 0$$

and hence that $\sigma_d^*(x)$ tends to a polynomial non-negative everywhere but vanishing on K

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Interval

Uniform measure on $K = [0,1] \subset \mathbb{R}$ has moments

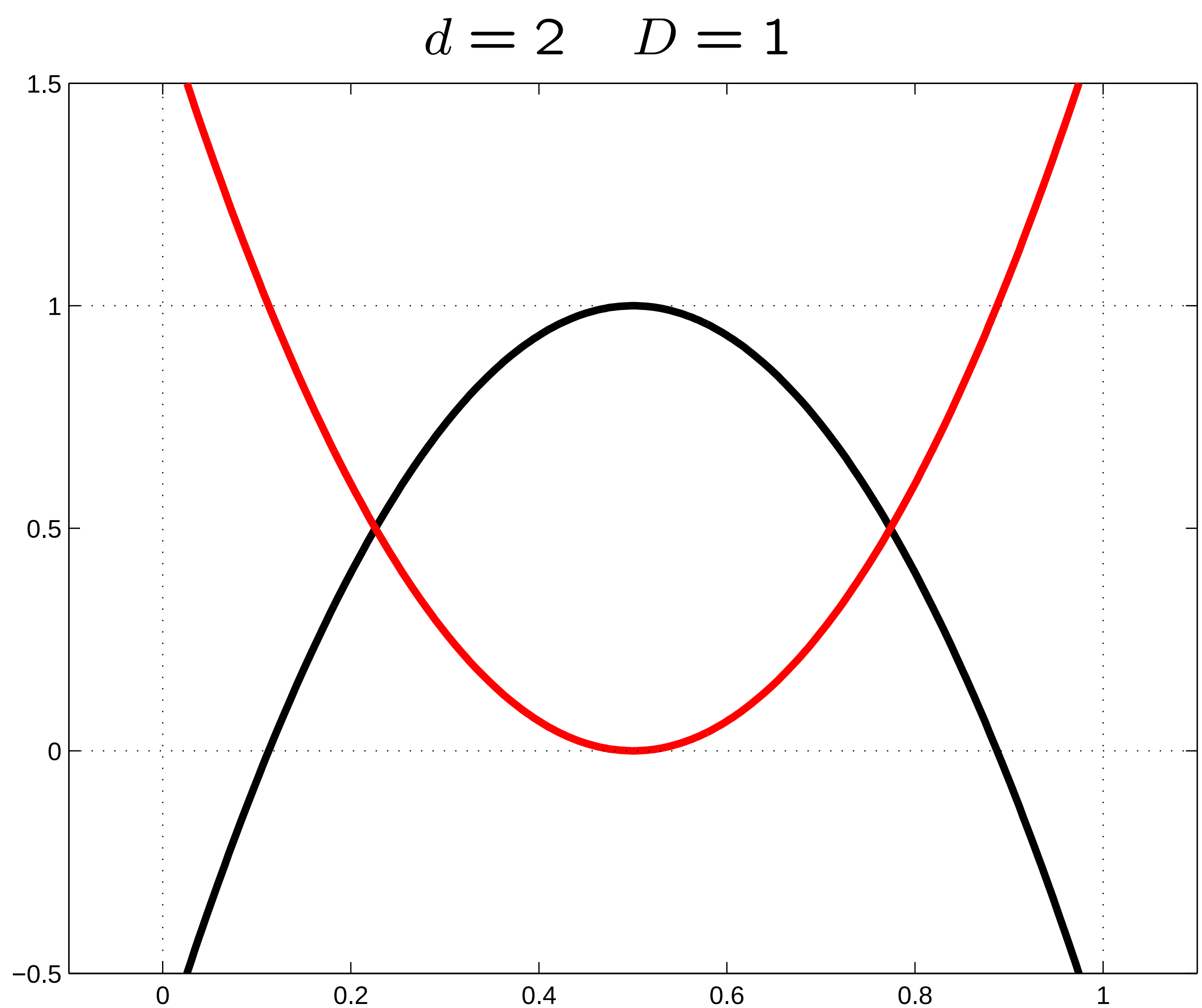
$$y_\alpha = \int_0^1 x^\alpha dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

We try to model K as $K_d^* = \{x \,:\, g_d^*(x) \geq 0\} \subset K$
for increasing values of degree $d = \deg g_d^*(x)$ and
for increasing values of order D in the SDP hierarchy

In black we represent $g_d^*(x)$

In red we represent non-negative polynomial $\sigma_d^*(x)$
which should be as small as possible in K

$$d = 2$$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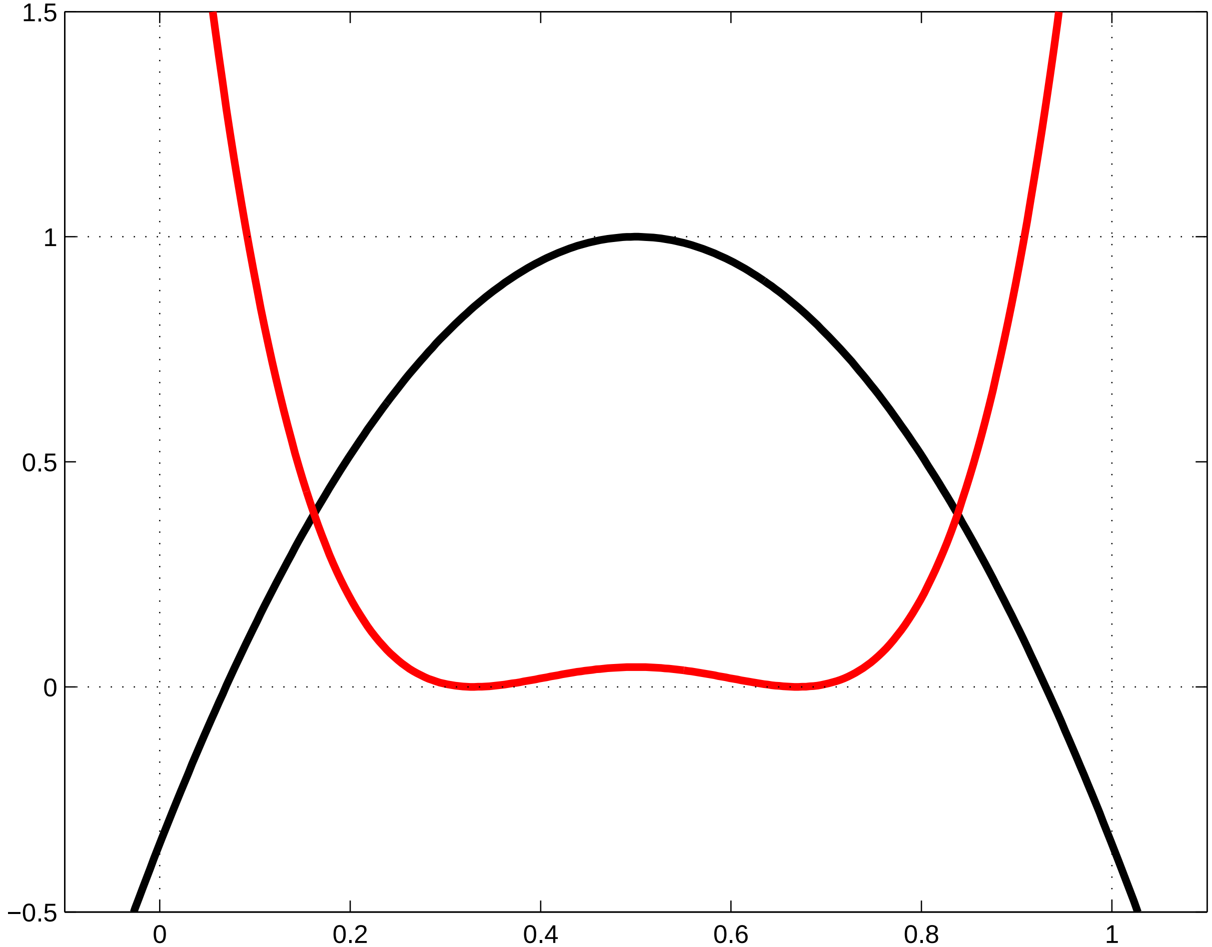
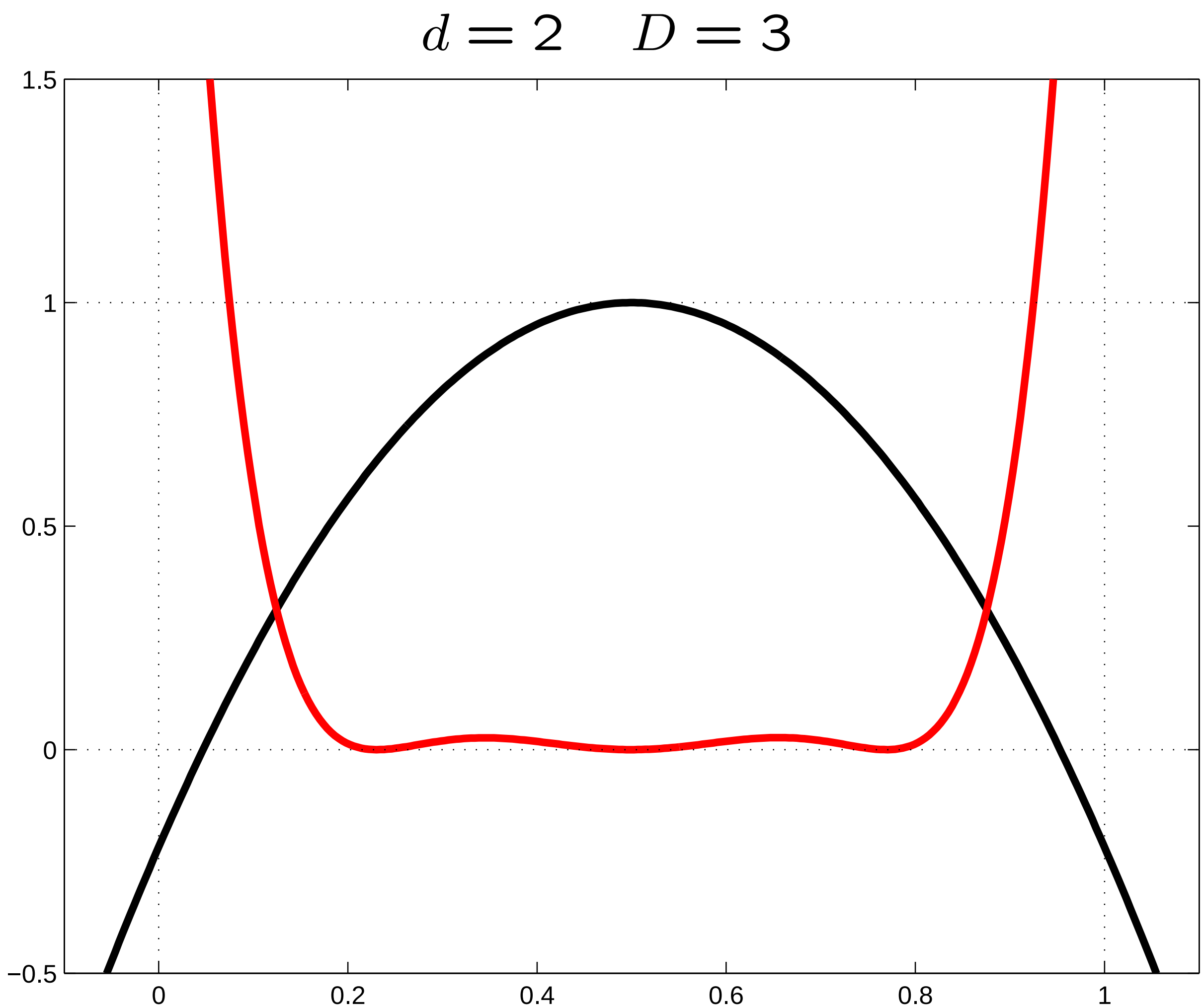
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1

$d = 2 \quad D = 2$

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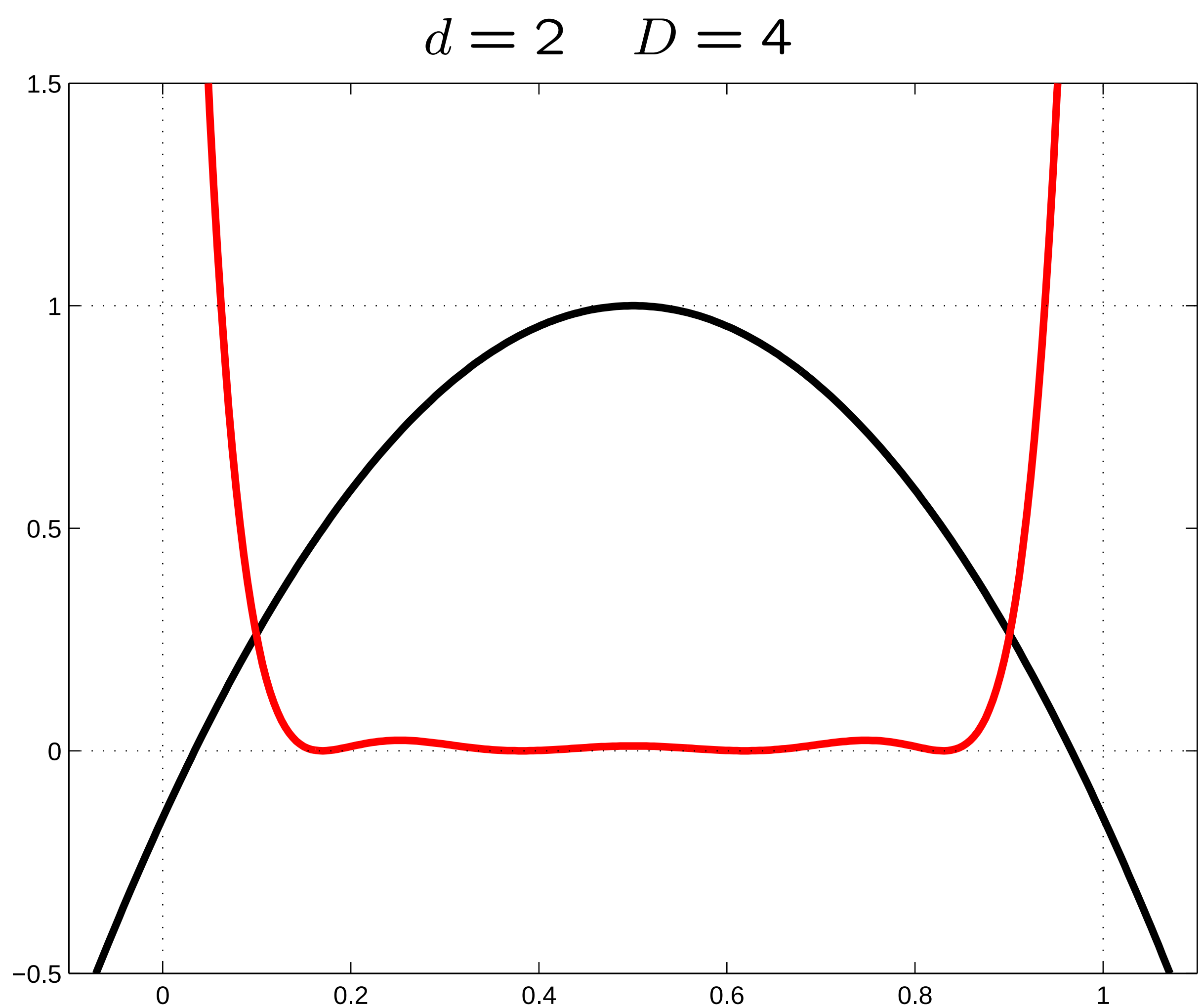
0.2
0.4
0.6
0.8
1



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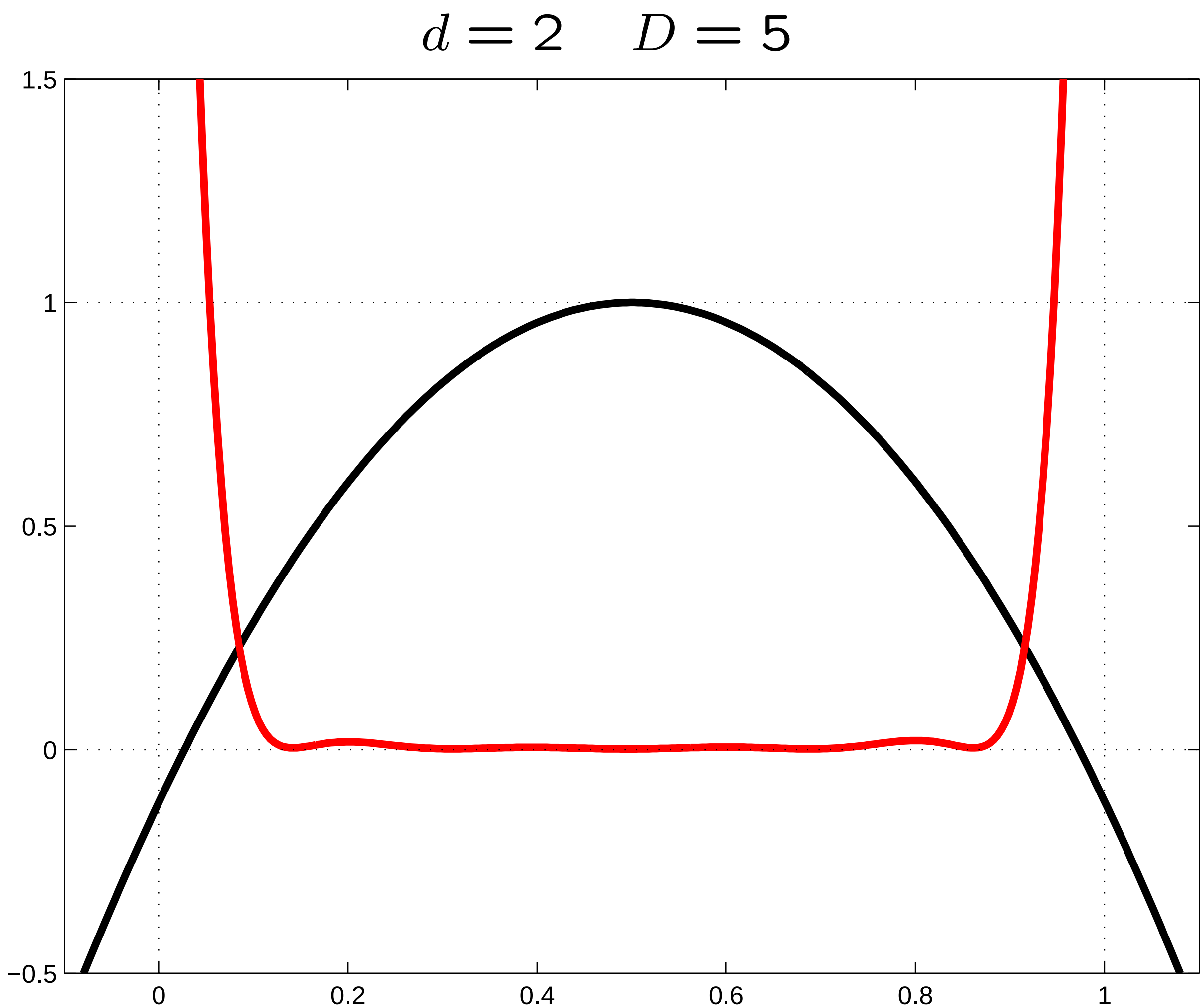
0.2
0.4
0.6
0.8
1



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0.5
0
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0.4
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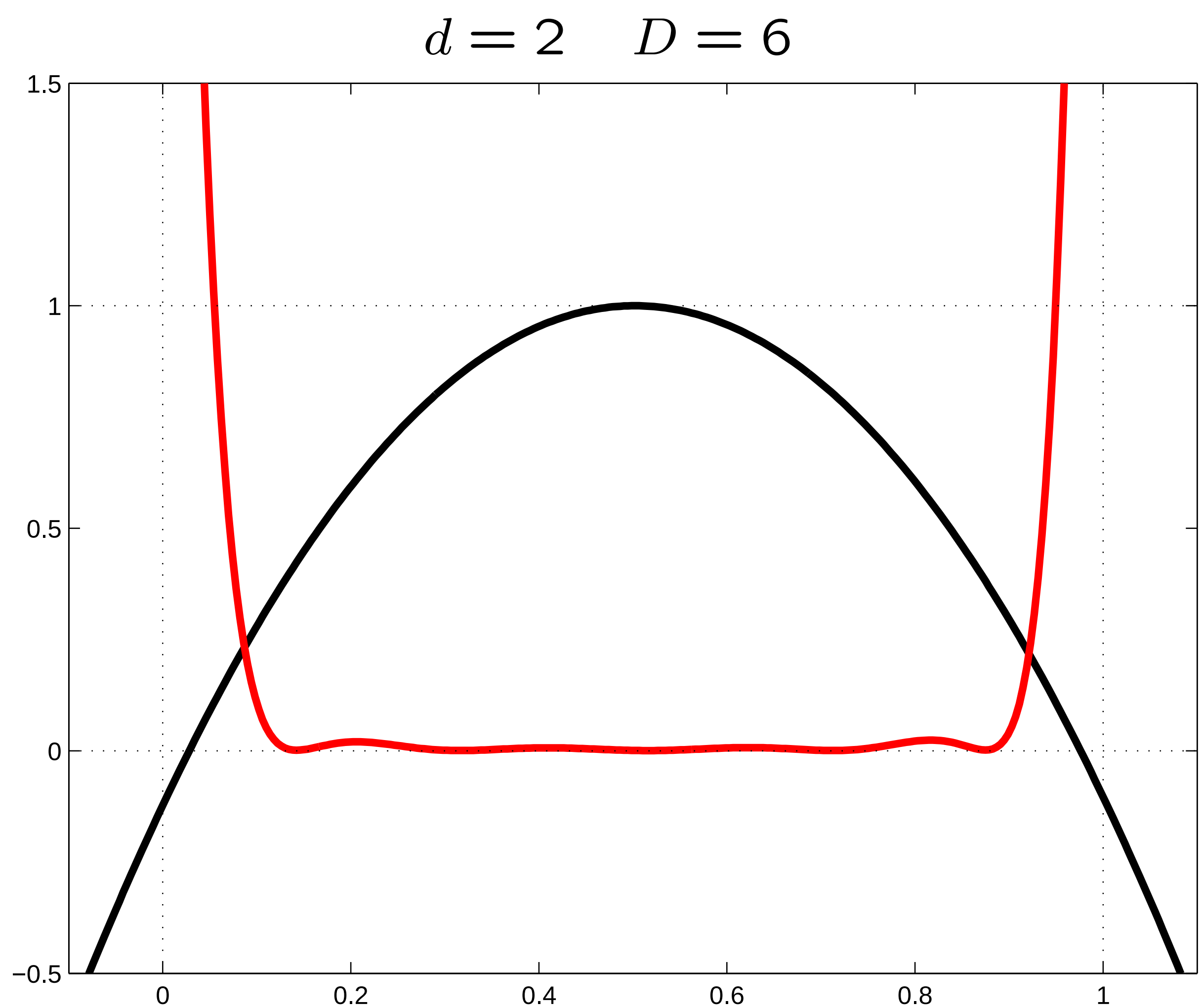
0.8
1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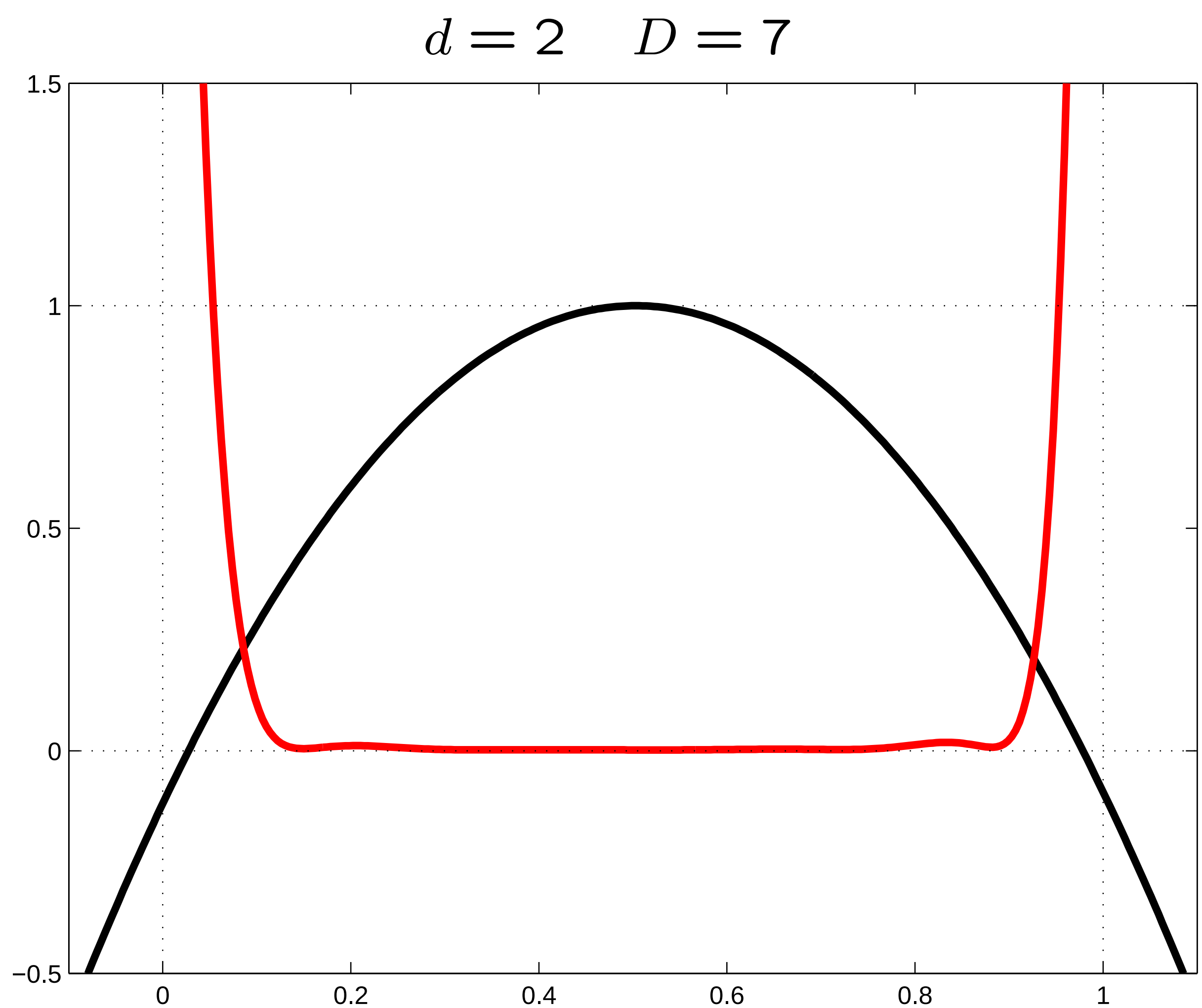
0
−0.5
0
0.2
0.4
0.6
0.8
1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1
0.5
0
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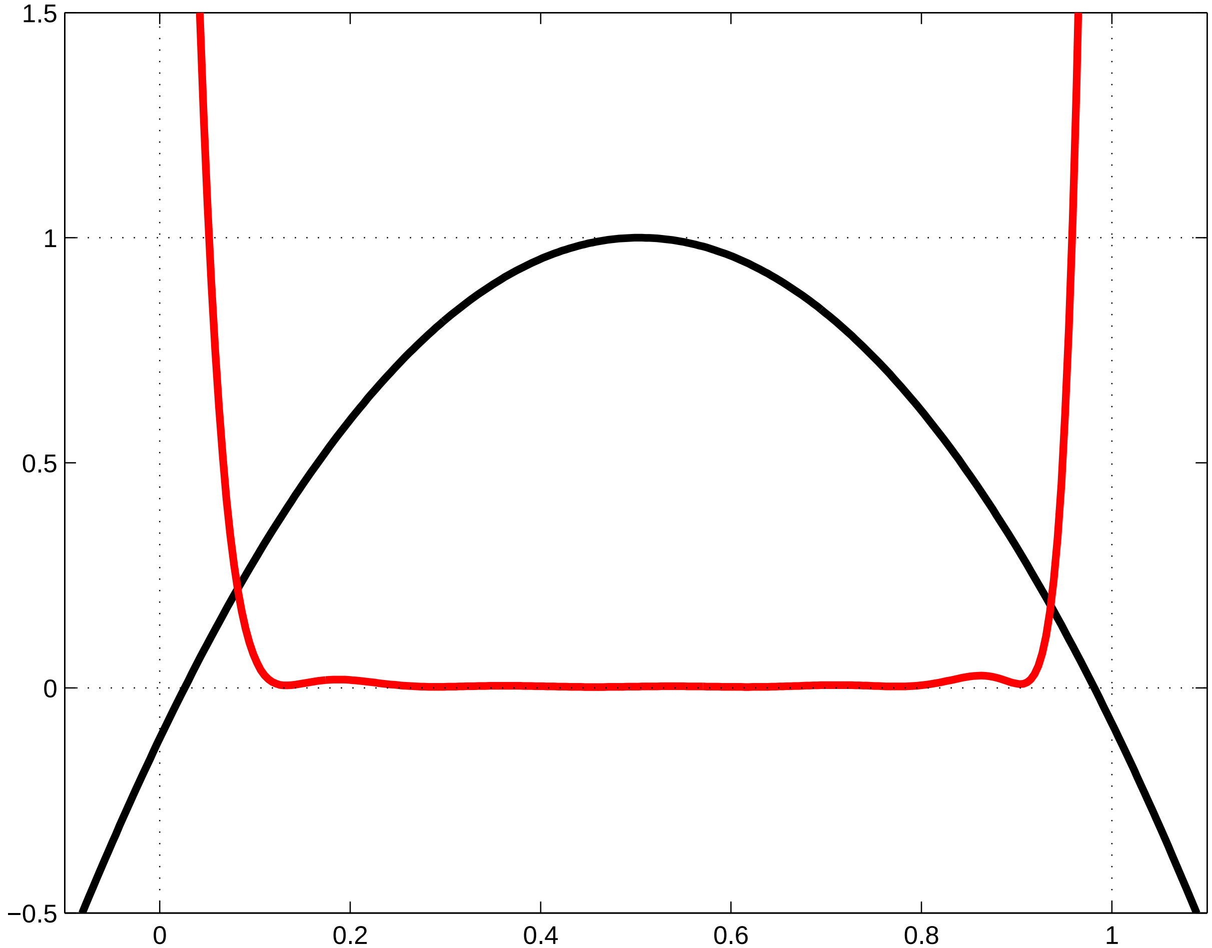
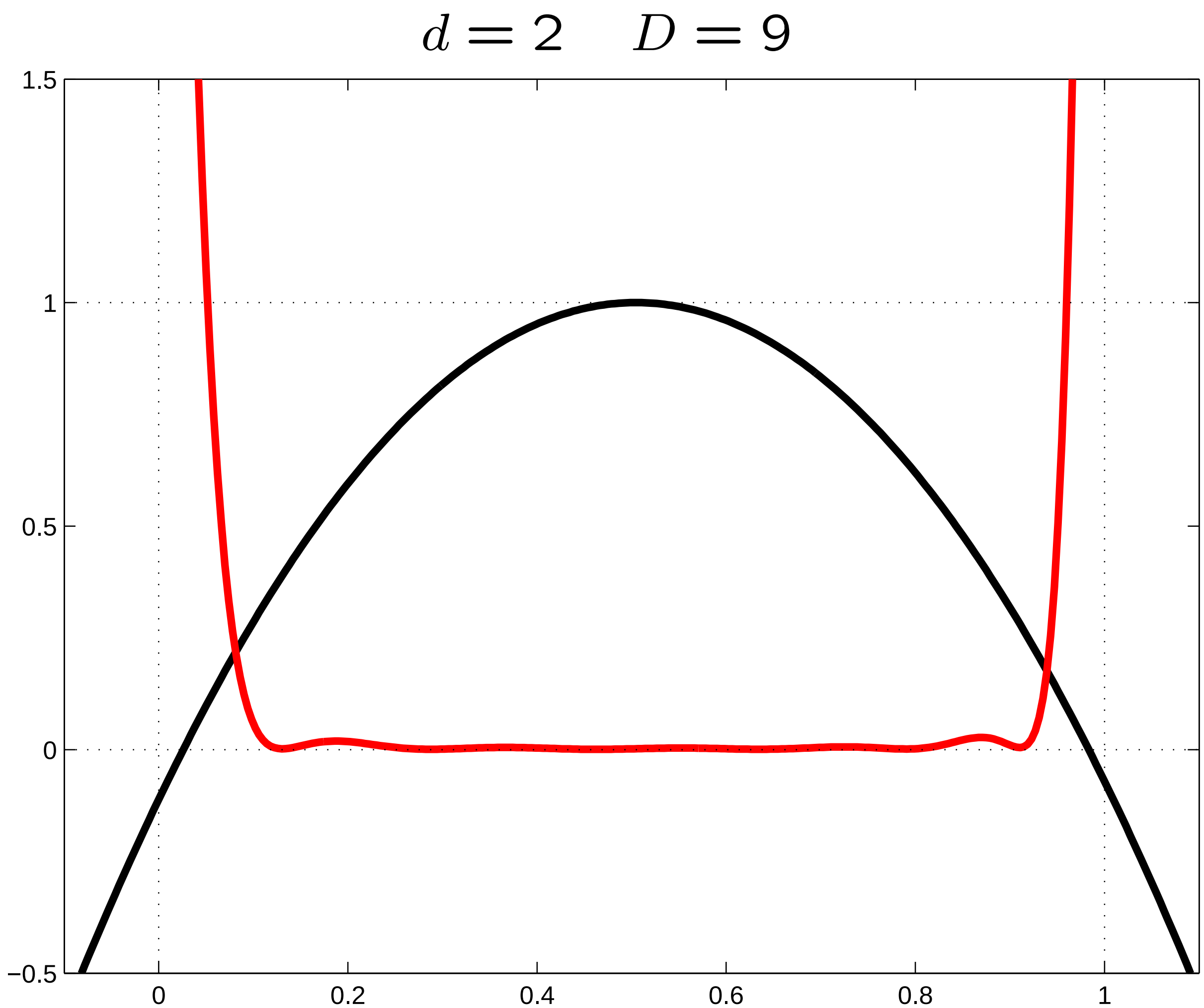
0
0.2
0.4
0.6
0.8
1

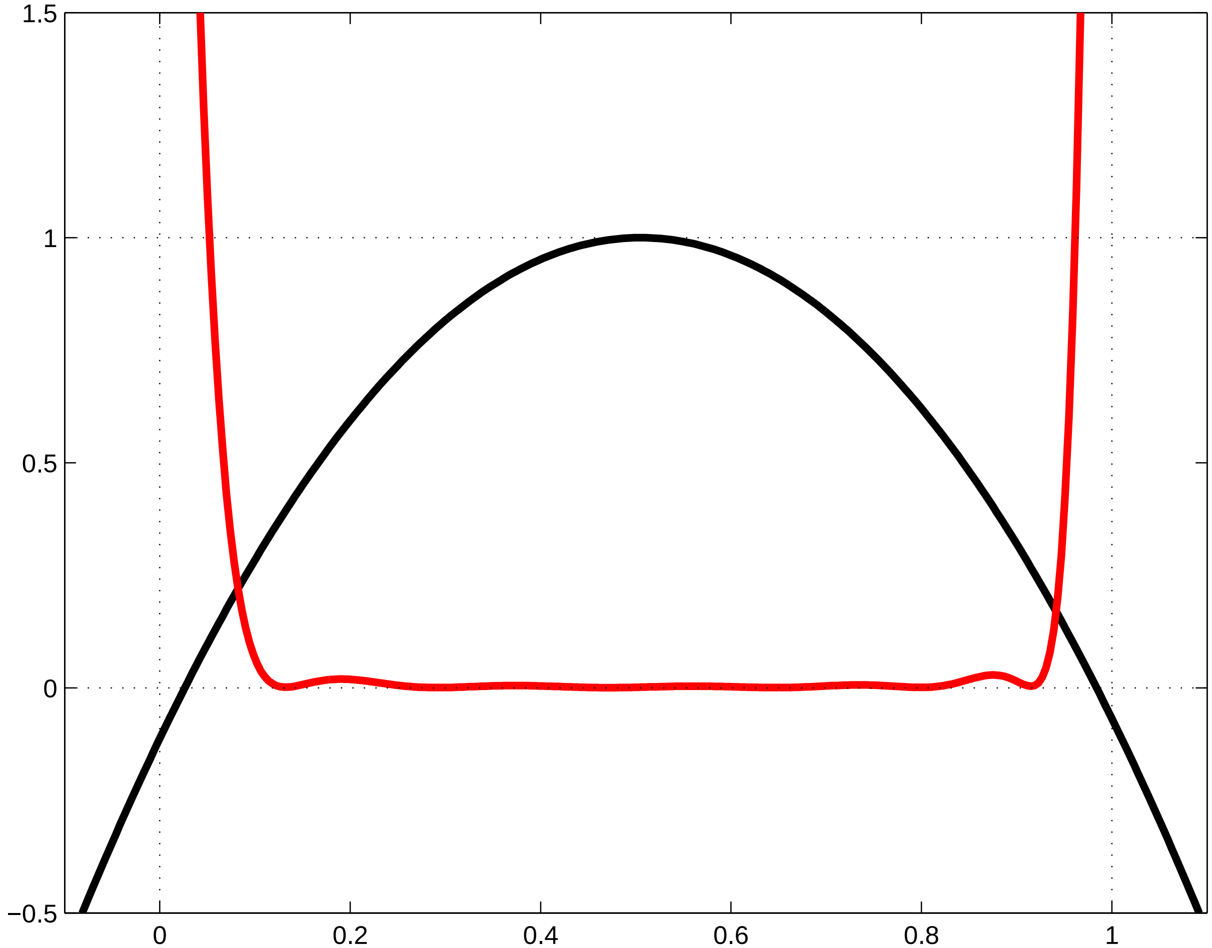
$d = 2 \quad D = 8$



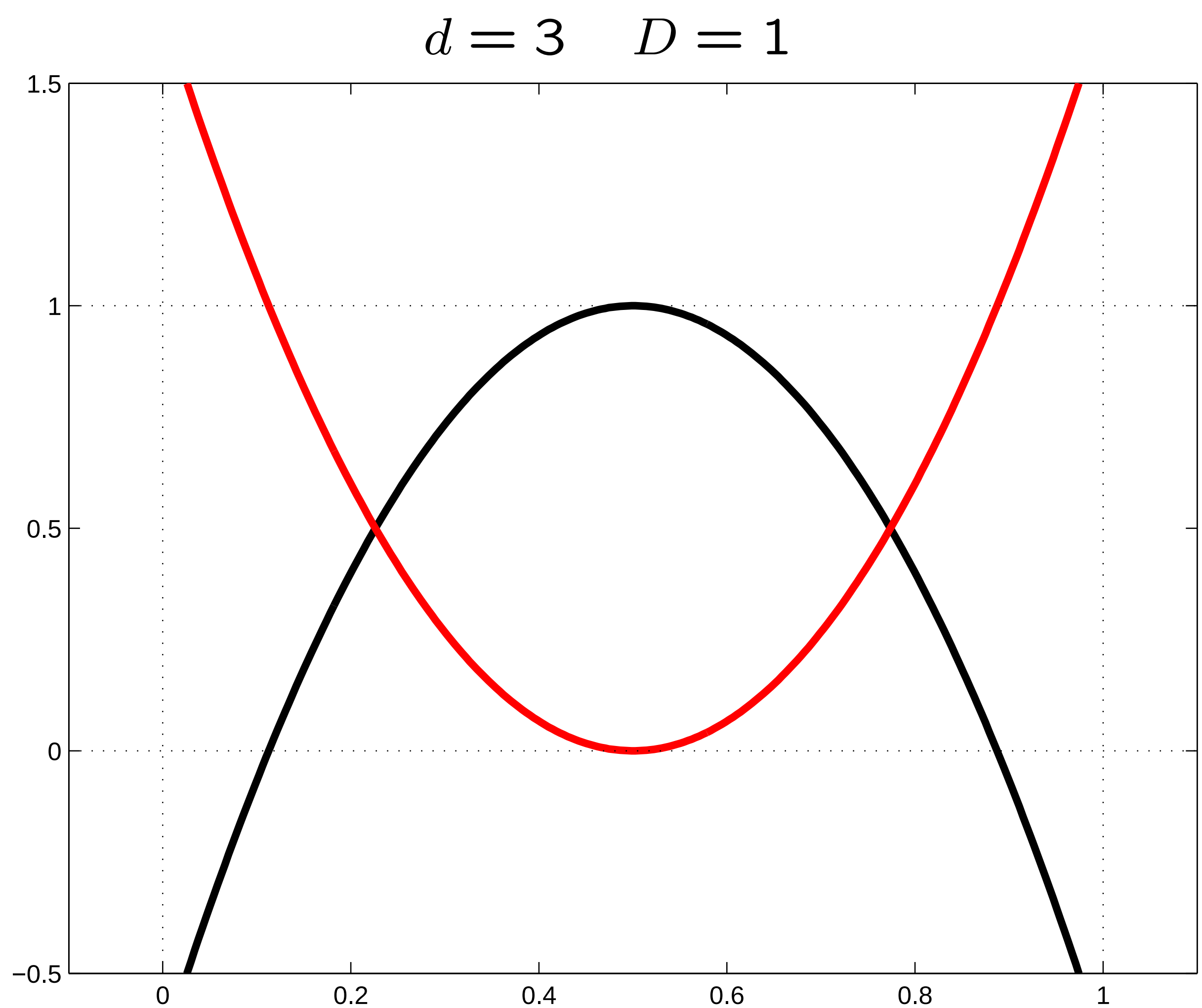


$d = 2 \quad D = 9$
1.5
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1

$d = 2 \quad D = 10$



$$d = 3$$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1
0.5
0
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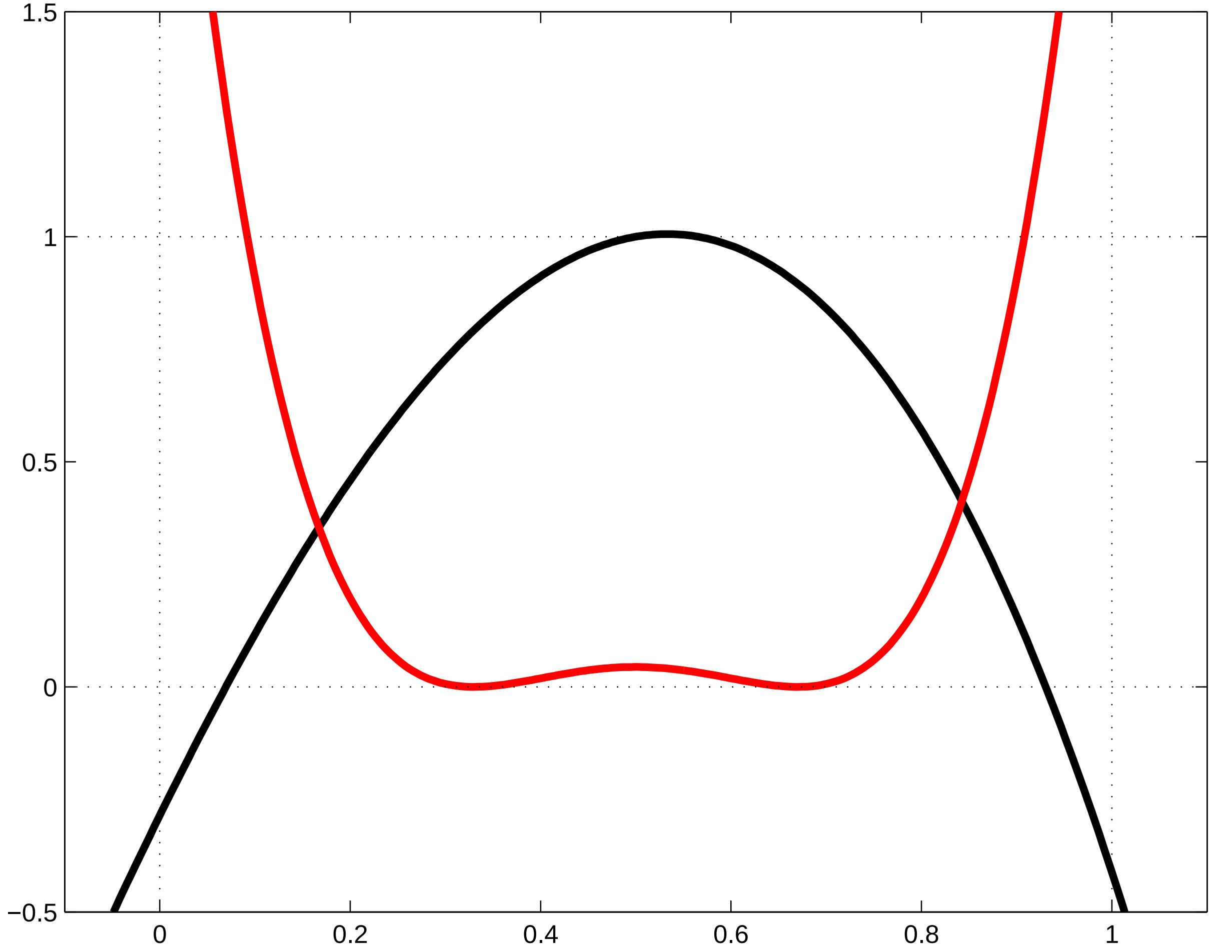
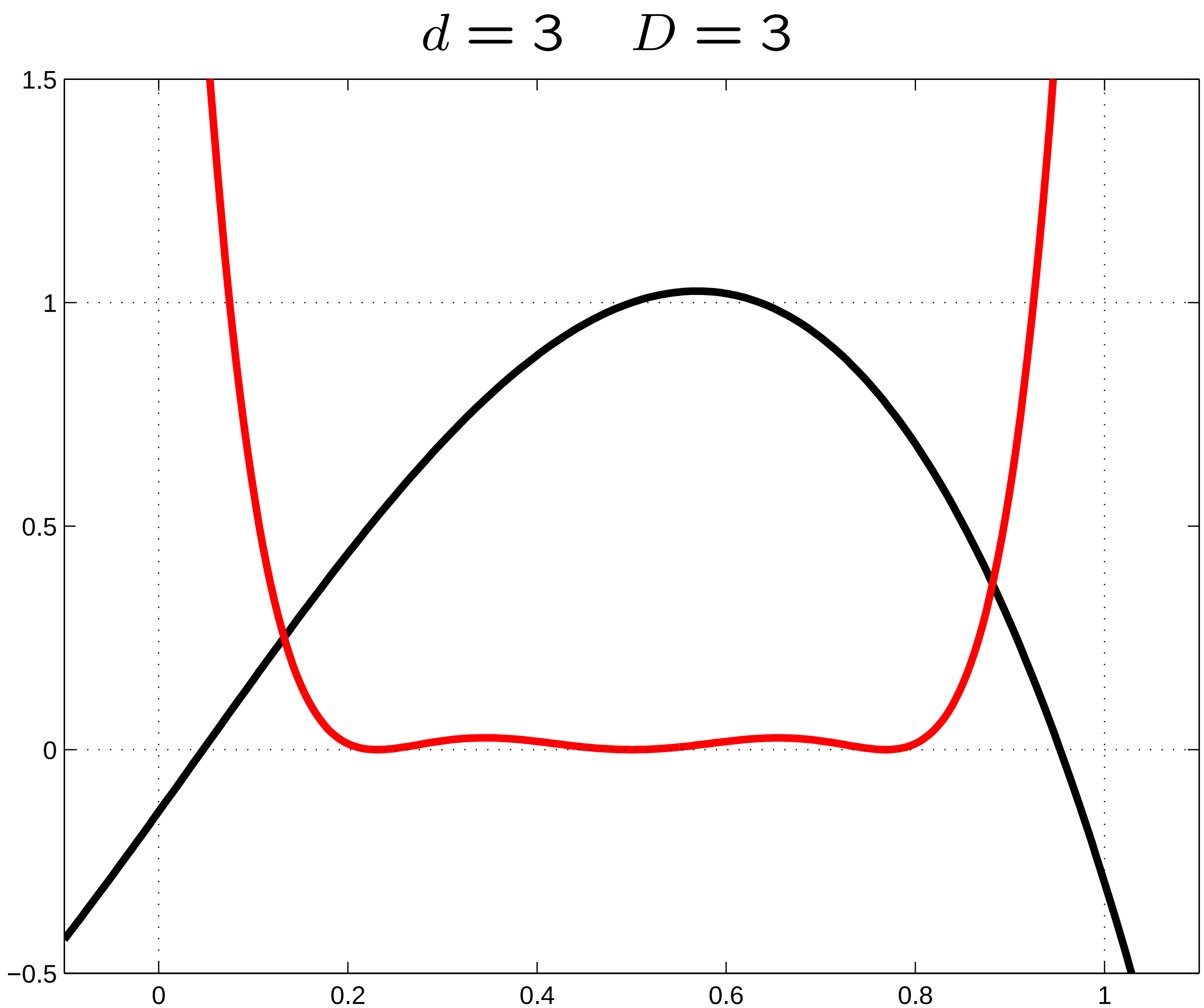
0
0.2
0.4
0.6
0.8
1

$d = 3 \quad D = 2$

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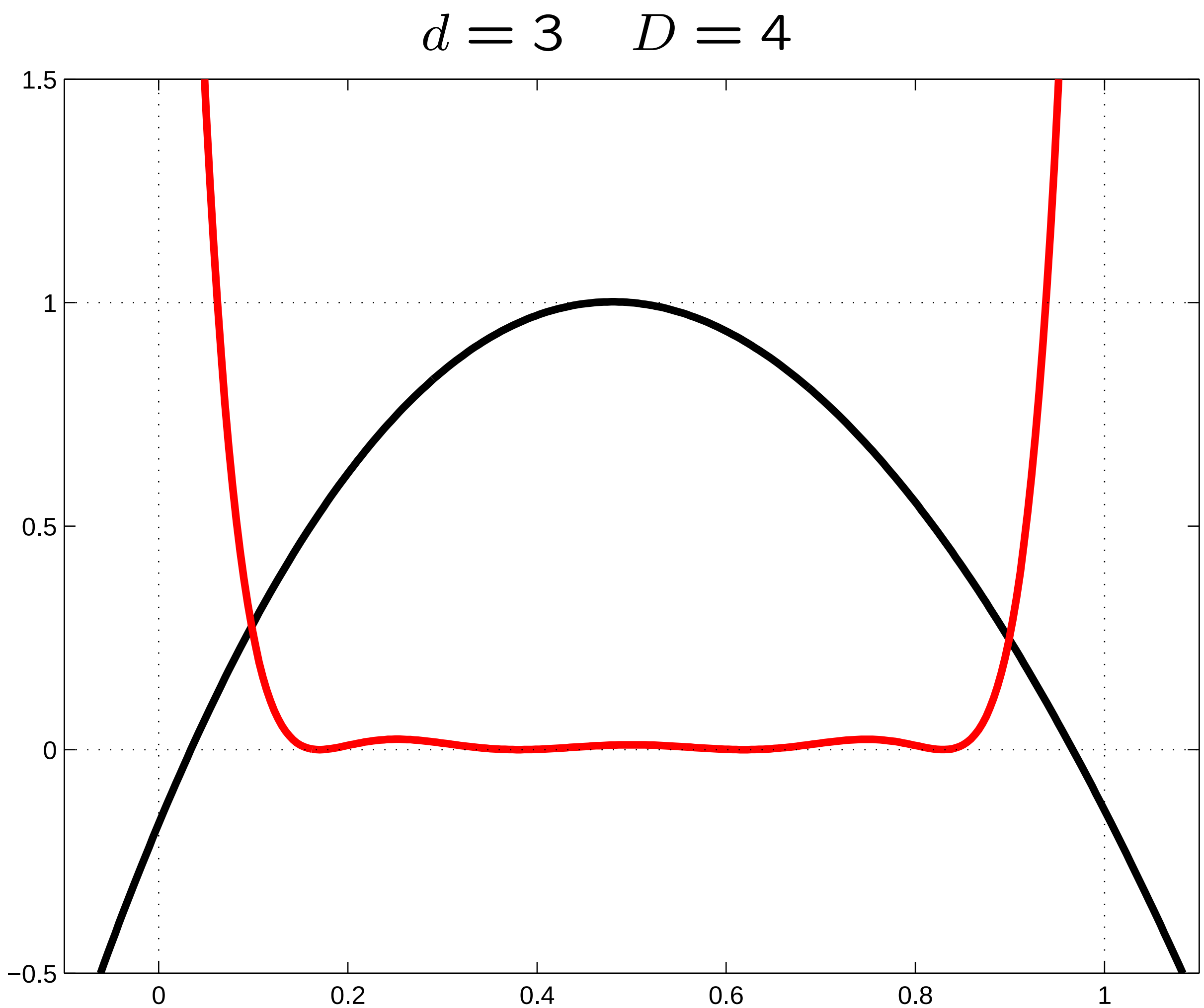
0.2
0.4
0.6
0.8
1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0.4
0.6
0.8
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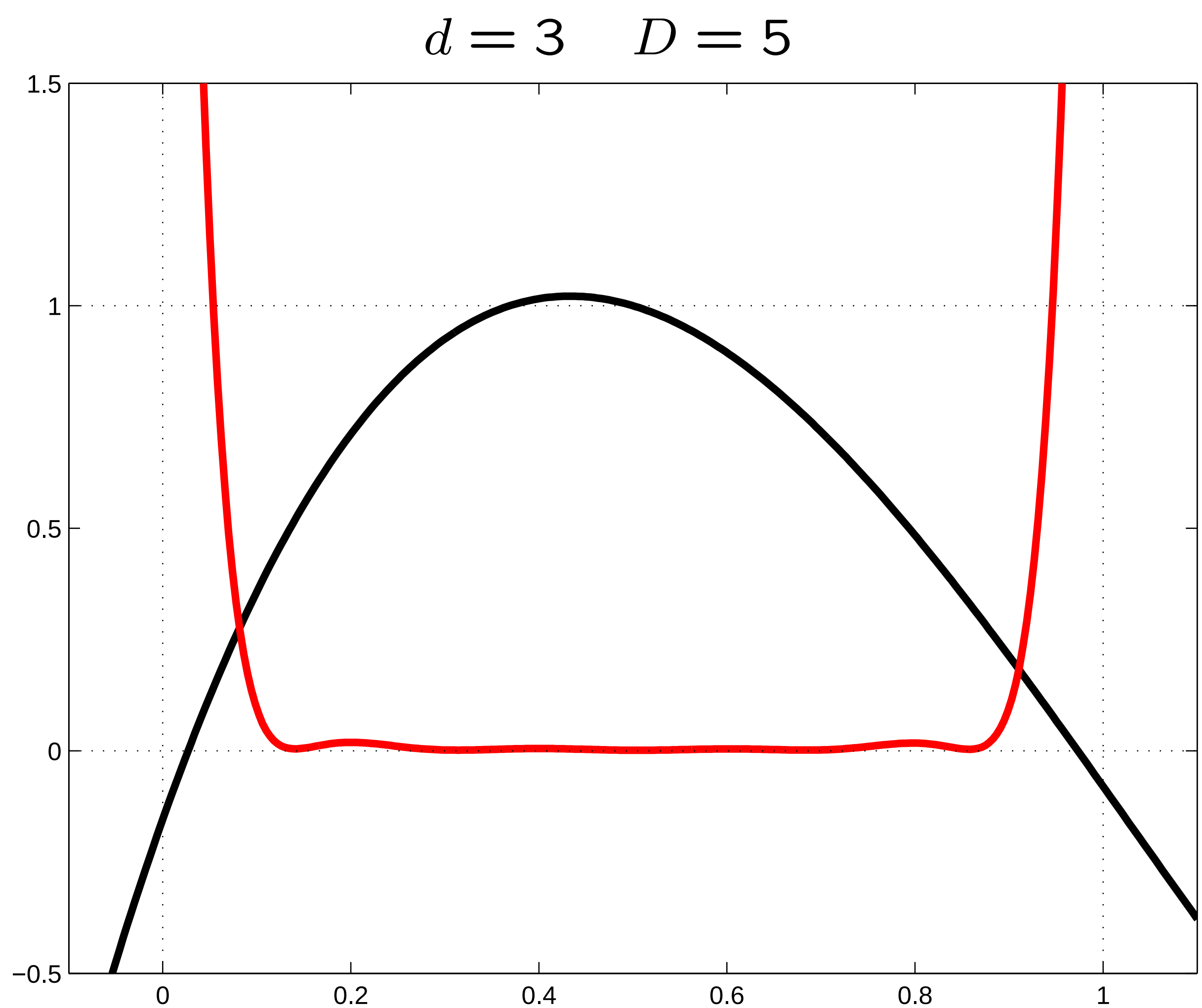

$d = 3 \quad D = 5$
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1
0.5
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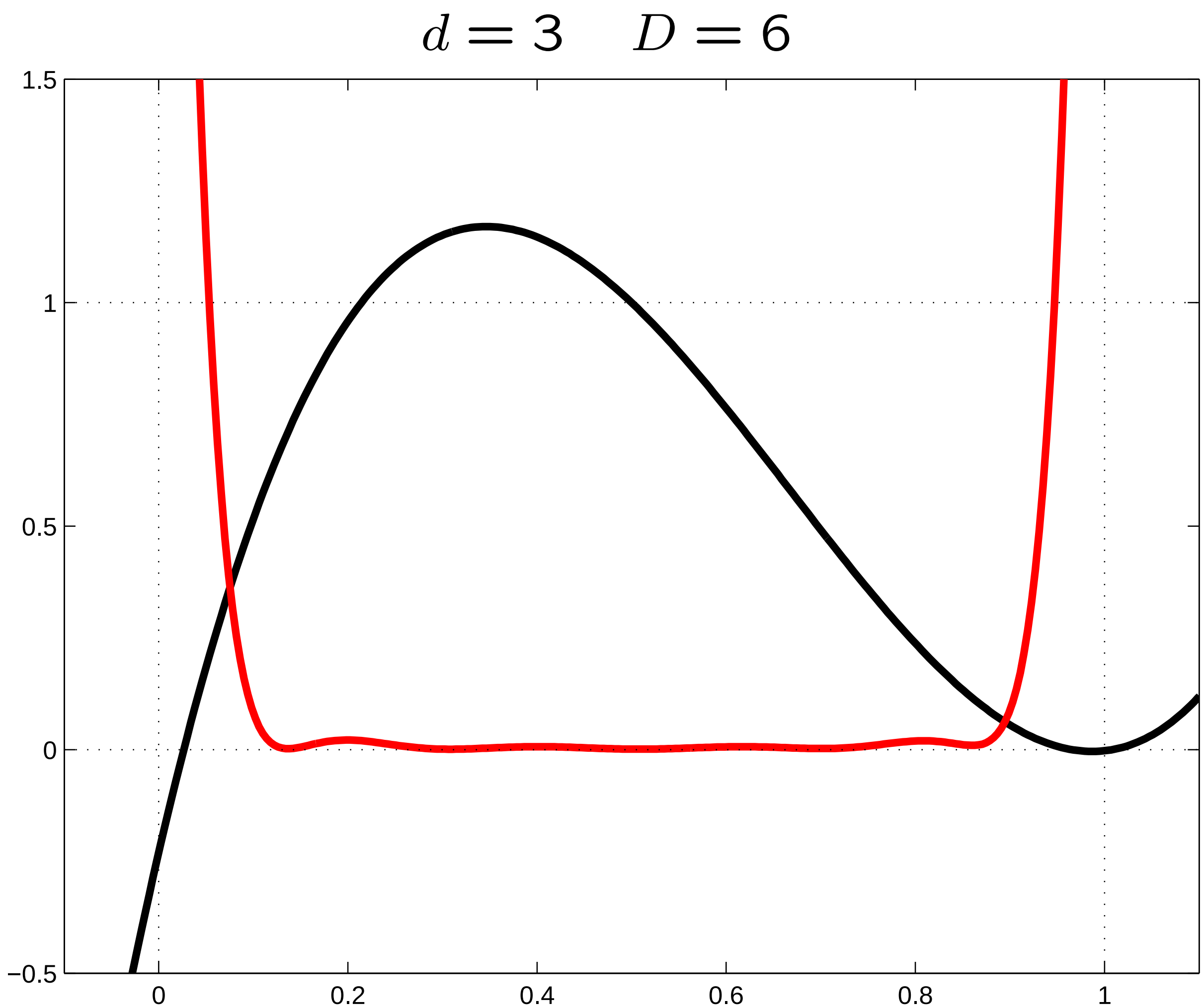
−0.5
0
0.2
0.4
0.6
0.8
1



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1
0.5
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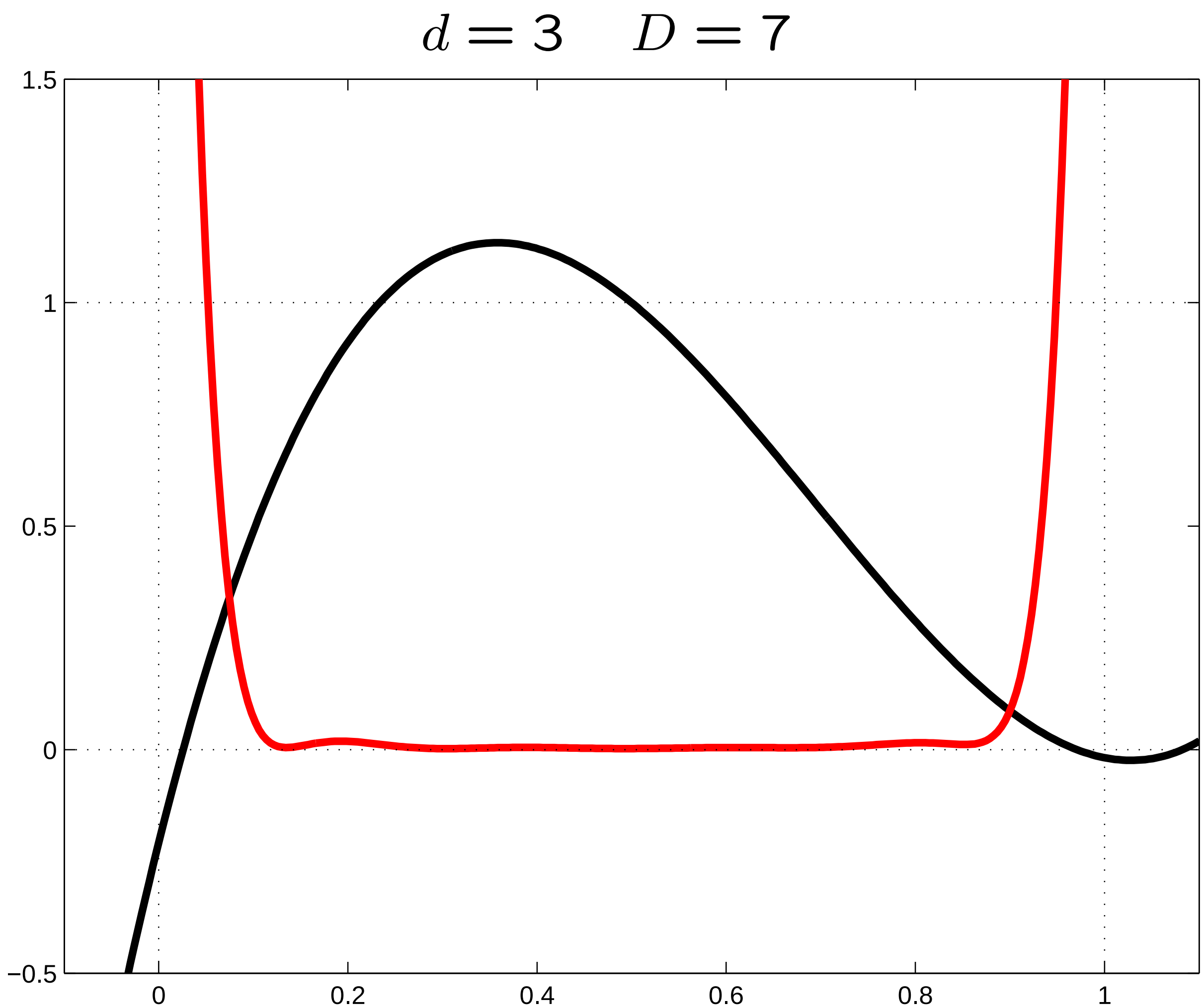
−0.5
0
0.2
0.4
0.6
0.8
1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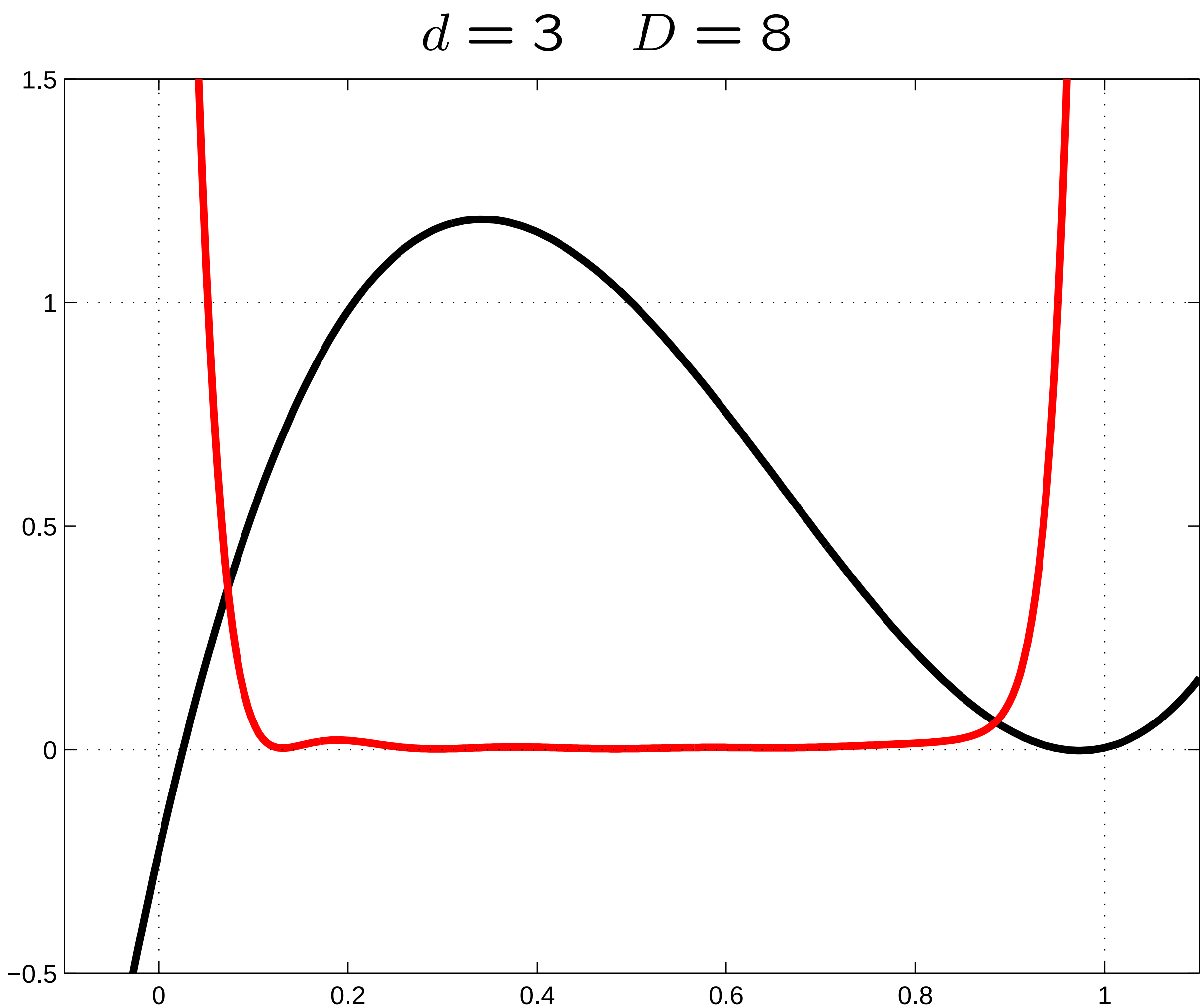
0.4
0.6
0.8
1



$d = 3 \quad D = 8$
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1
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0
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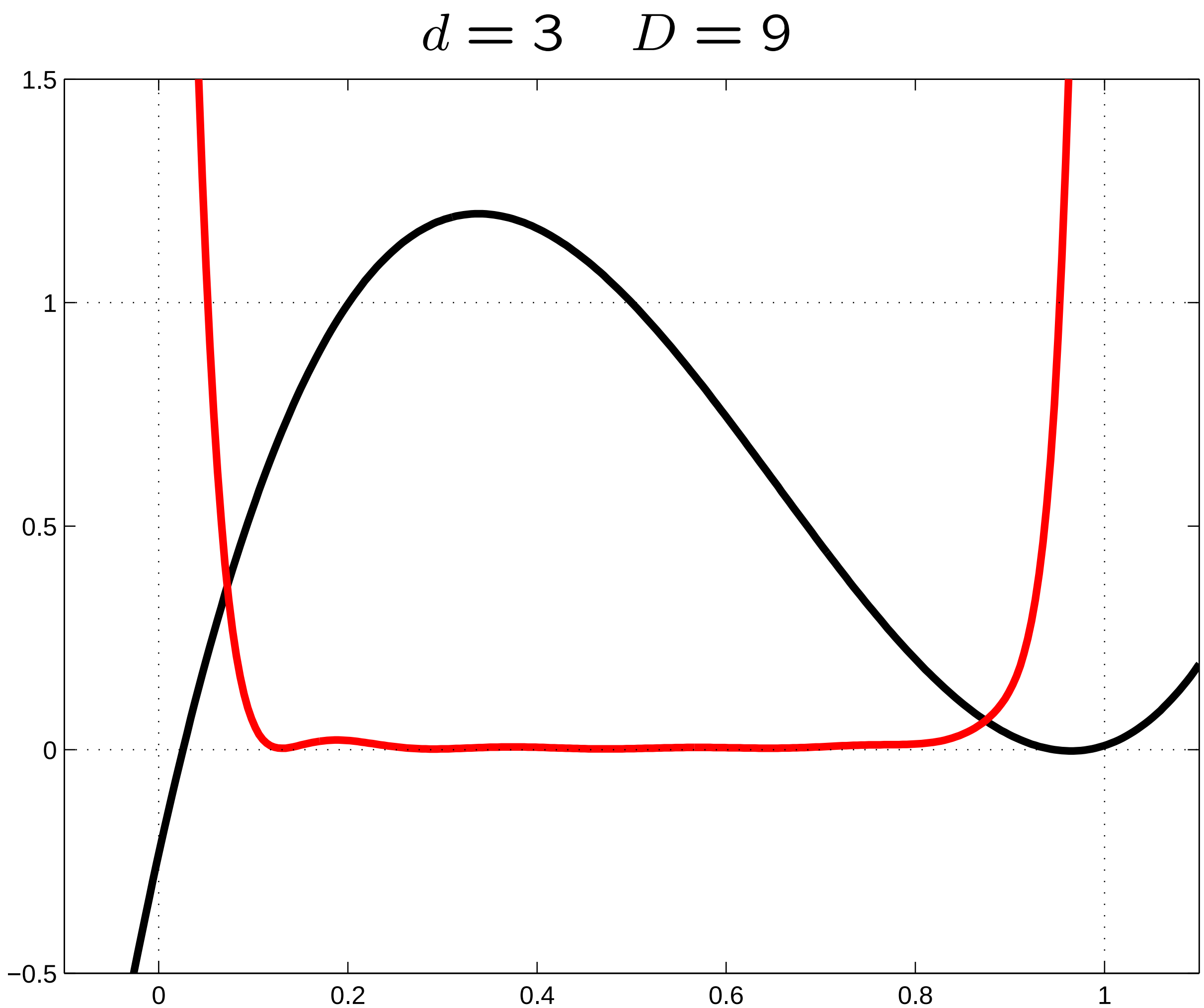
0.2
0.4
0.6
0.8
1



$d = 3 \quad D = 9$
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1
0.5
0
−0.5
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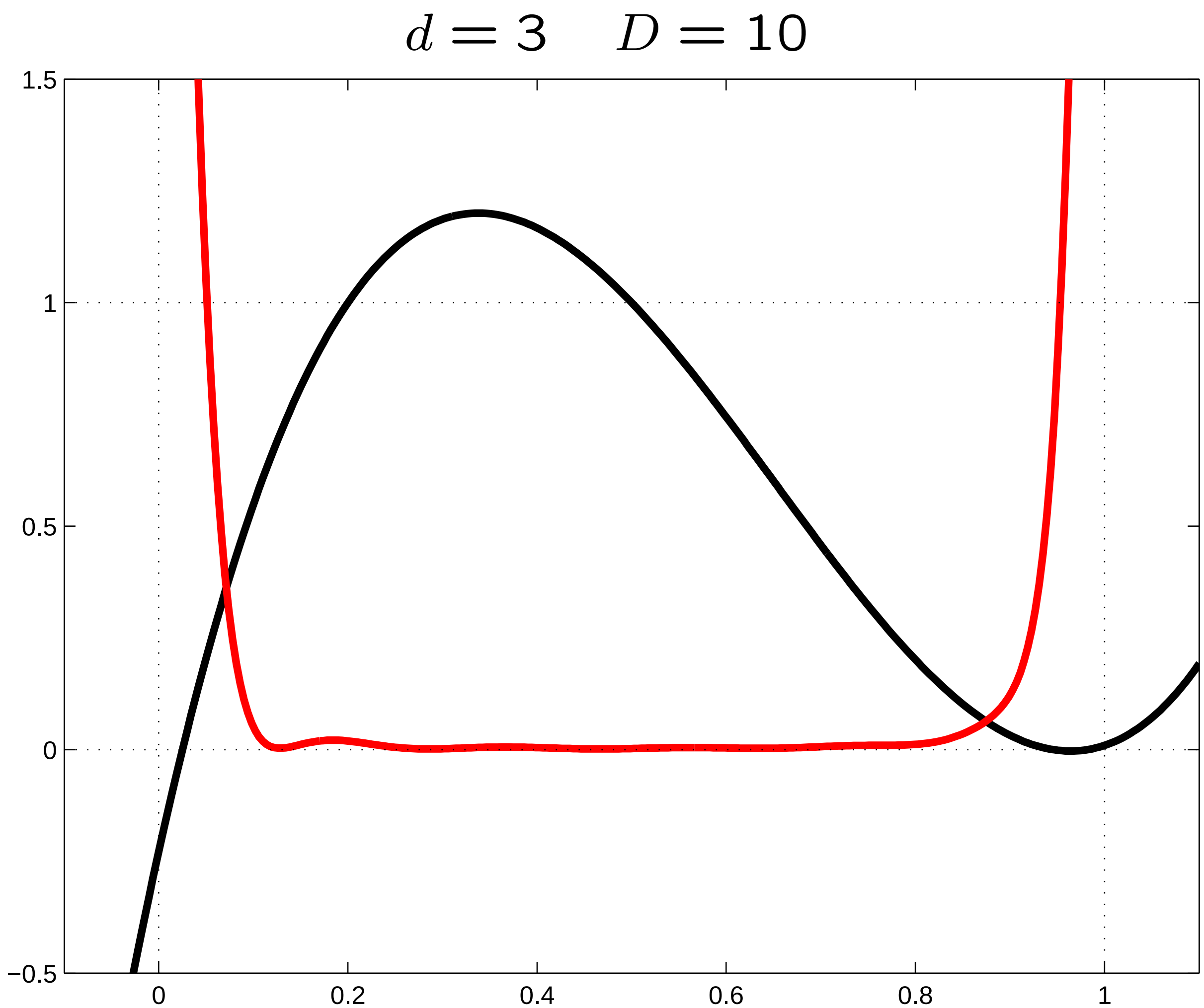
0.2
0.4
0.6
0.8
1



$d = 3 \quad D = 10$
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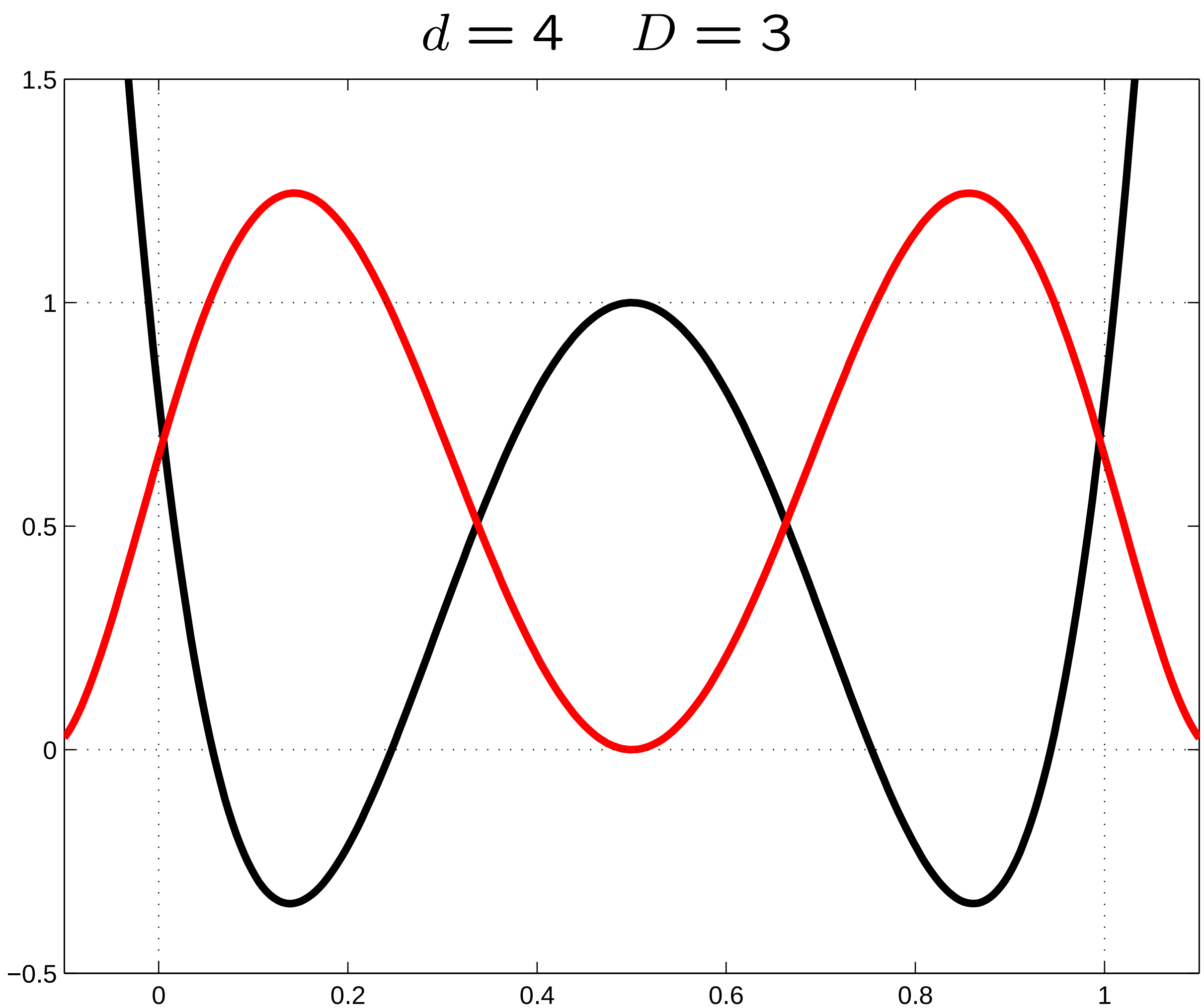
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1

$$d = 4$$



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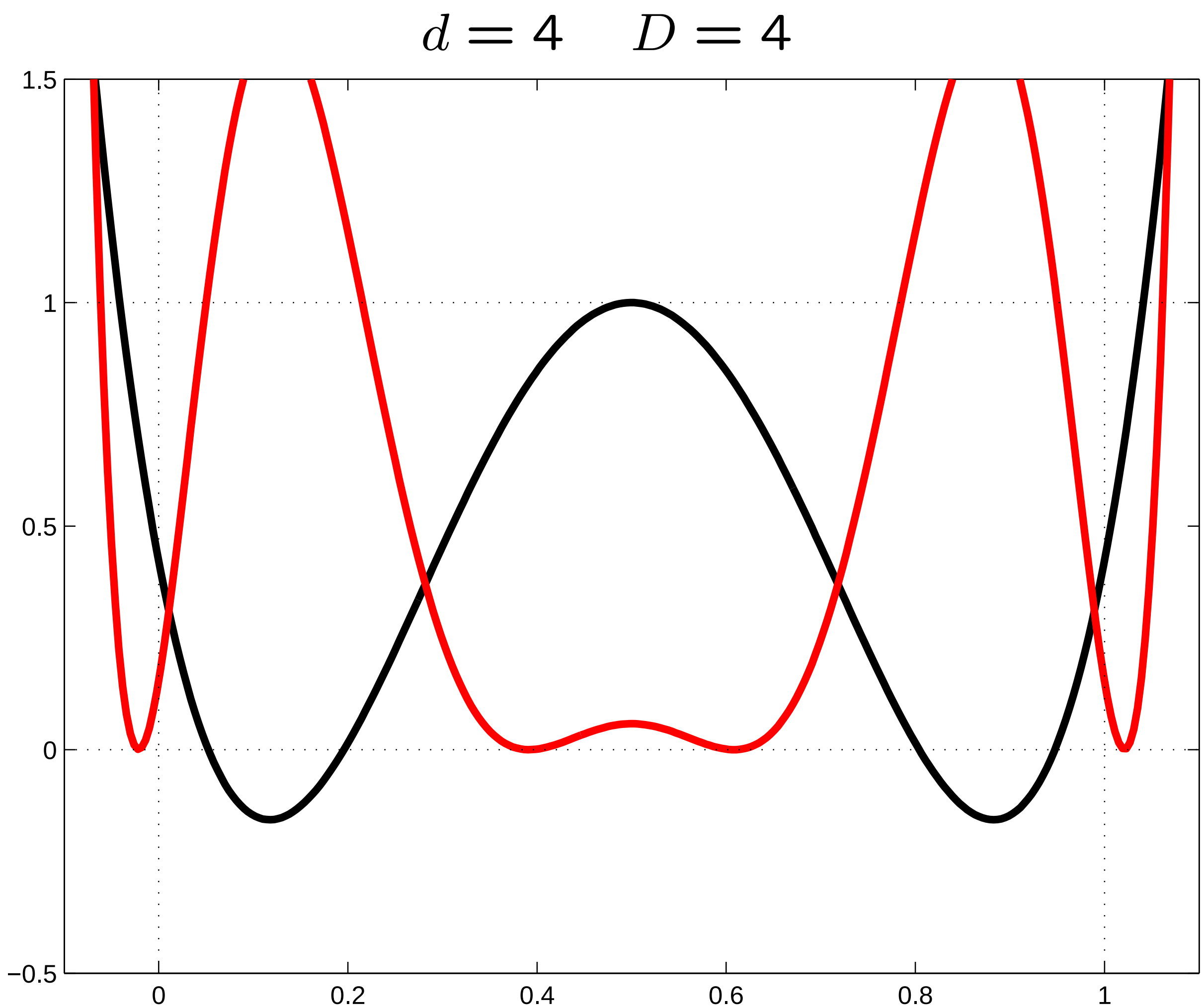
0
−0.5
0
0.2
0.4
0.6
0.8
1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0.2
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0.6
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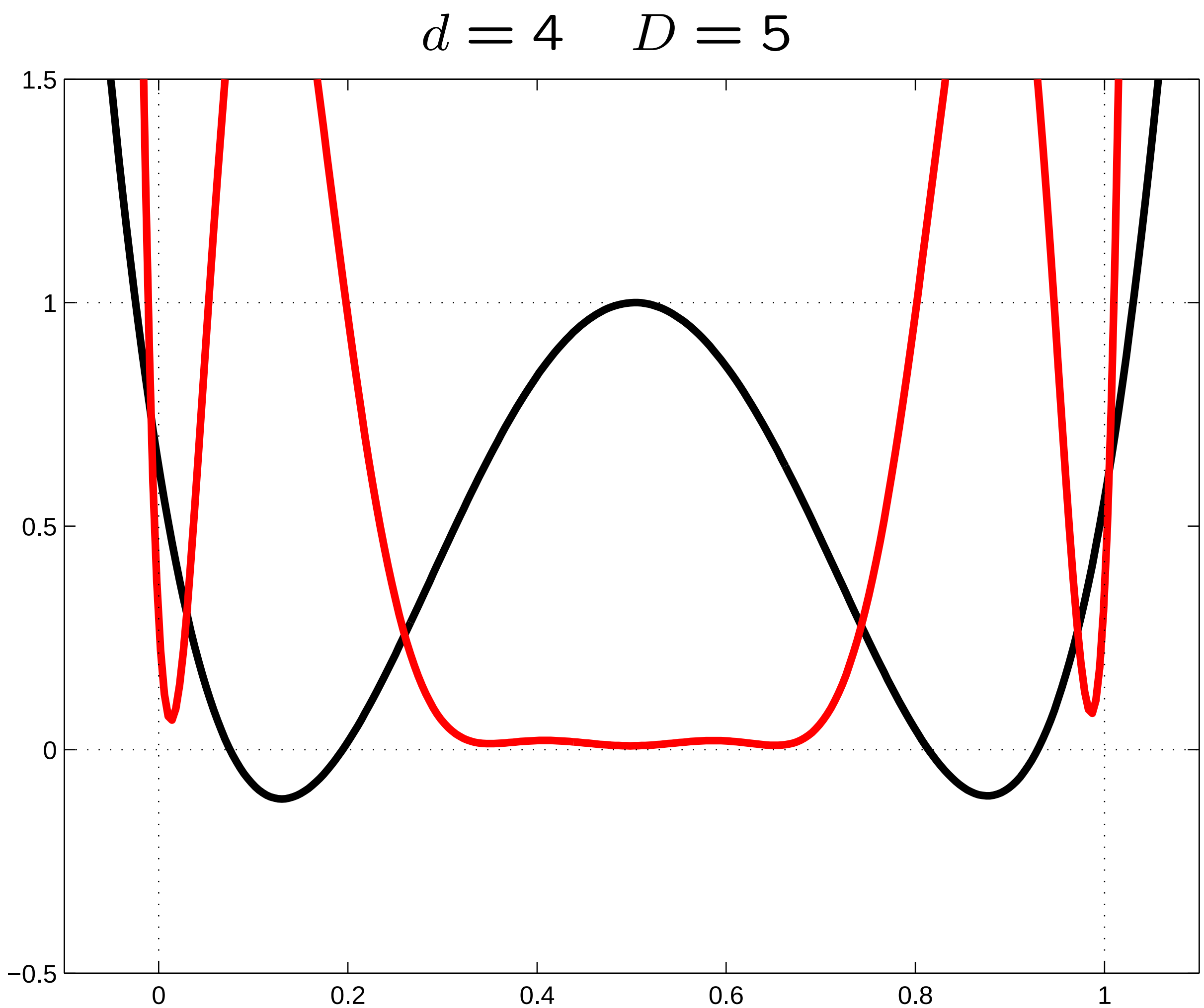
1



$d = 4 \quad D = 5$
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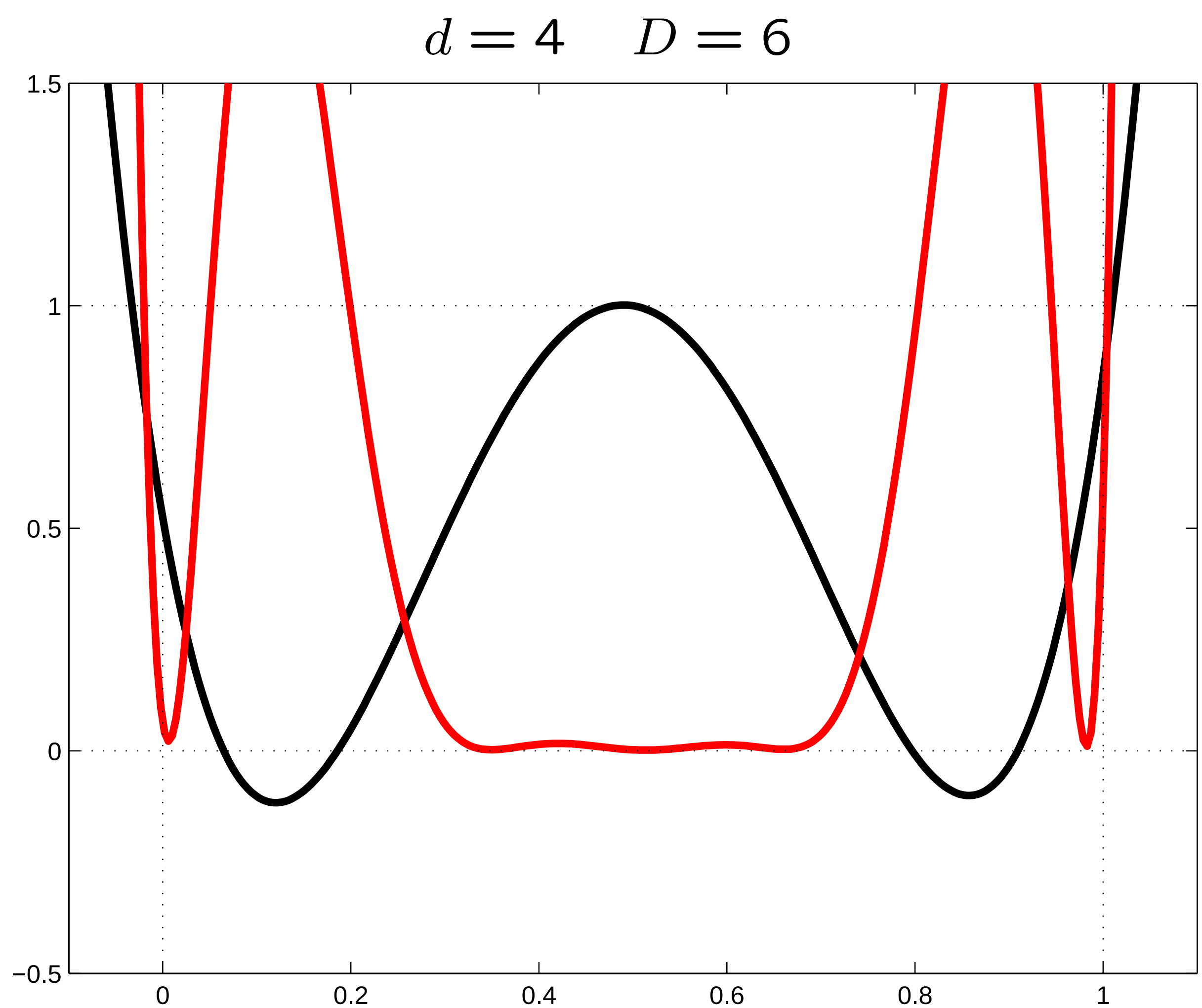
0
0.2
0.4
0.6
0.8
1


$d = 4 \quad D = 6$
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0.5
0
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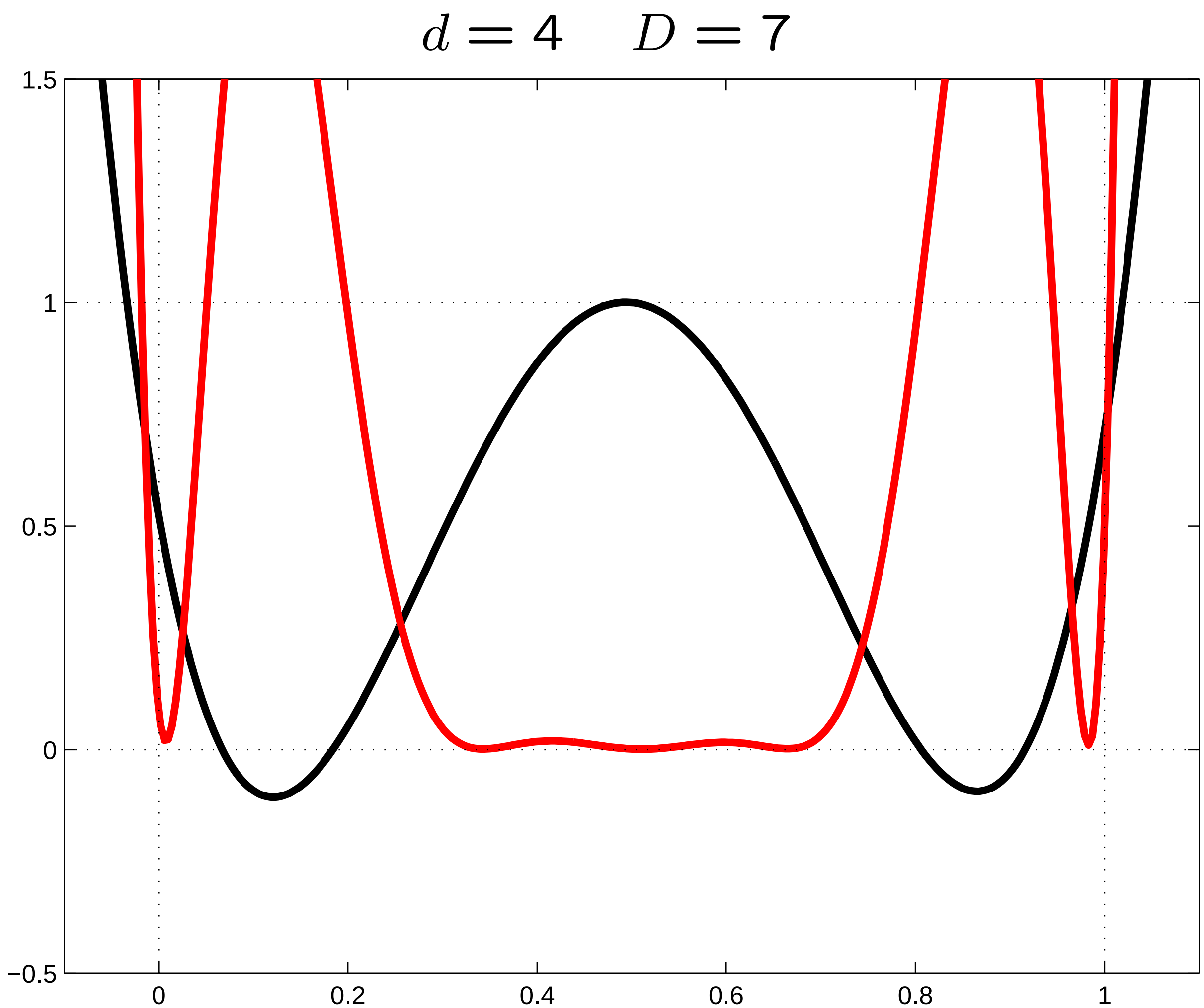
0
0.2
0.4
0.6
0.8
1



$d = 4 \quad D = 7$
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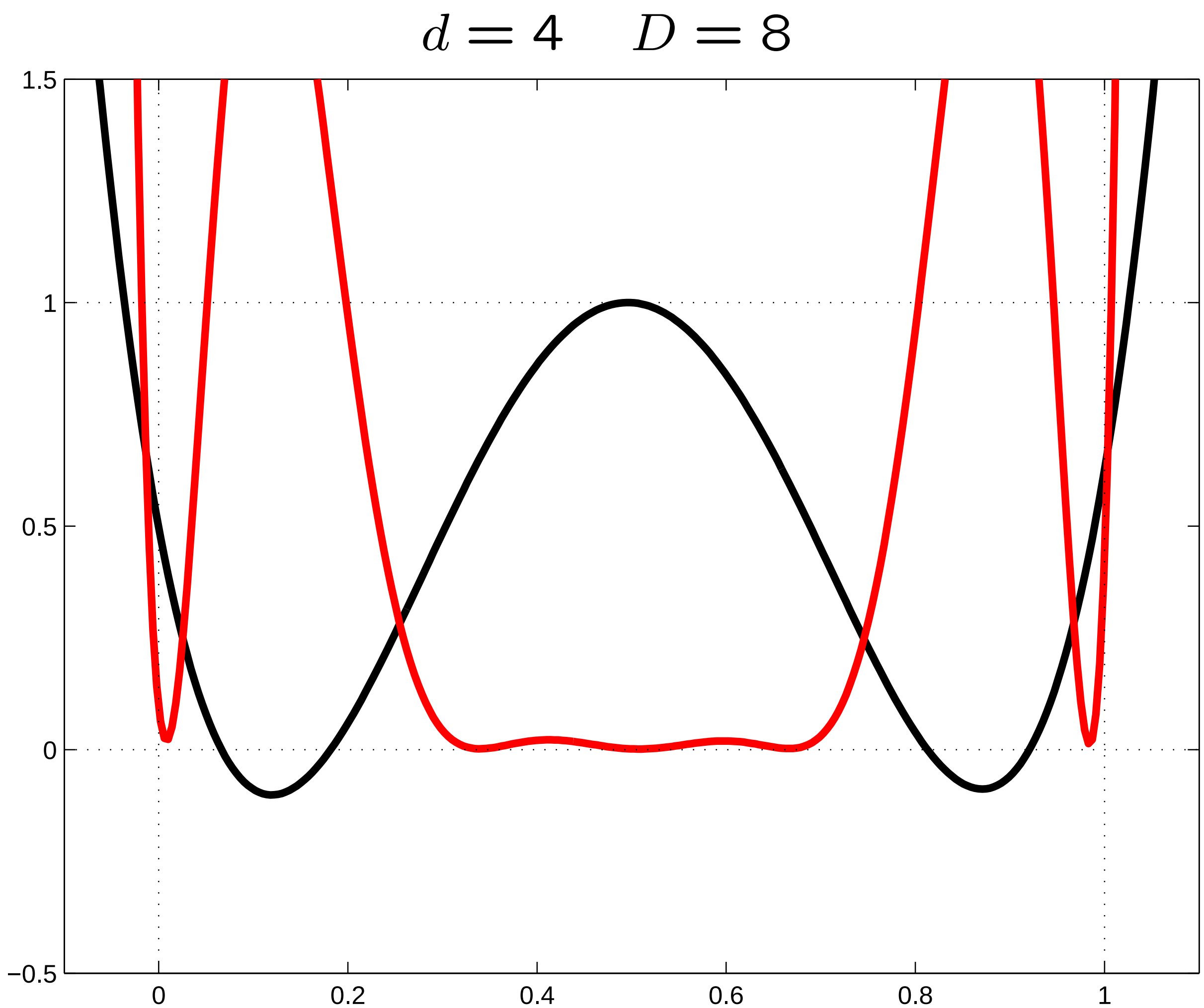
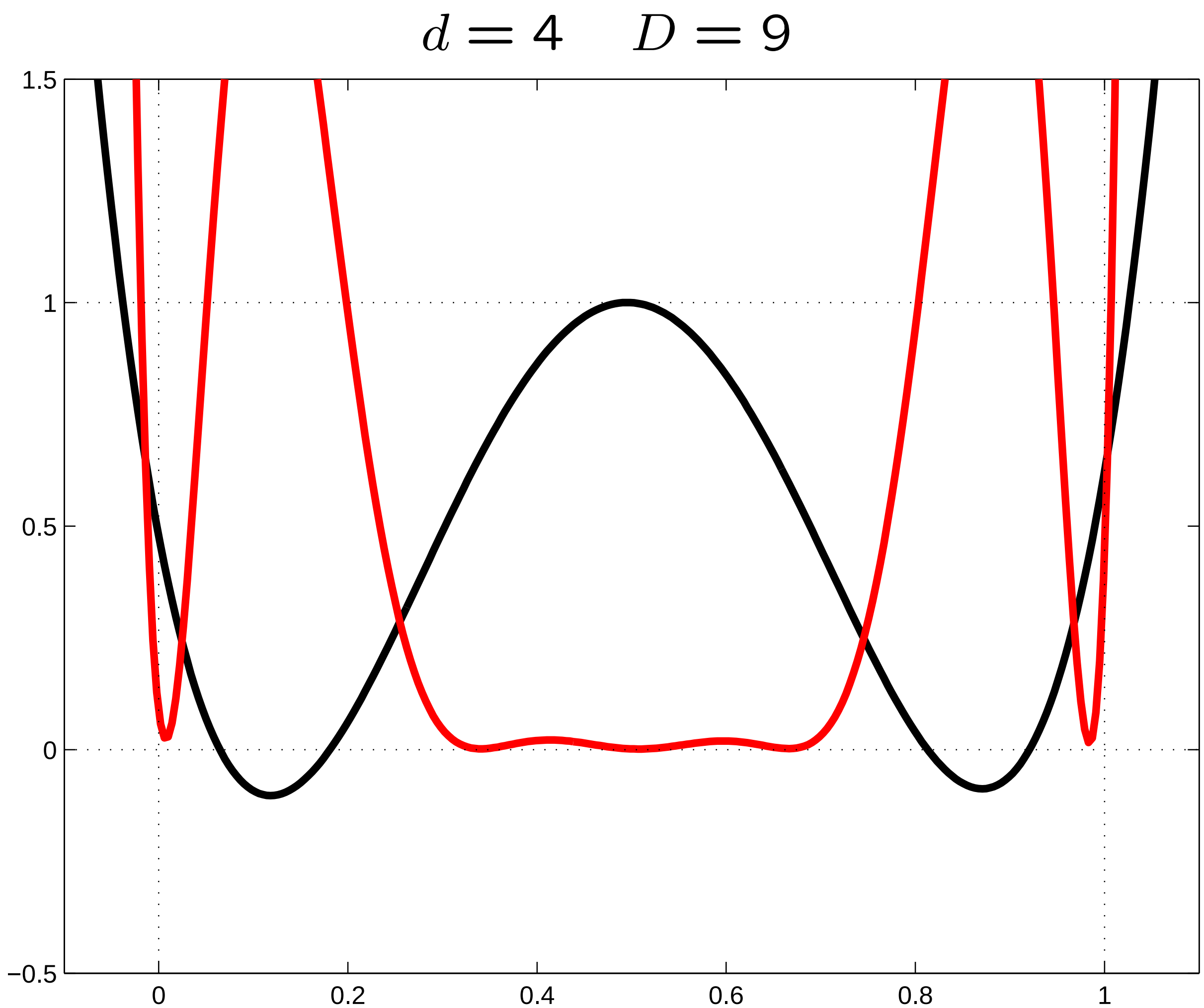
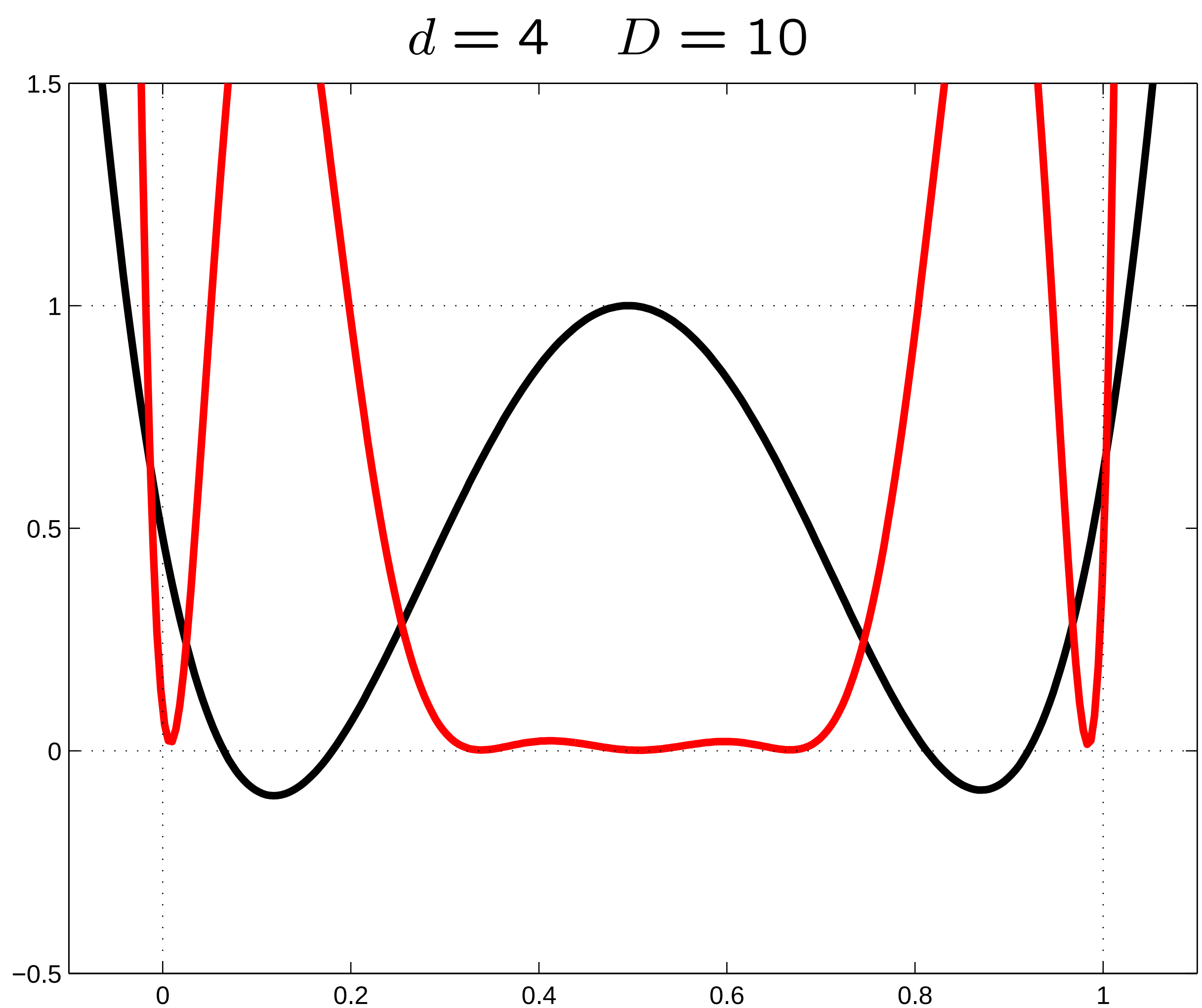
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1



$d = 4 \quad D = 8$
1.5
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1


$d = 4 \quad D = 9$
1.5
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1



$d = 4 \quad D = 10$
1.5
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1

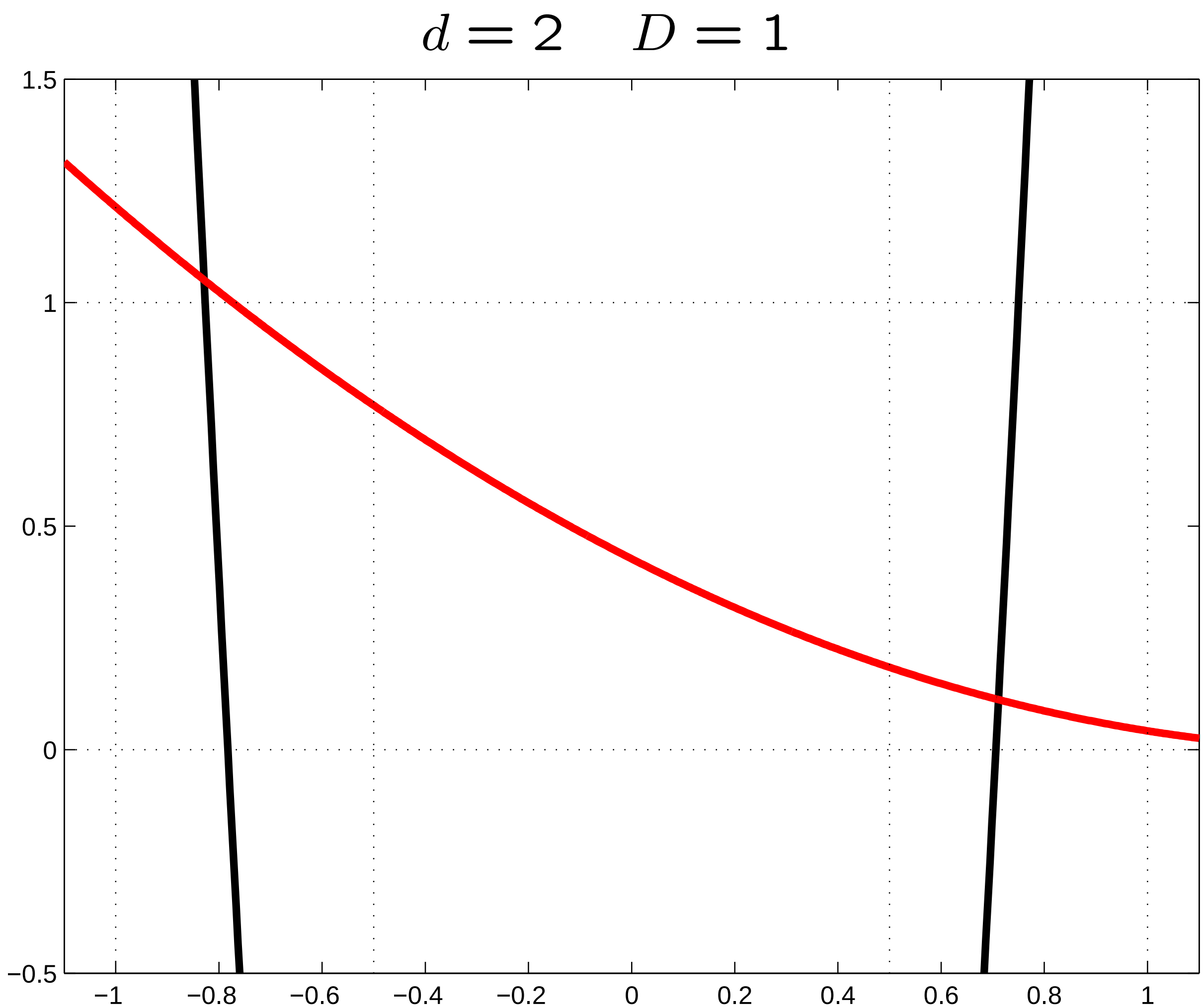
Union of two intervals

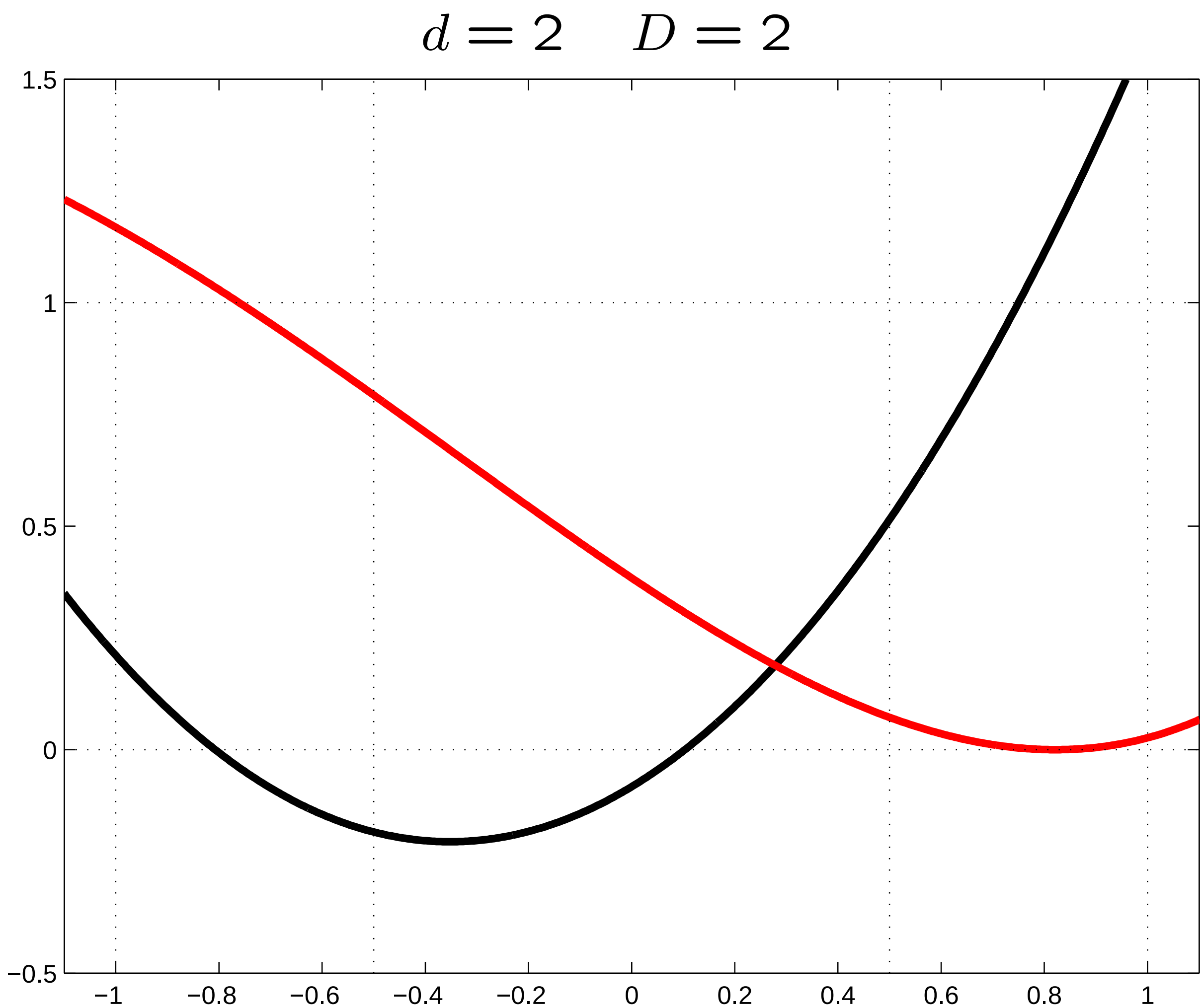
Uniform measure on $K = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1] \subset \mathbb{R}$ has moments

$$y_\alpha = \int_0^1 x^\alpha dx = \frac{-(-1)^{\alpha+1} + (-\frac{1}{2})^{\alpha+1} - (\frac{1}{2})^{\alpha+1} + (1)^{\alpha+1}}{\alpha + 1}$$

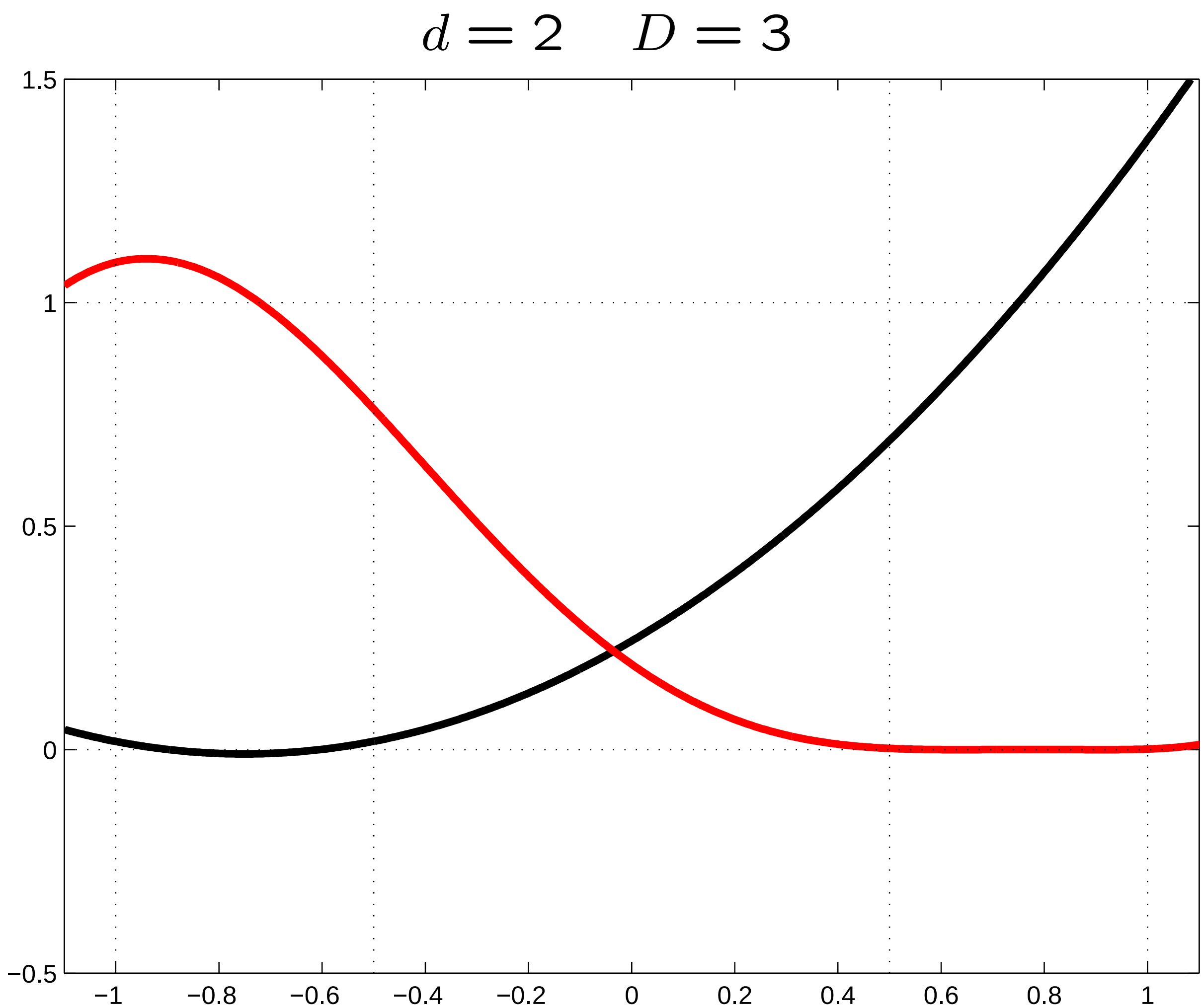
$$y = [1, 0, \frac{7}{12}, 0, \frac{31}{80}, 0, \frac{127}{448}, \cdots]$$

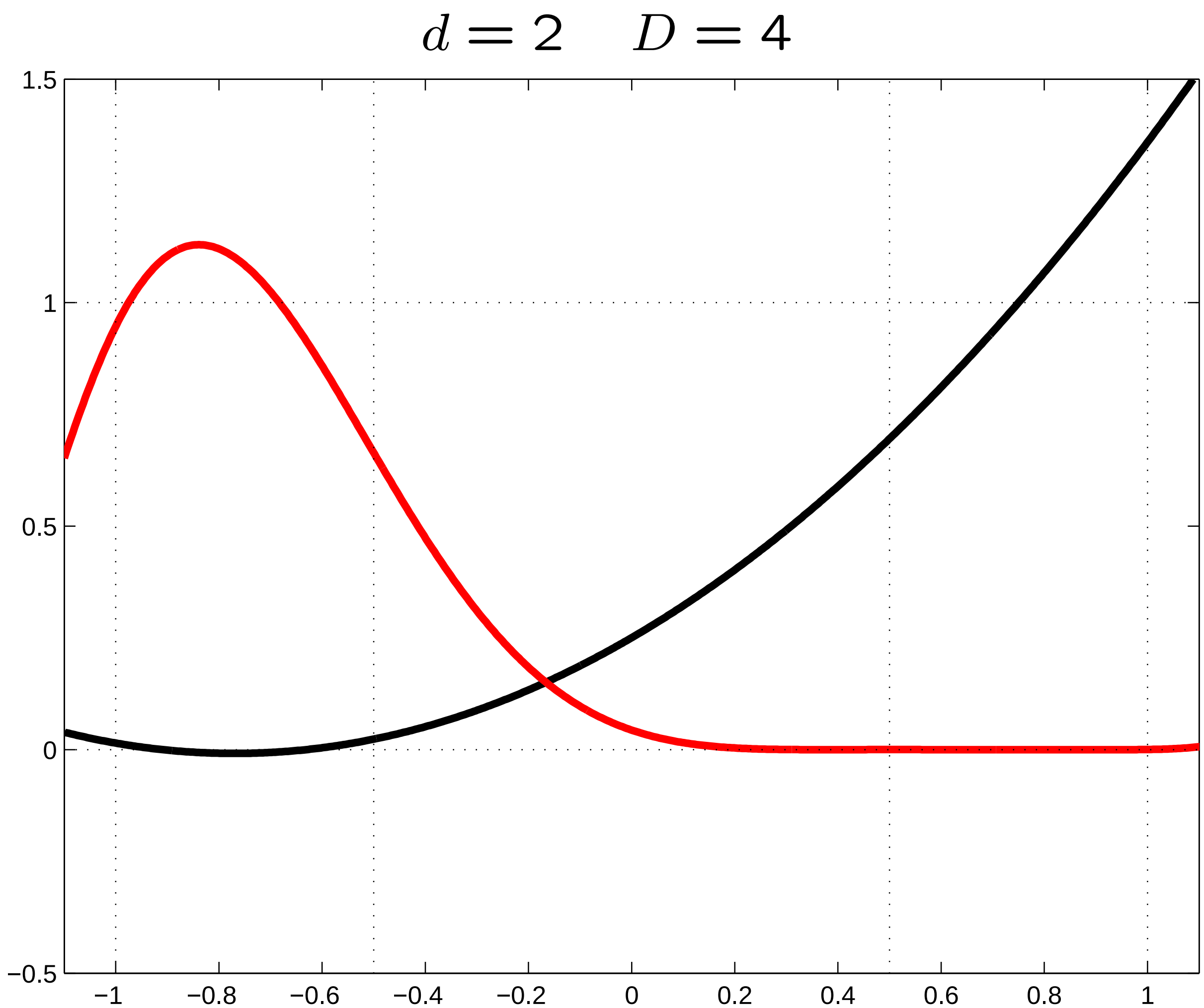
$$d = 2$$

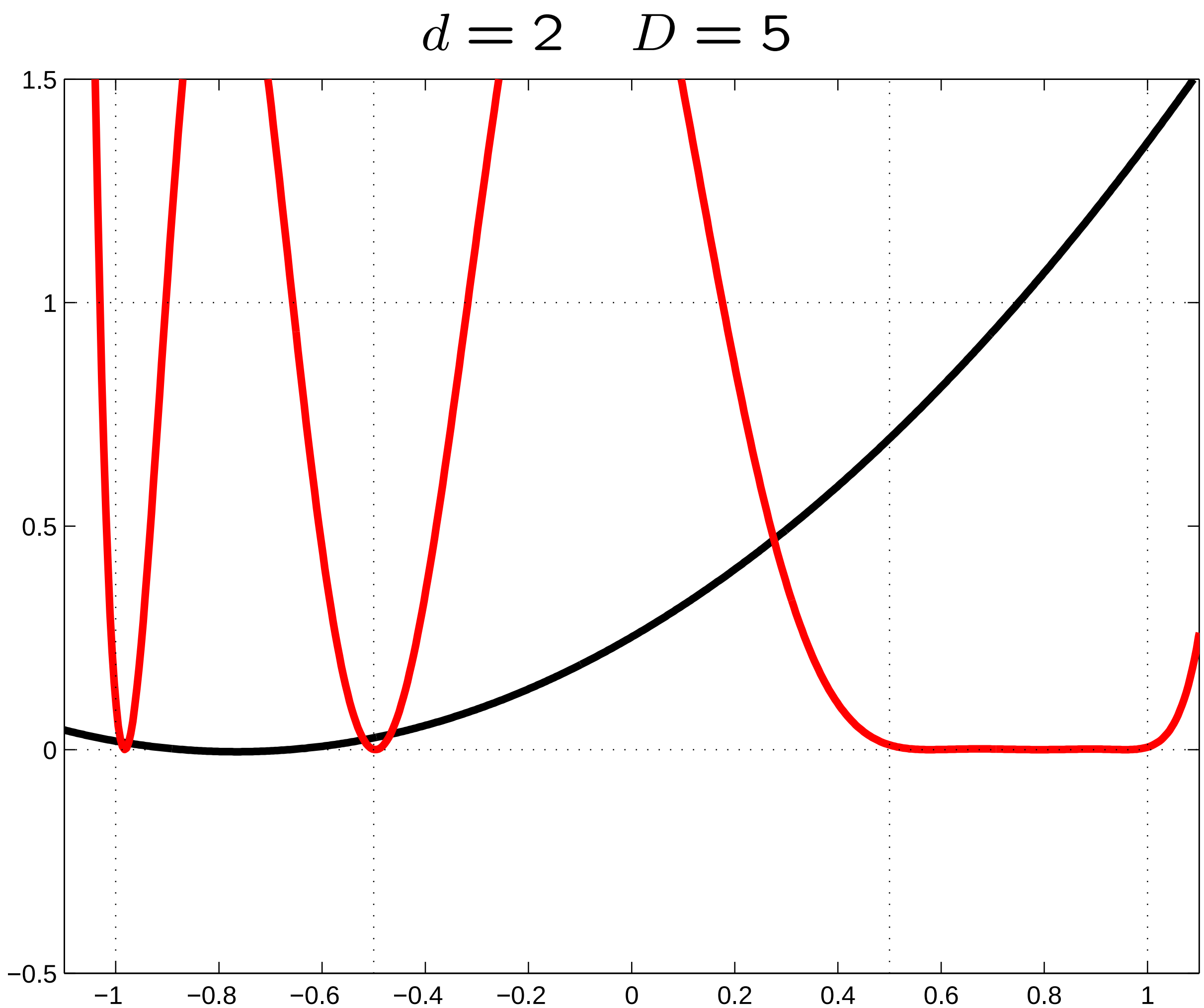

$d = 2 \quad D = 1$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

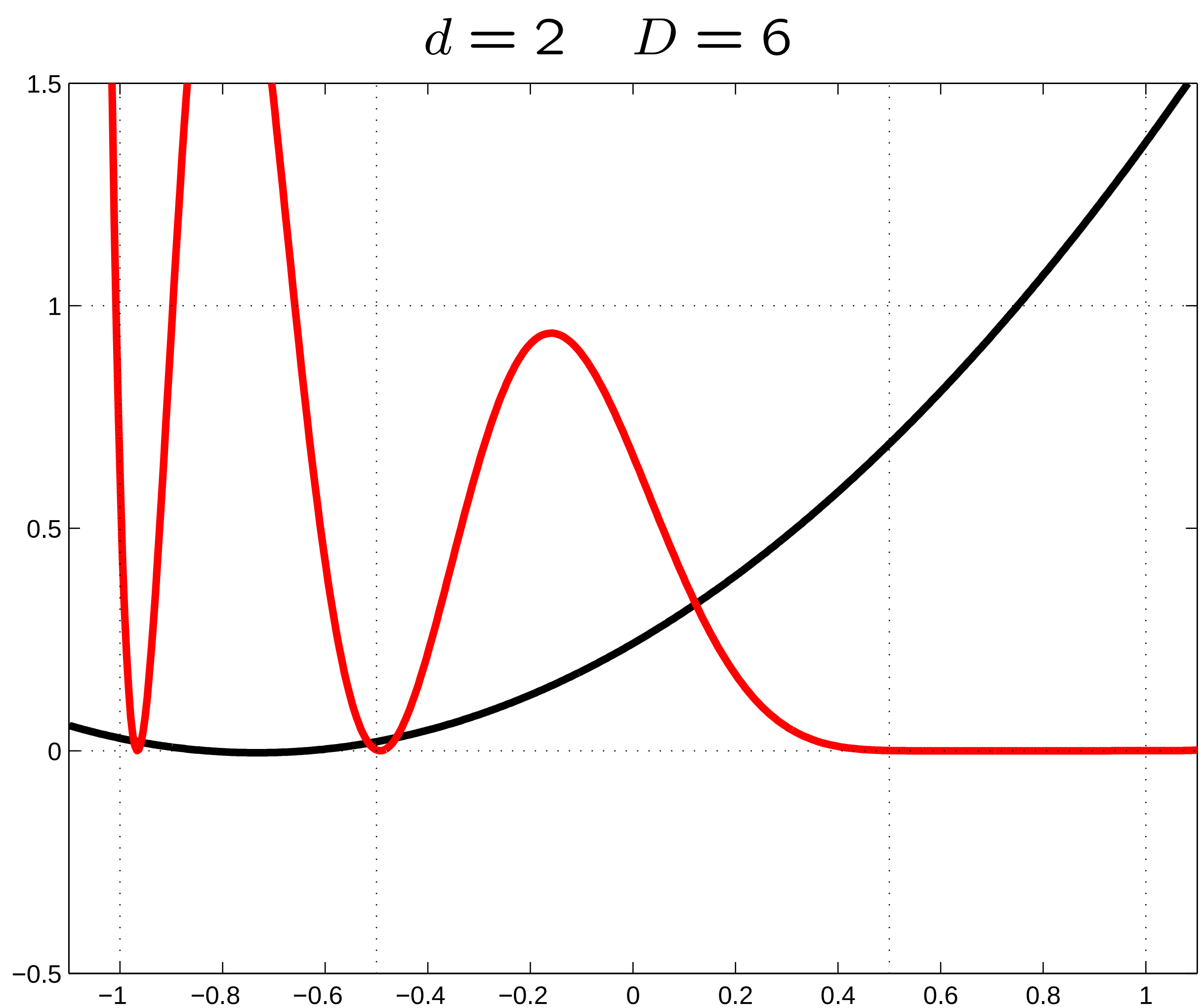


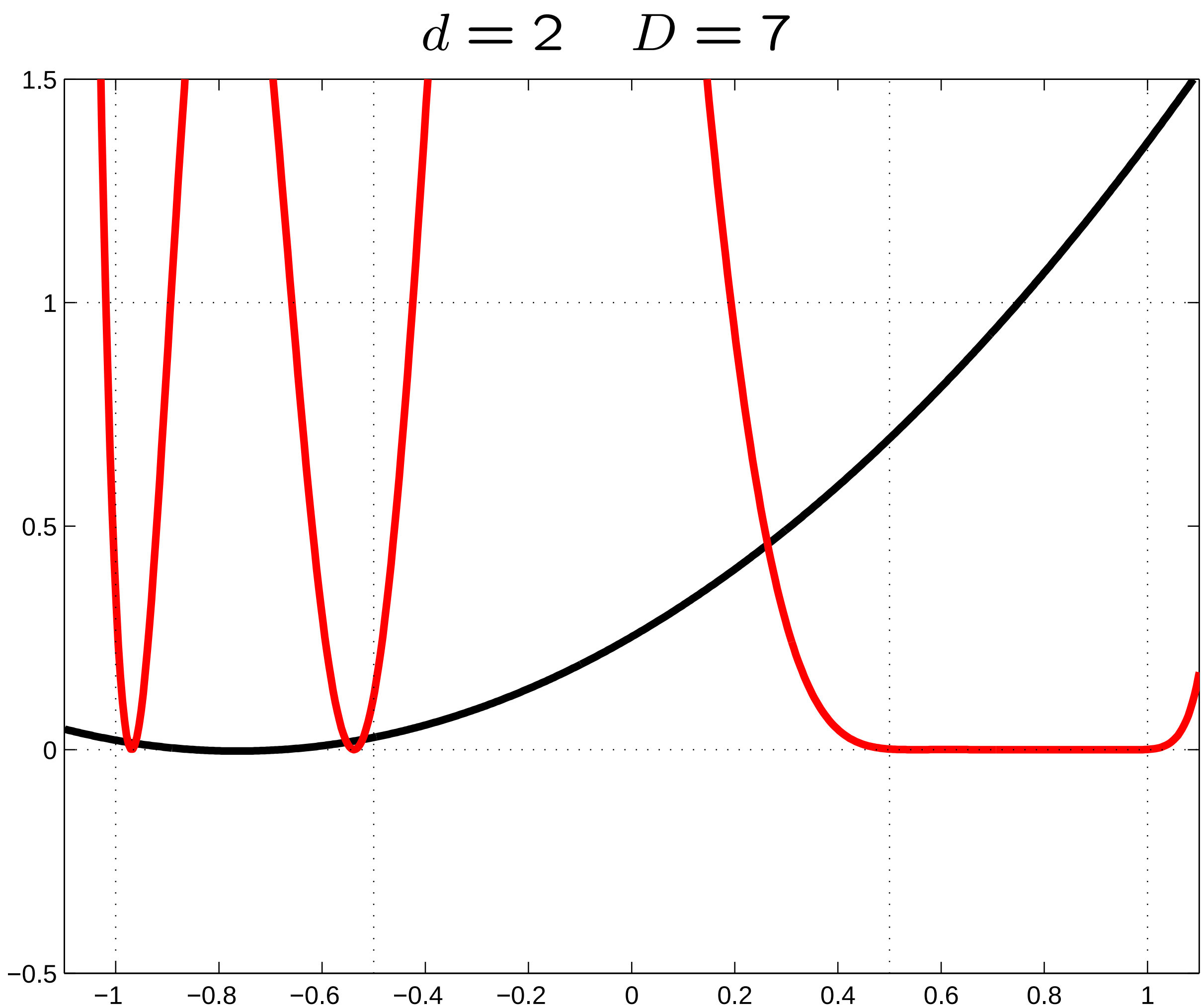
$d = 2 \quad D = 2$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

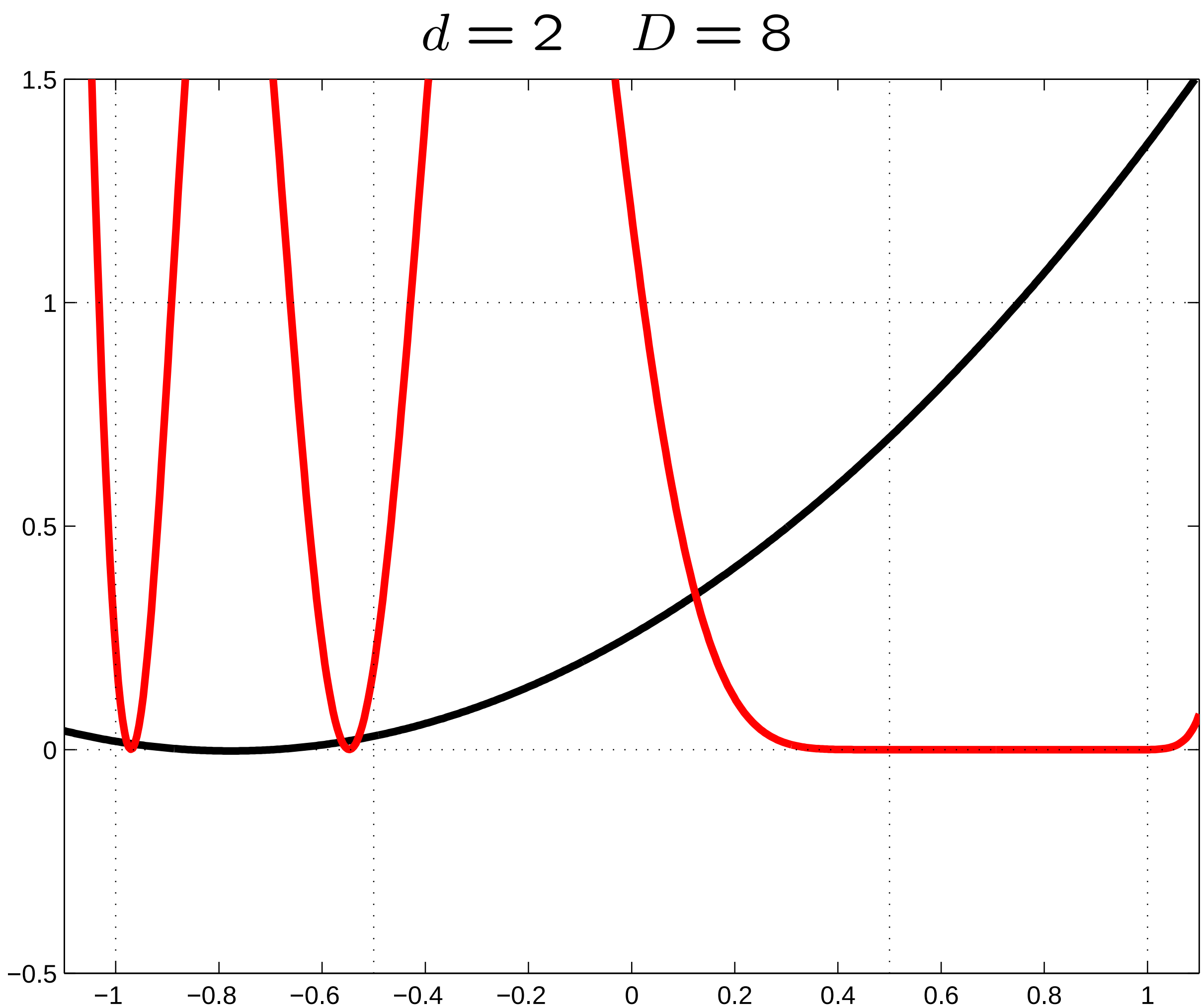

$d = 2 \quad D = 3$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

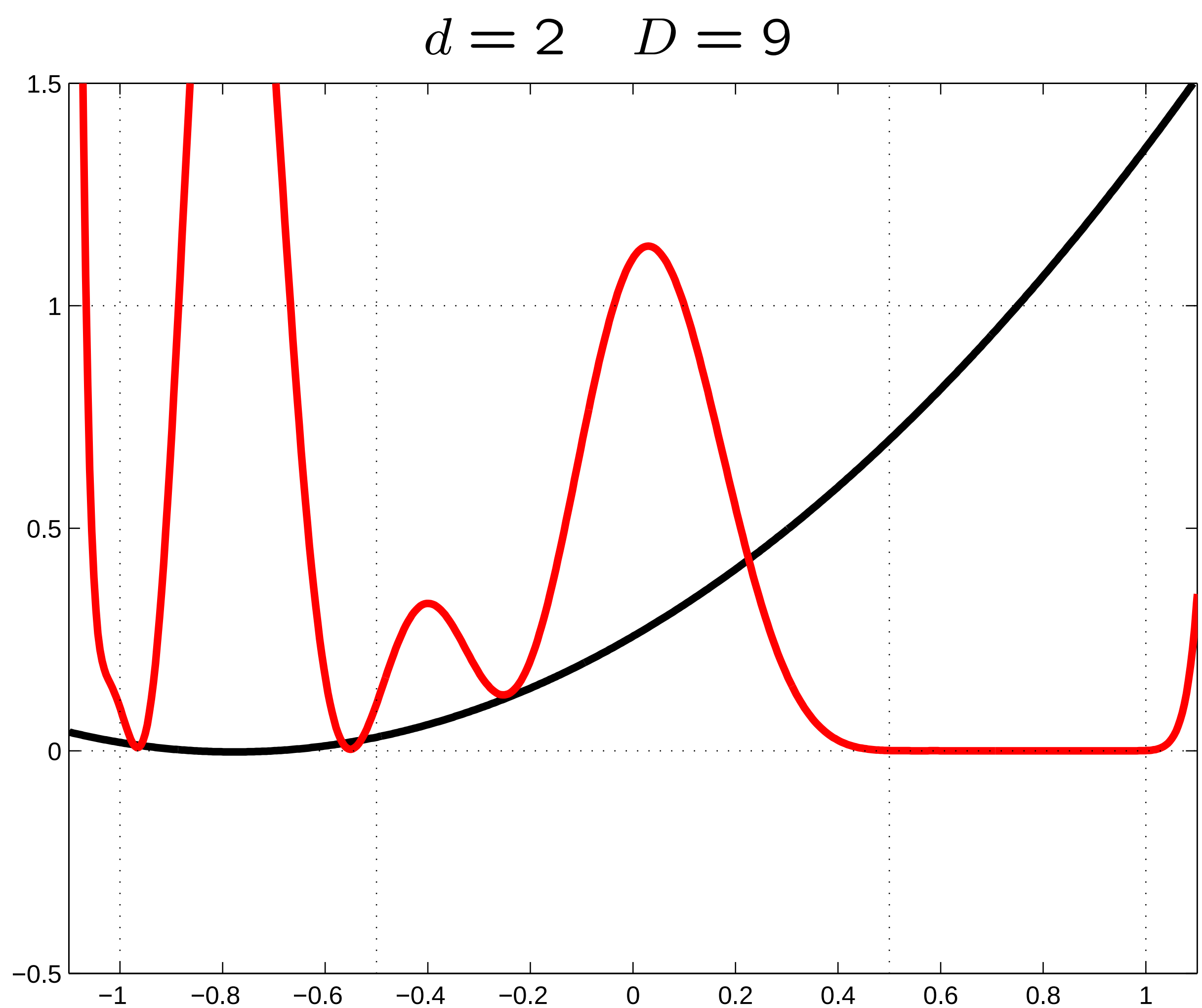

$d = 2 \quad D = 4$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


$d = 2 \quad D = 5$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

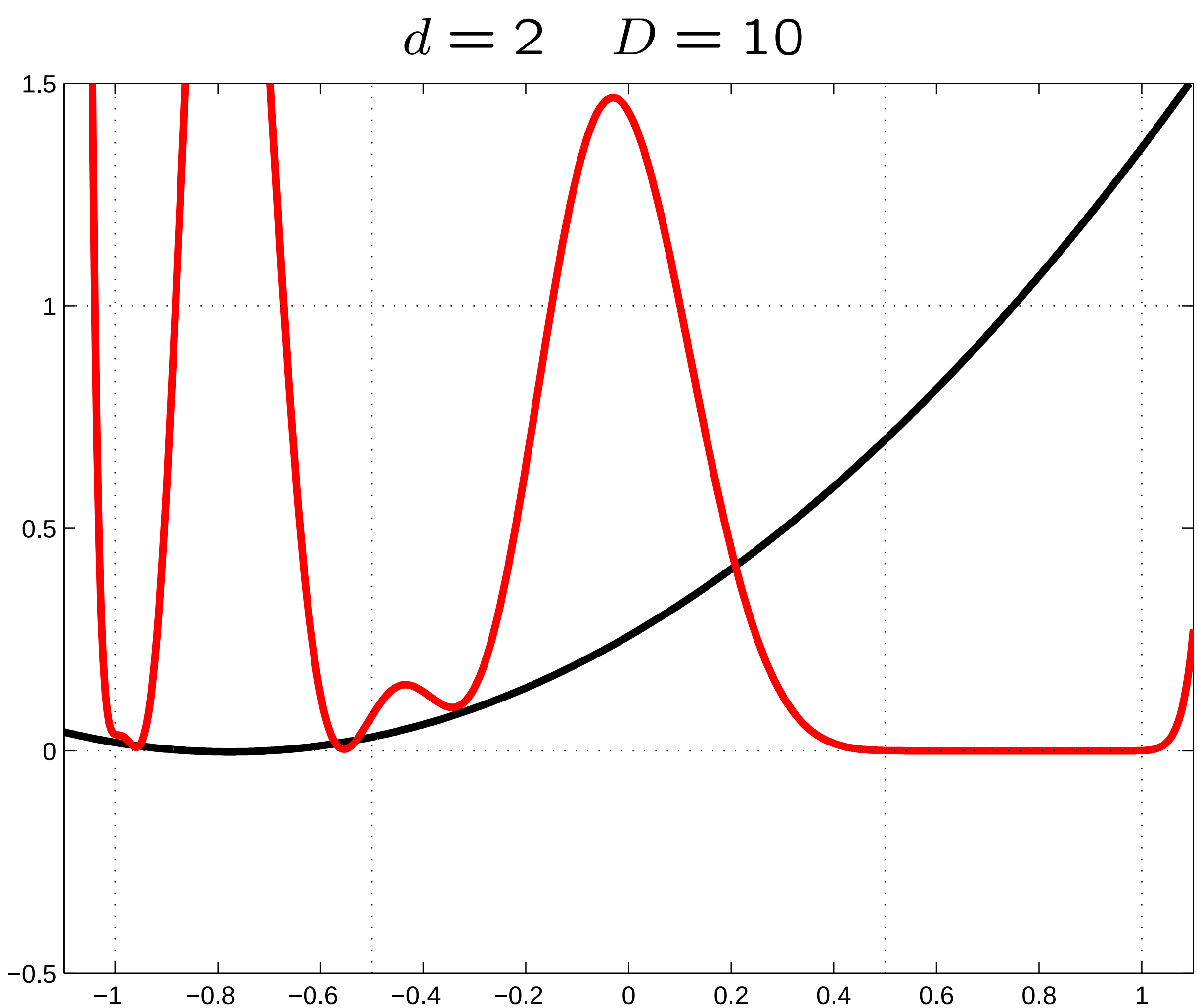

$d = 2 \quad D = 6$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


$d = 2 \quad D = 7$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

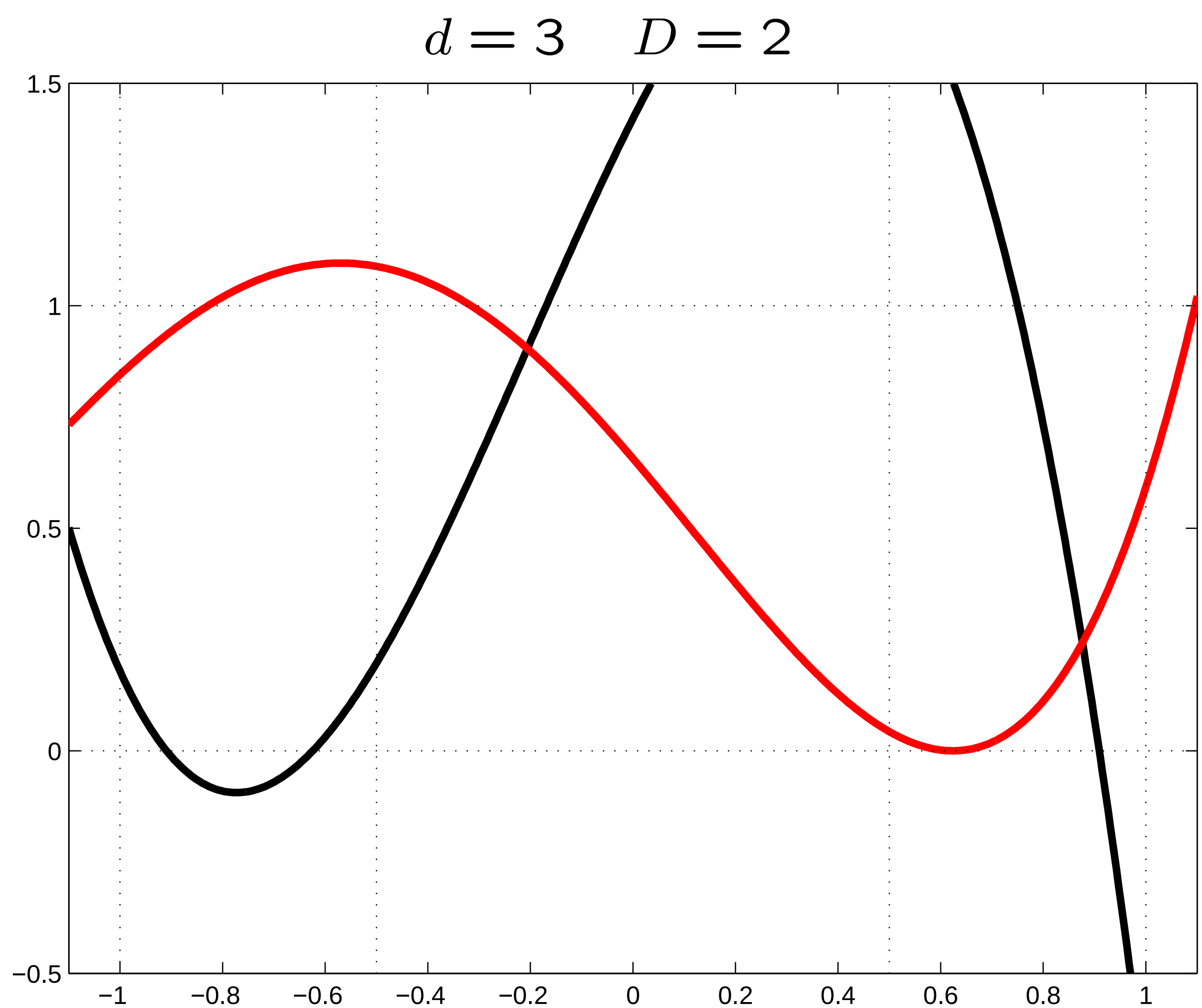

$d = 2 \quad D = 8$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

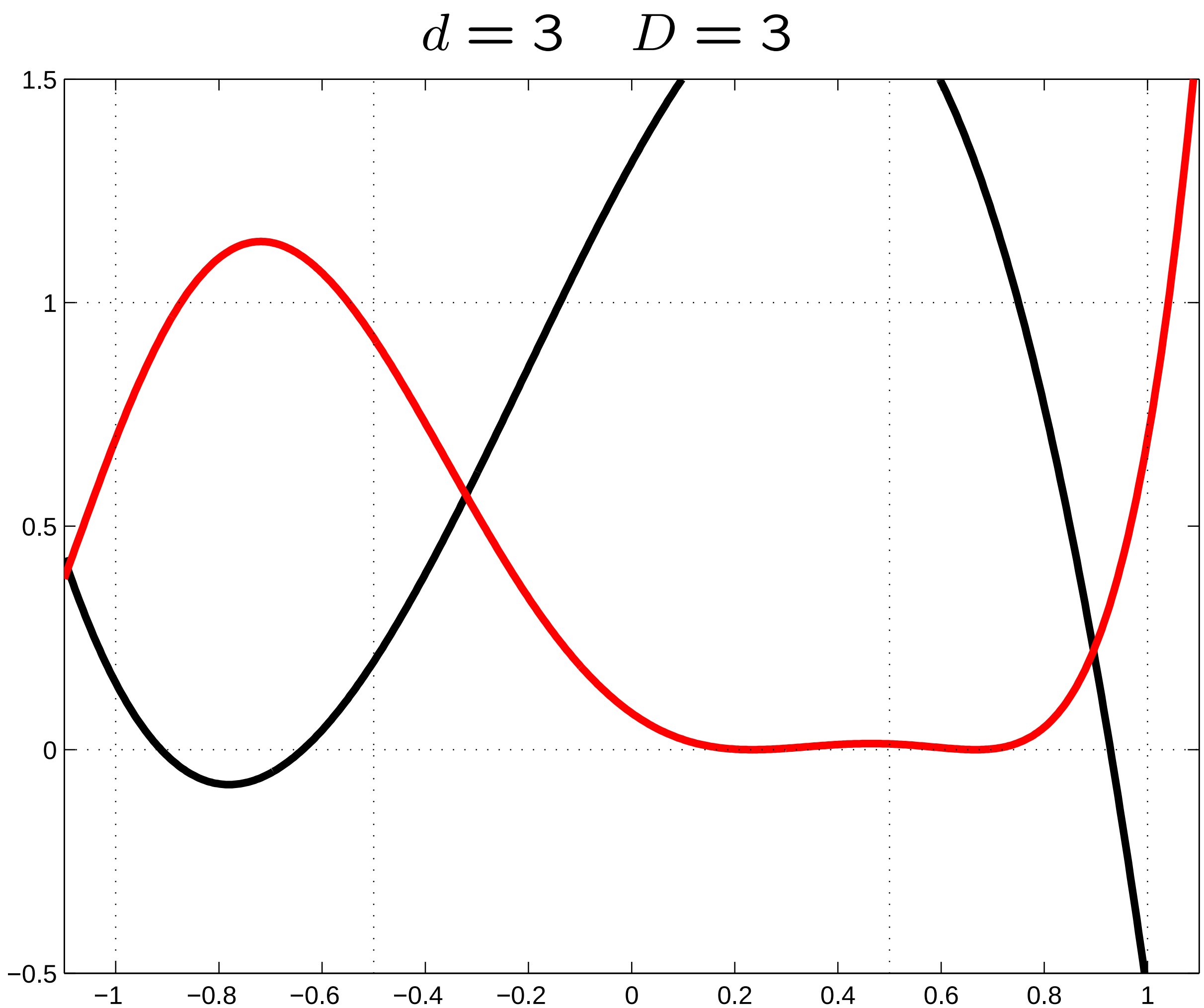


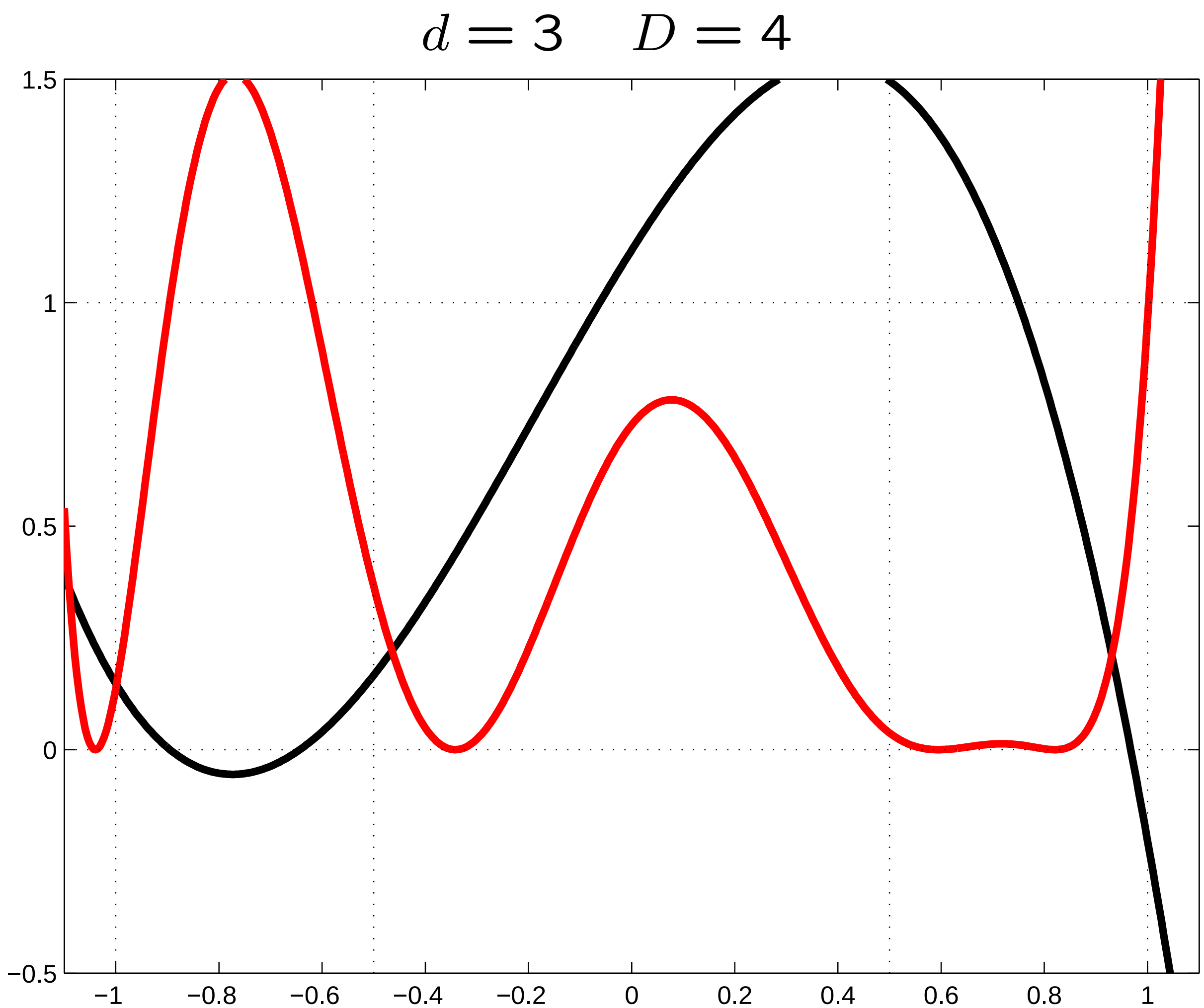
$d = 2 \quad D = 9$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


$d = 2 \quad D = 10$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

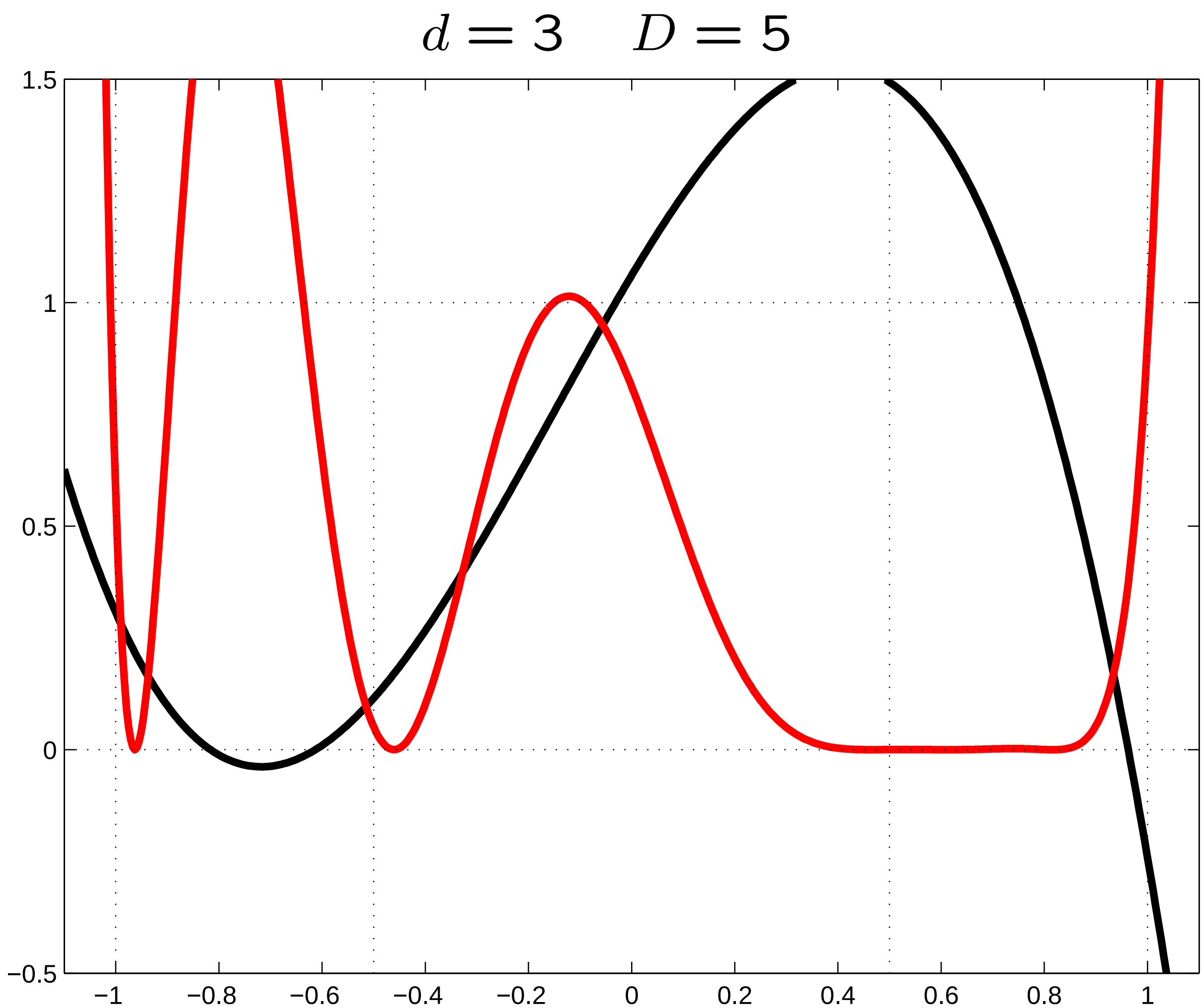
$$d = 3$$


$d = 3 \quad D = 2$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

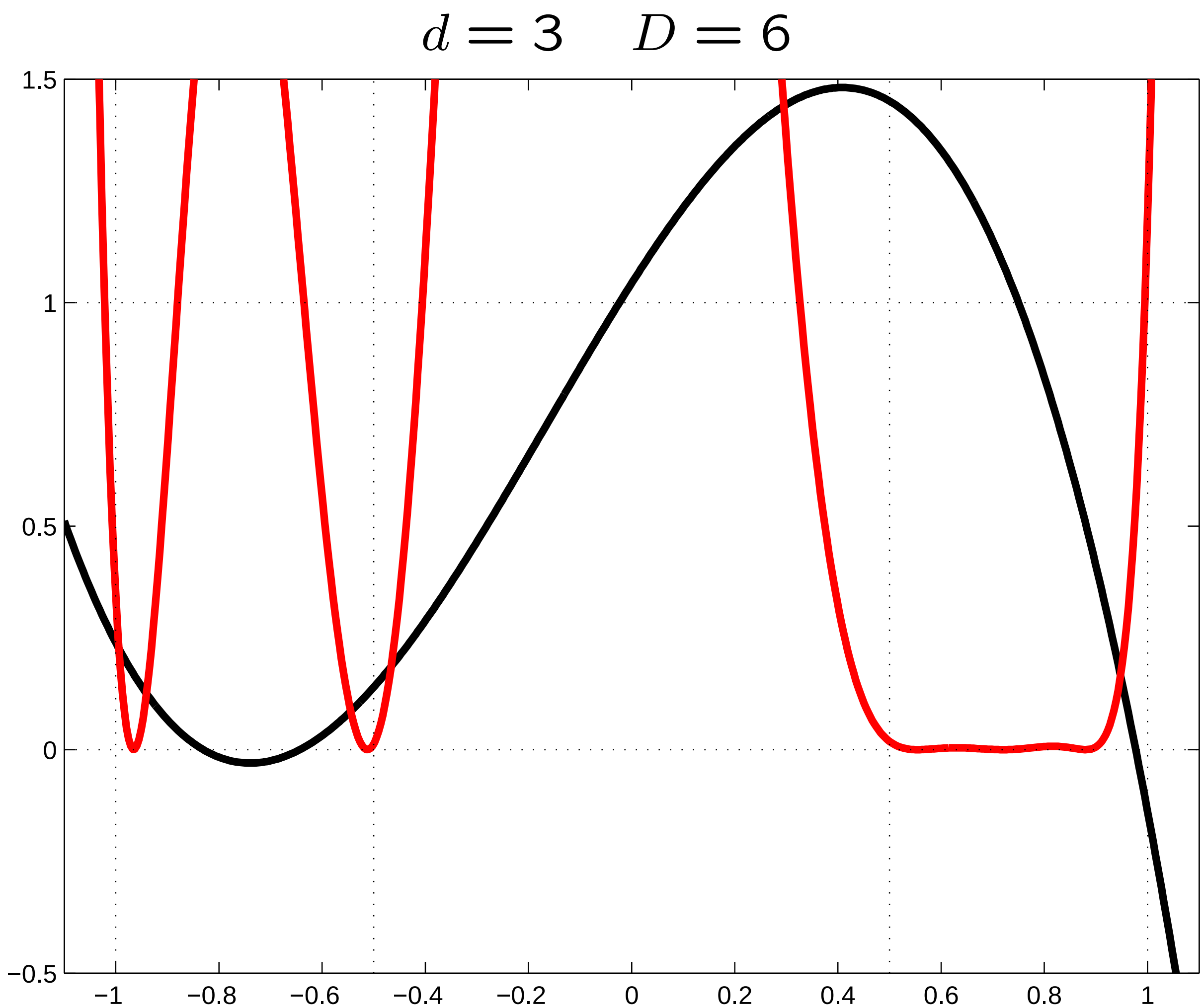

$d = 3 \quad D = 3$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



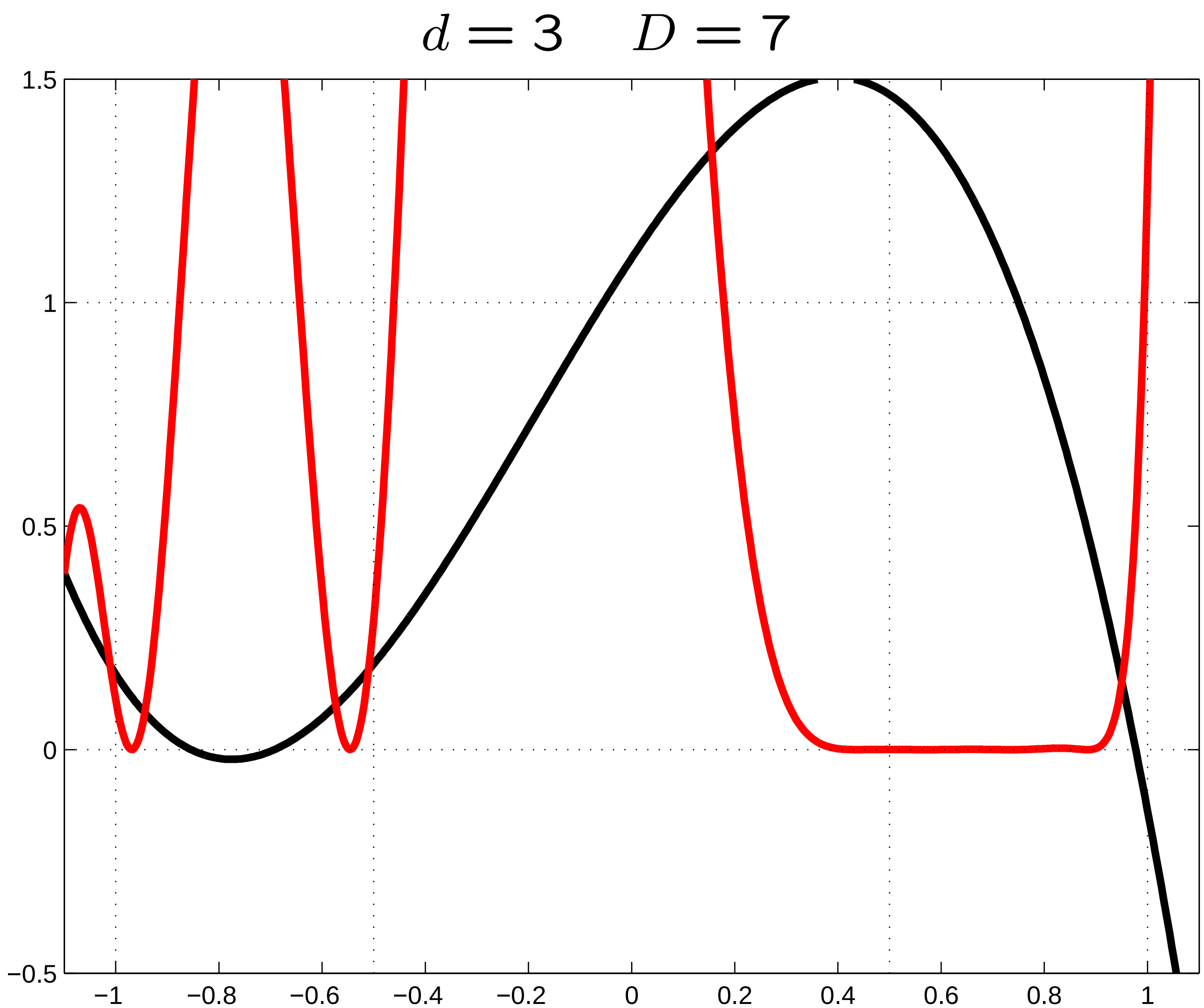
$d = 3 \quad D = 4$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



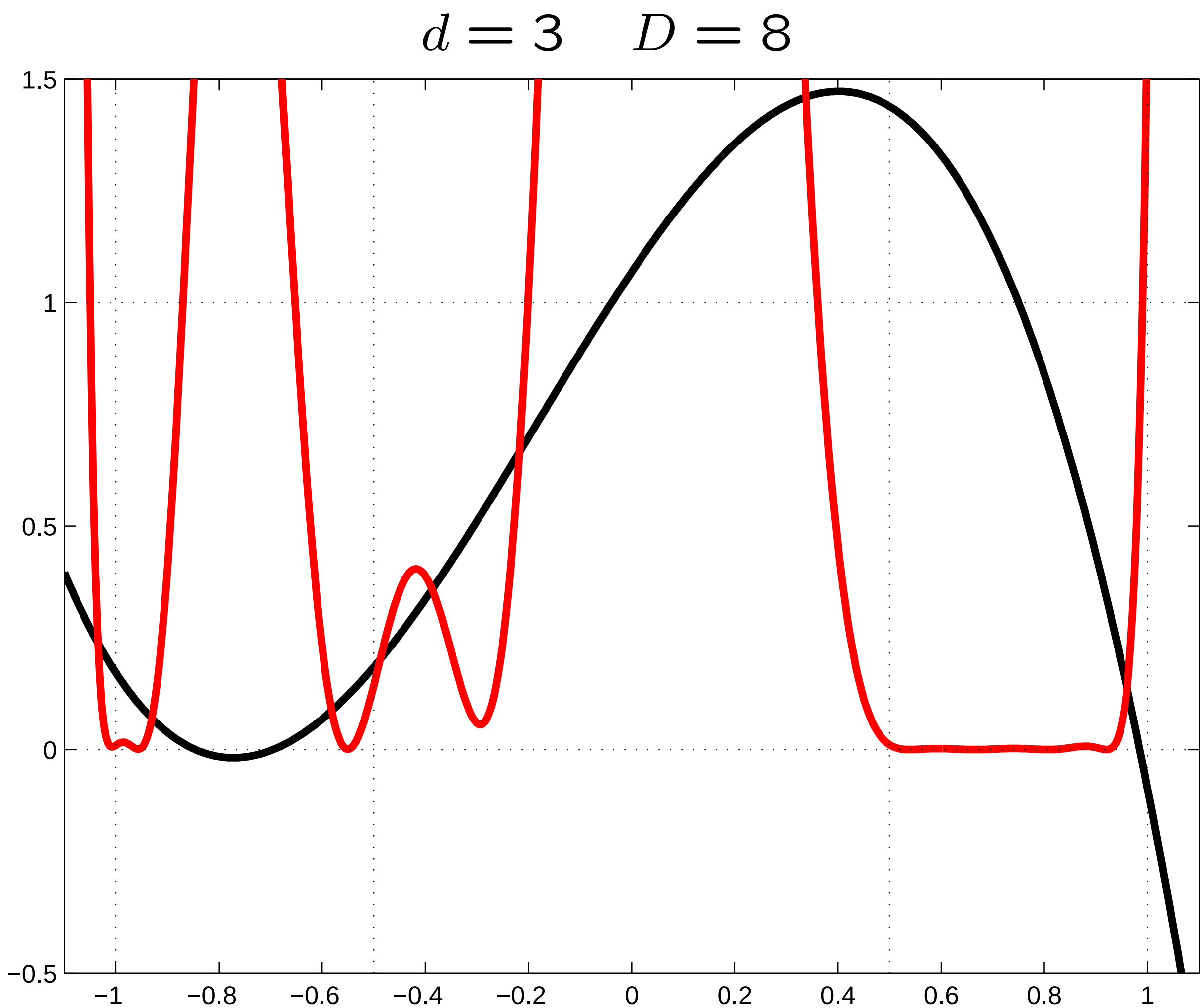
$d = 3 \quad D = 5$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

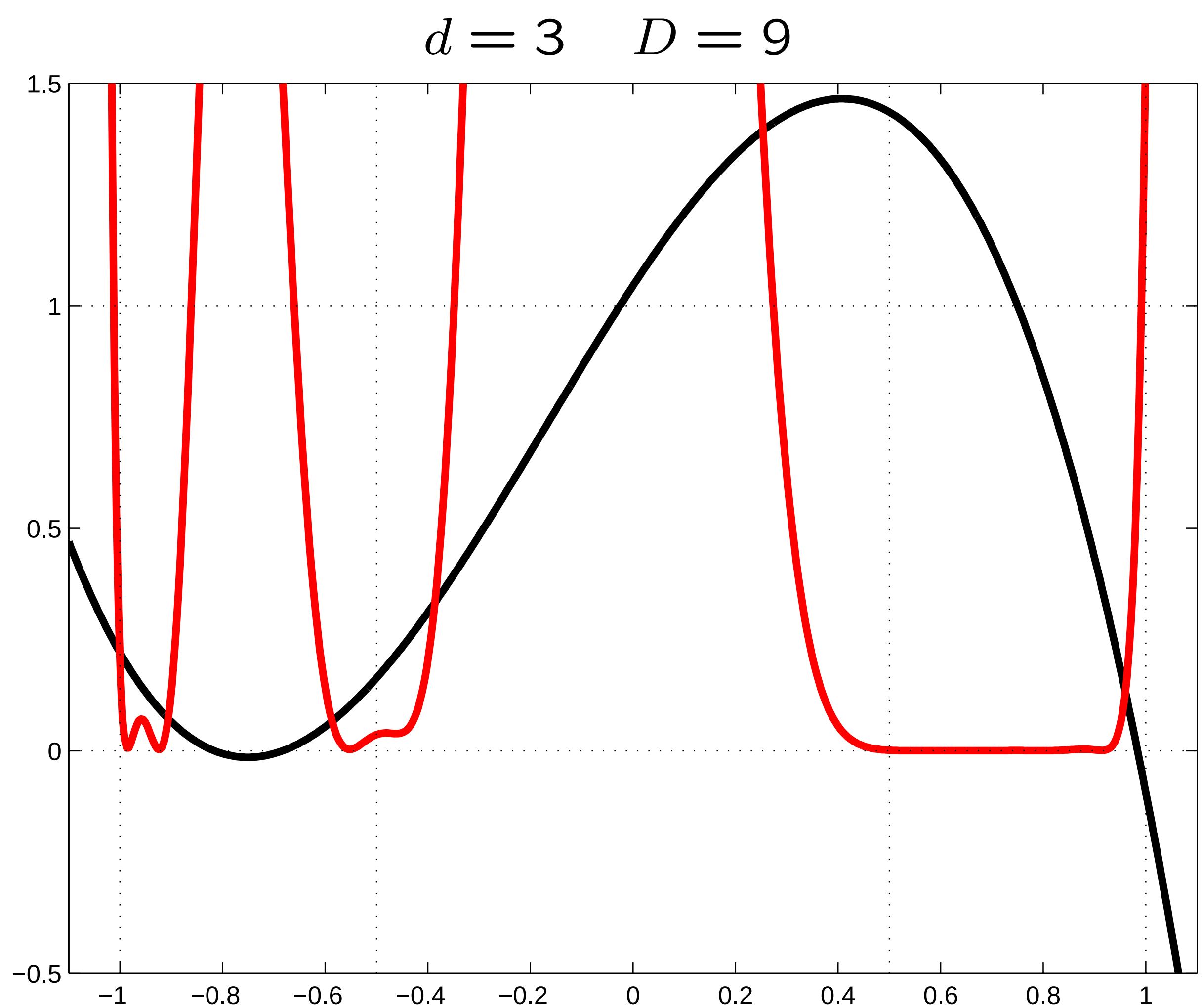


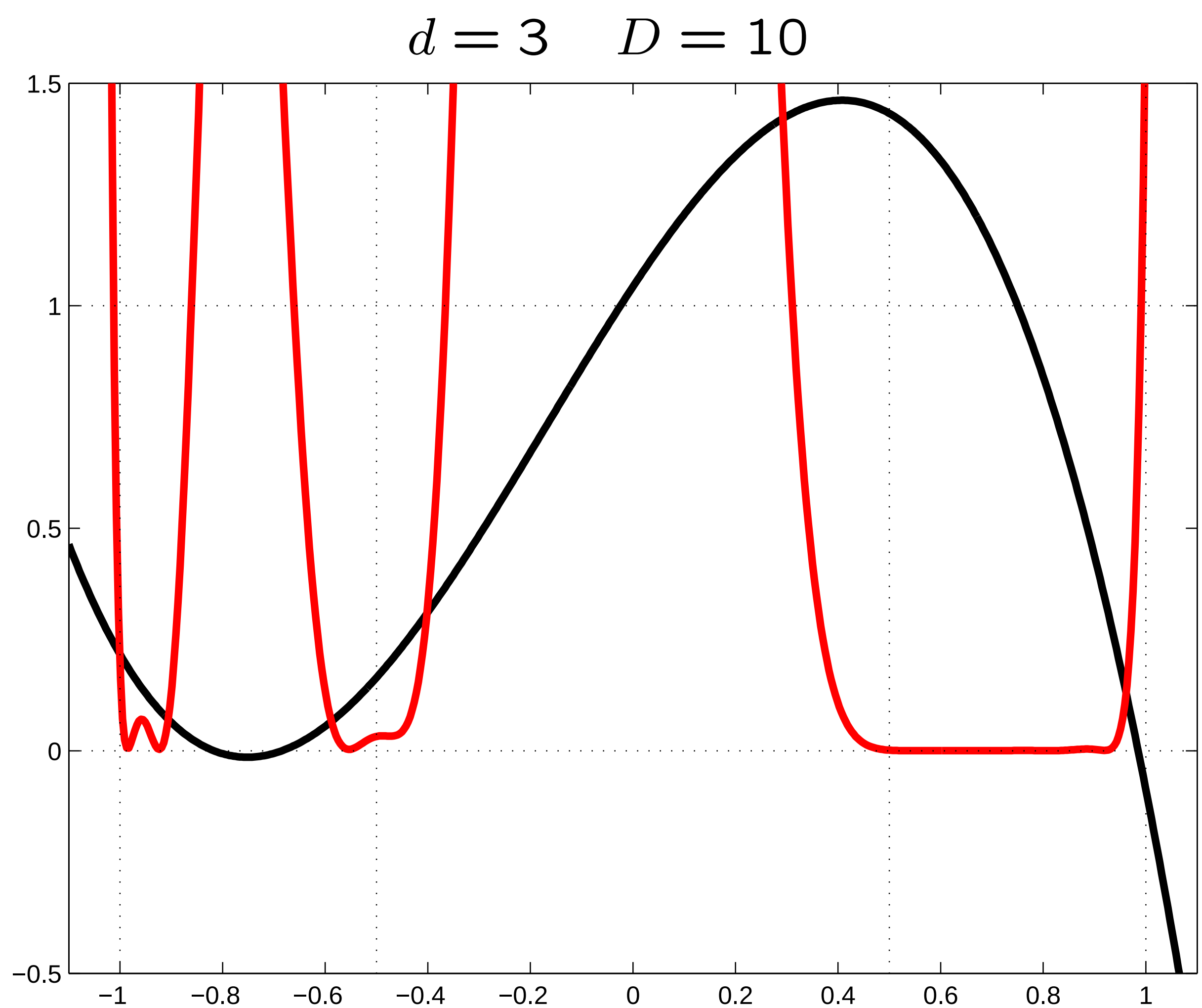
$d = 3 \quad D = 6$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



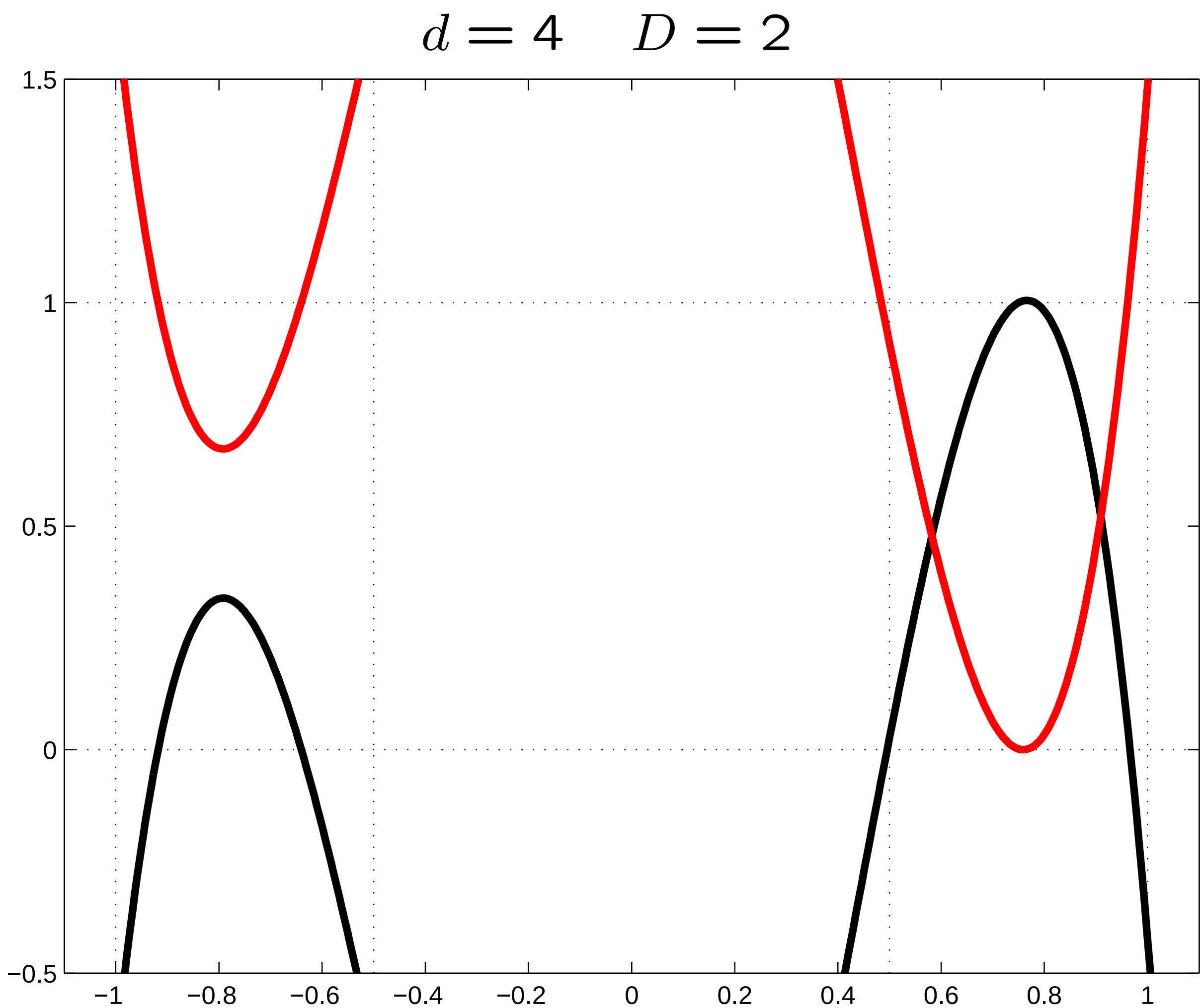
$d = 3 \quad D = 7$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

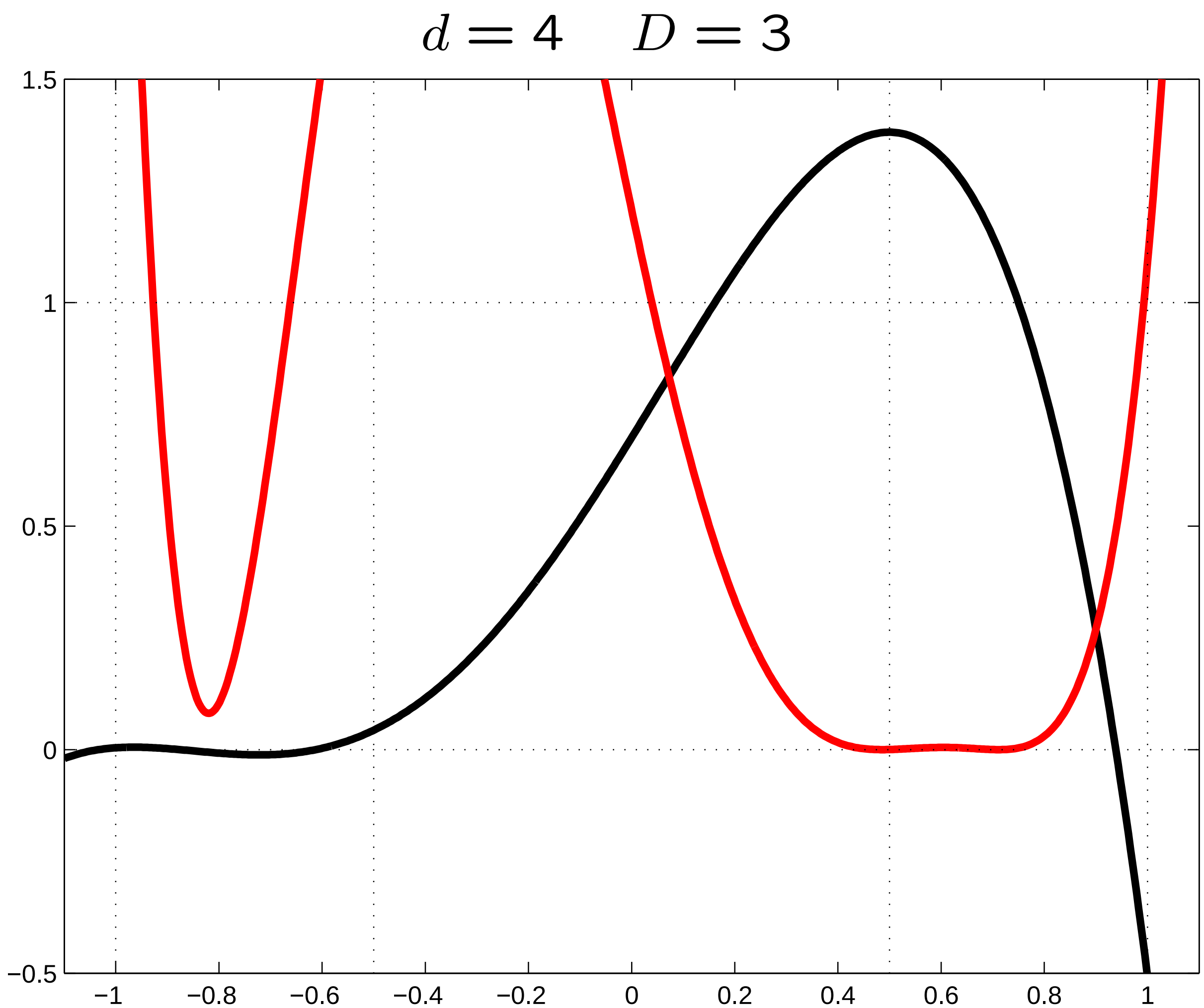

$d = 3 \quad D = 8$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


$d = 3 \quad D = 9$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

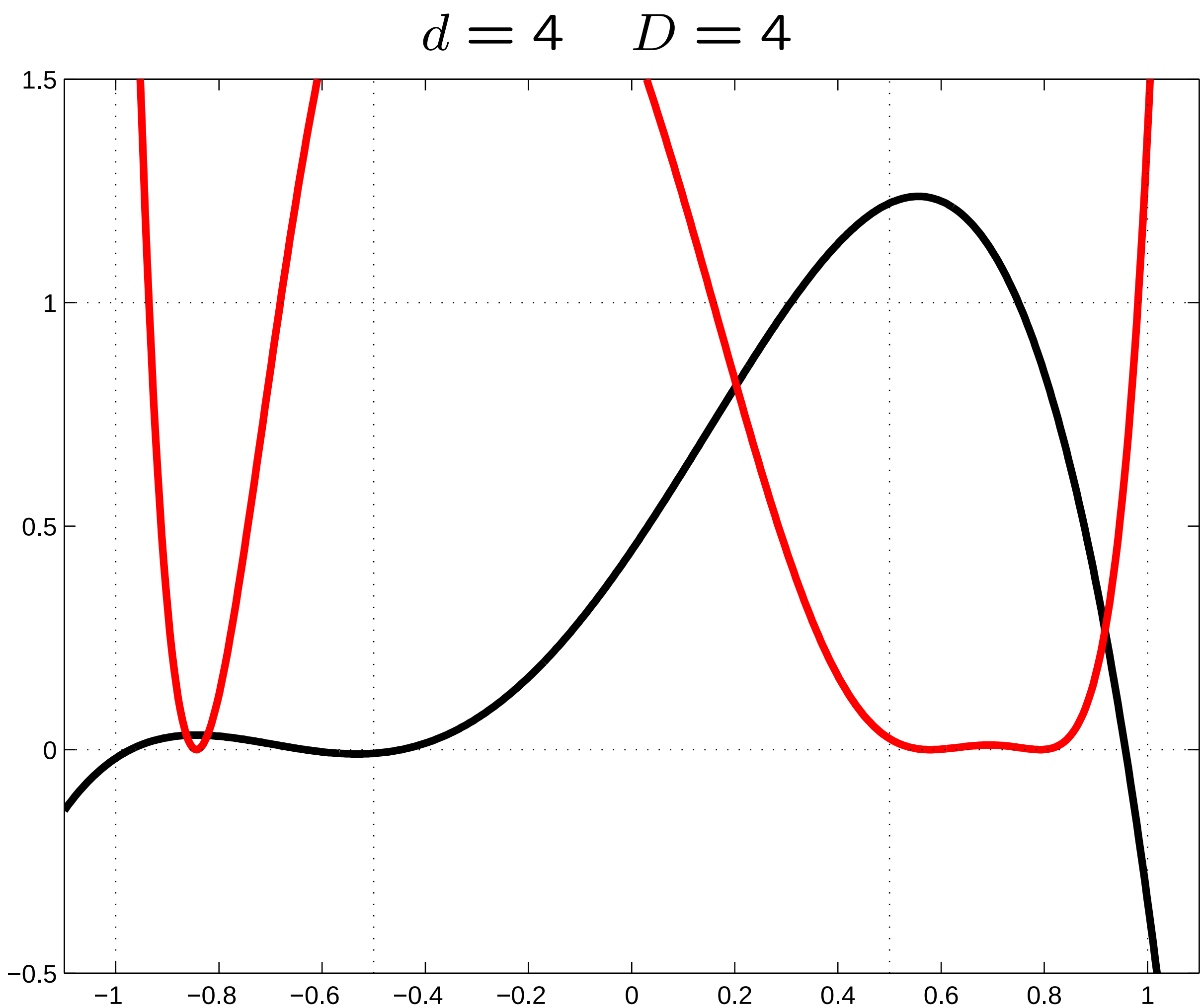

$d = 3 \quad D = 10$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

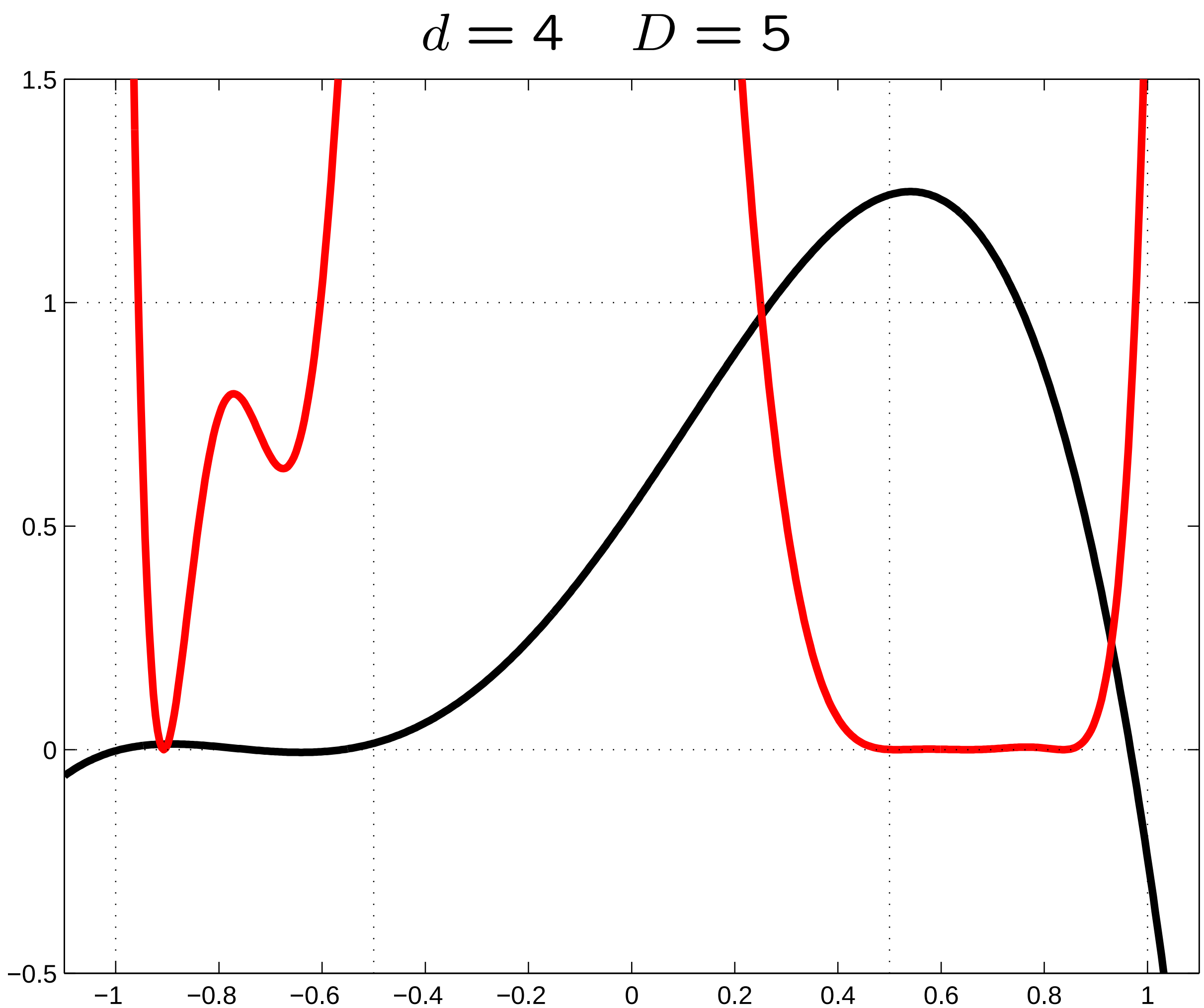
$$d = 4$$

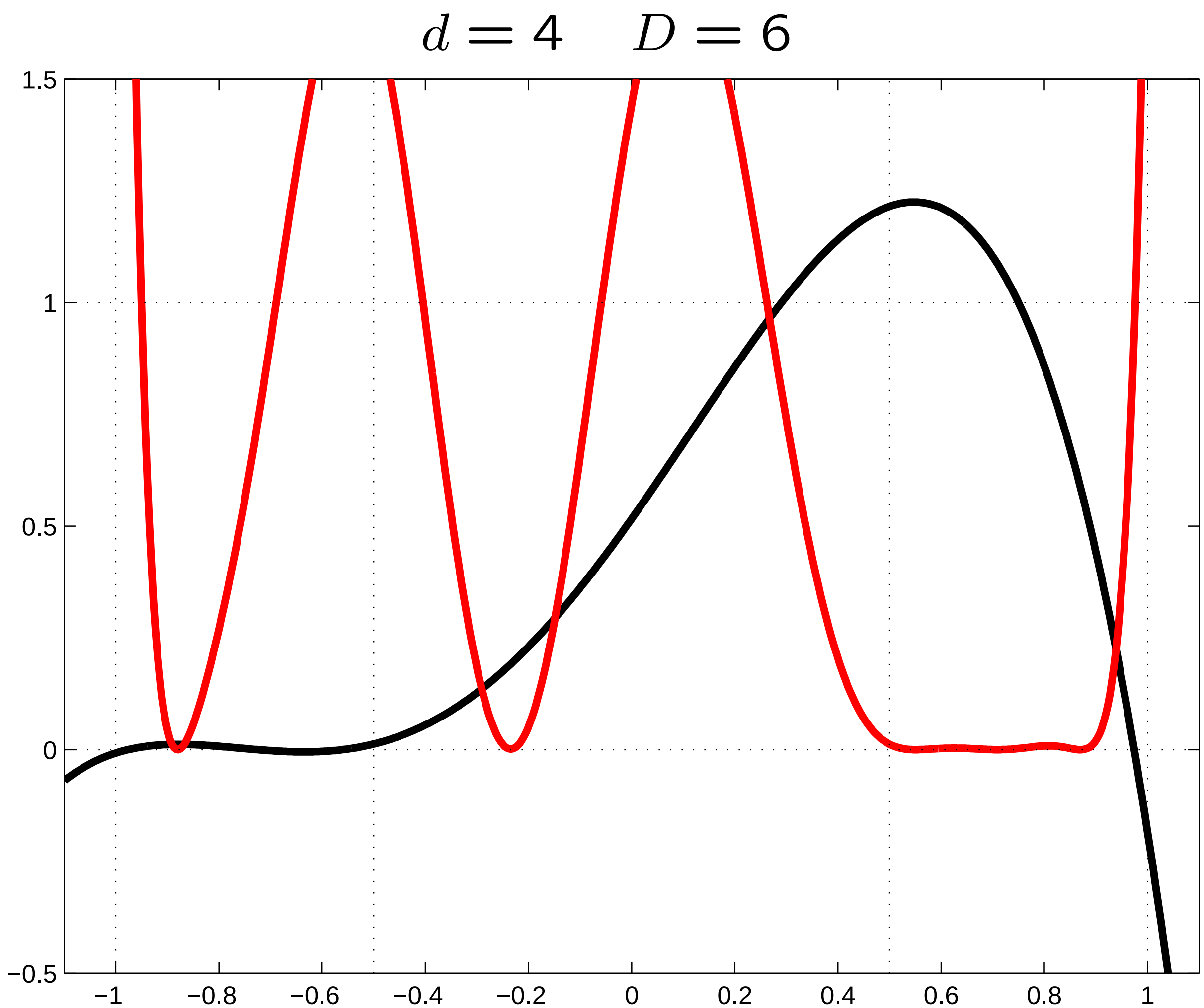

$d = 4 \quad D = 2$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



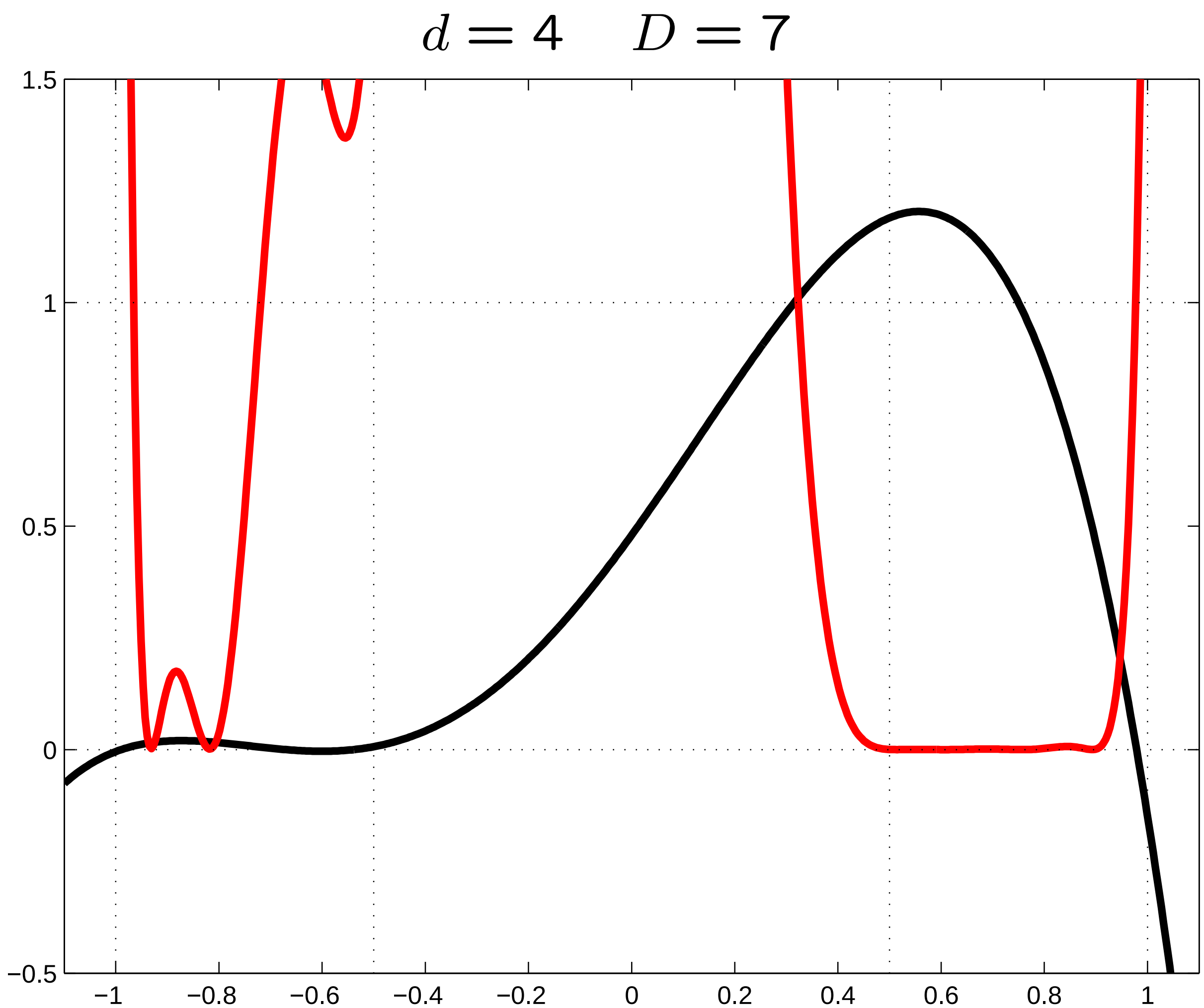
$d = 4 \quad D = 3$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

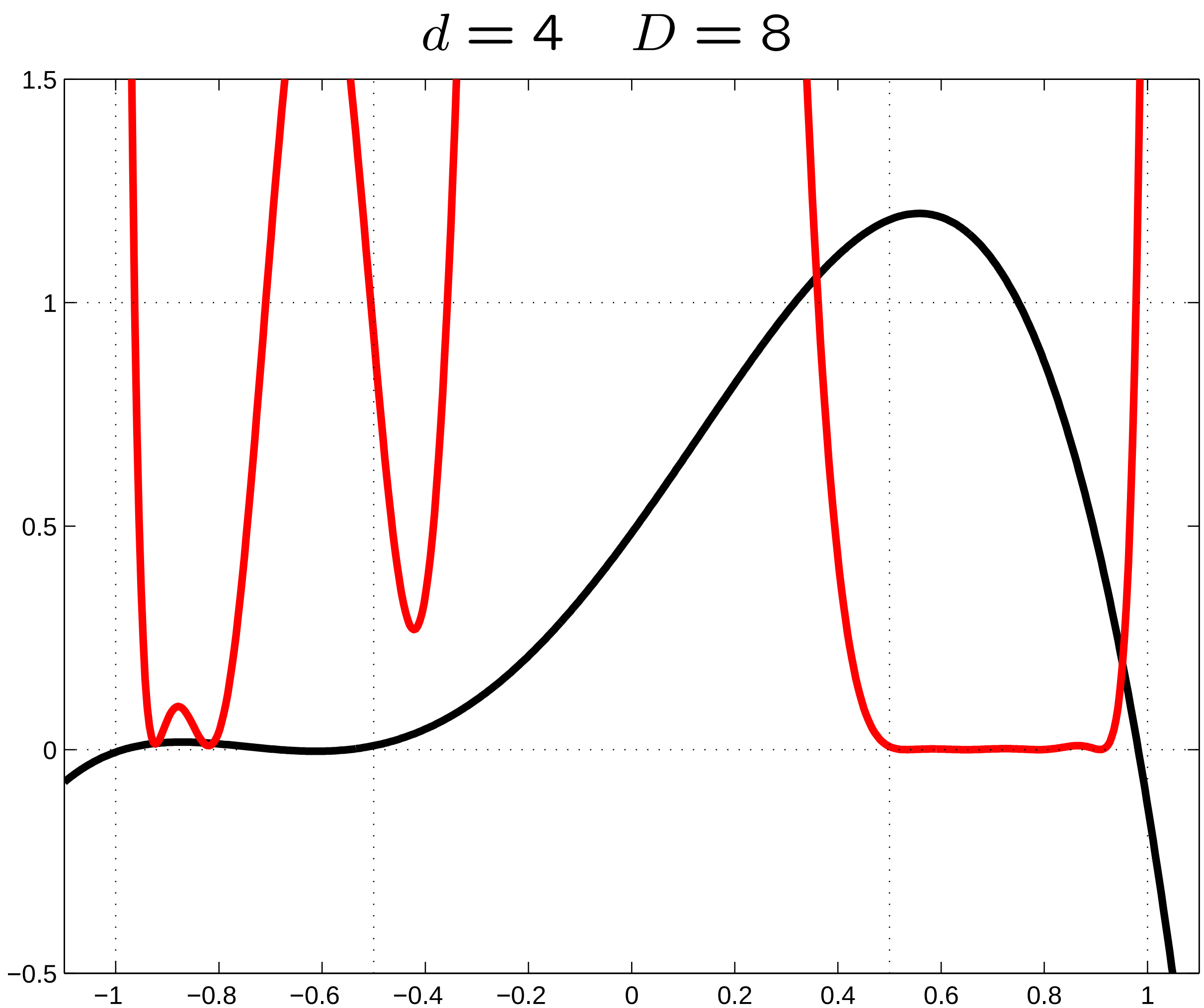

$d = 4 \quad D = 4$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


$d = 4 \quad D = 5$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

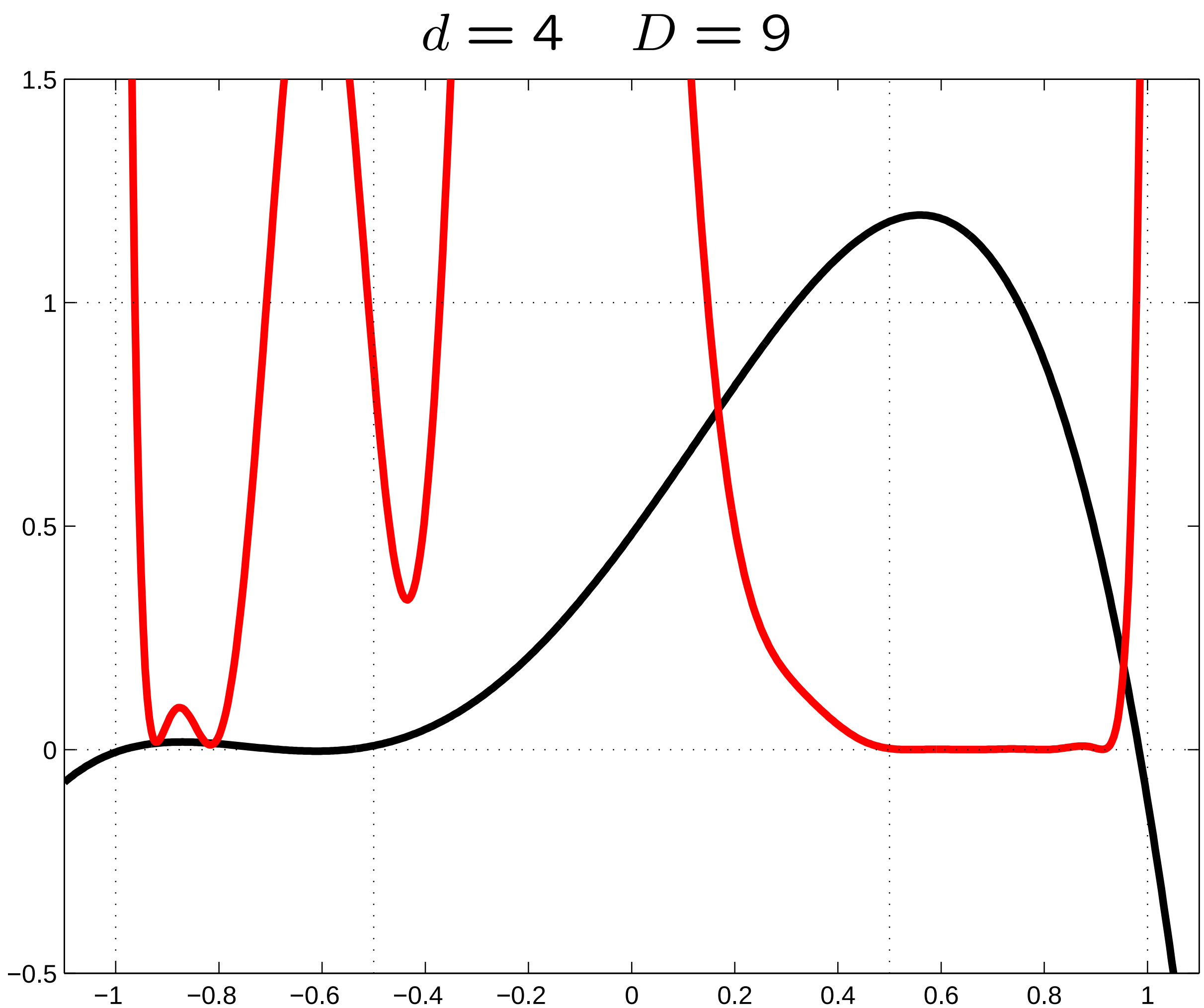


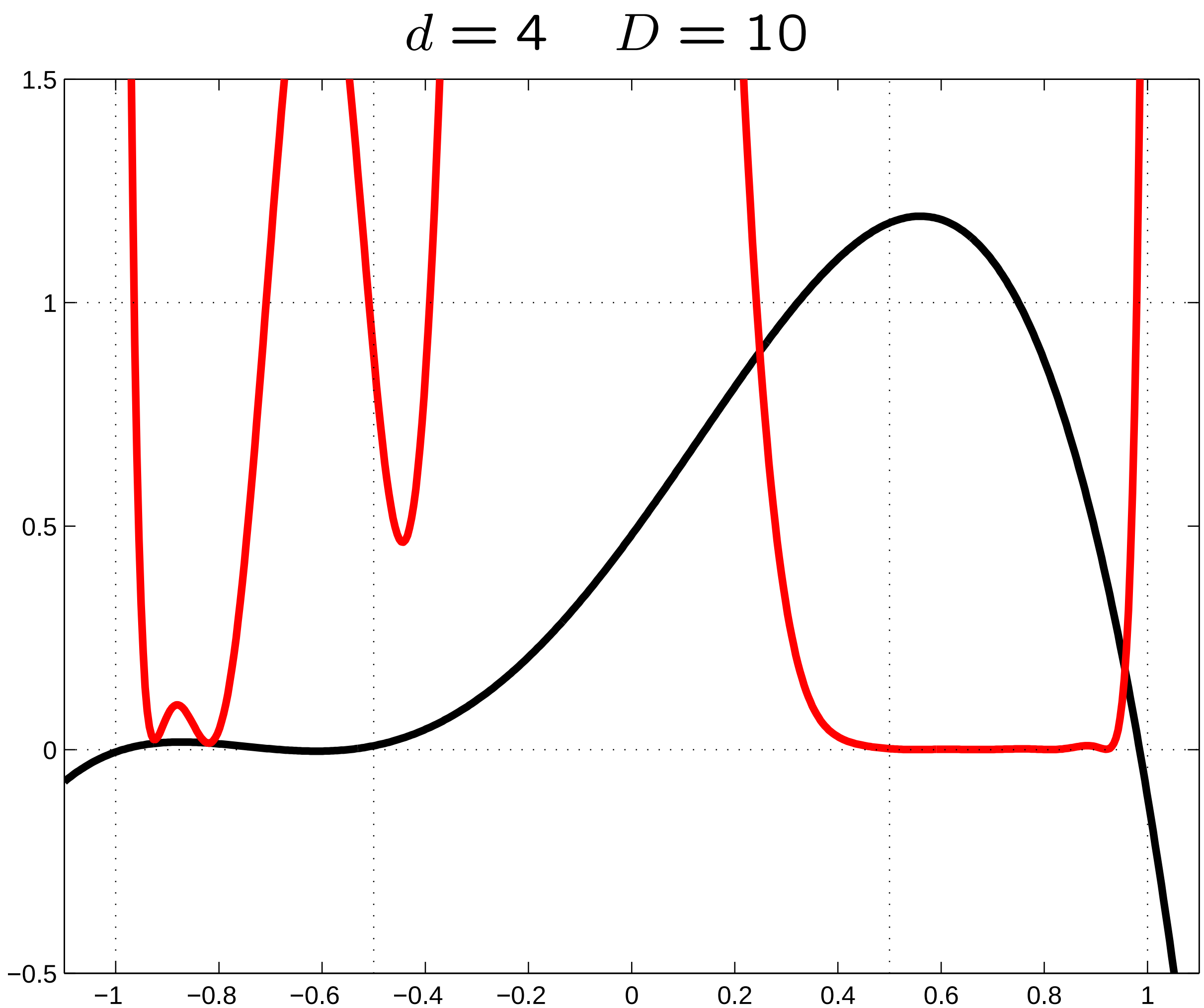
$d = 4 \quad D = 6$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


$d = 4 \quad D = 7$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



$d = 4 \quad D = 8$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

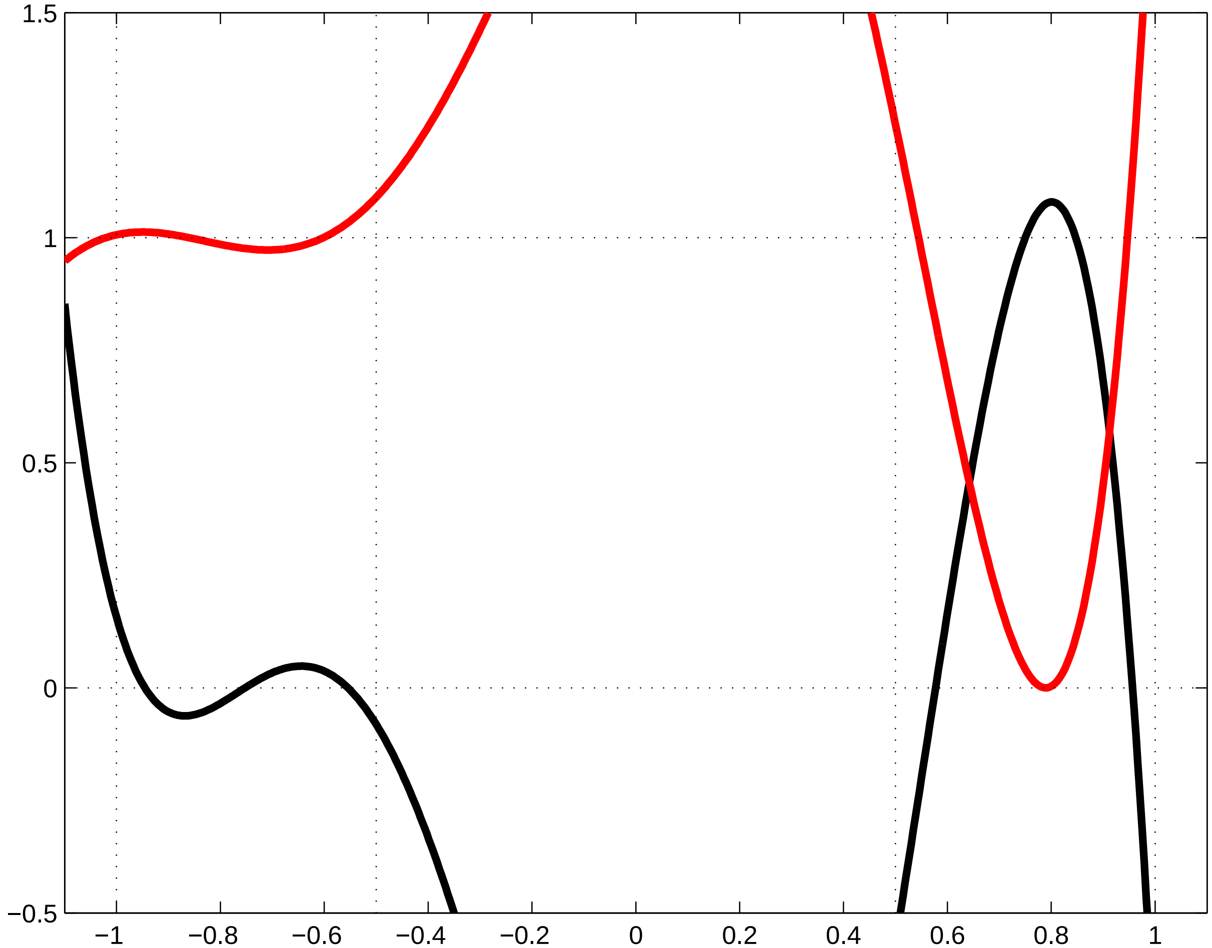

$d = 4 \quad D = 9$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

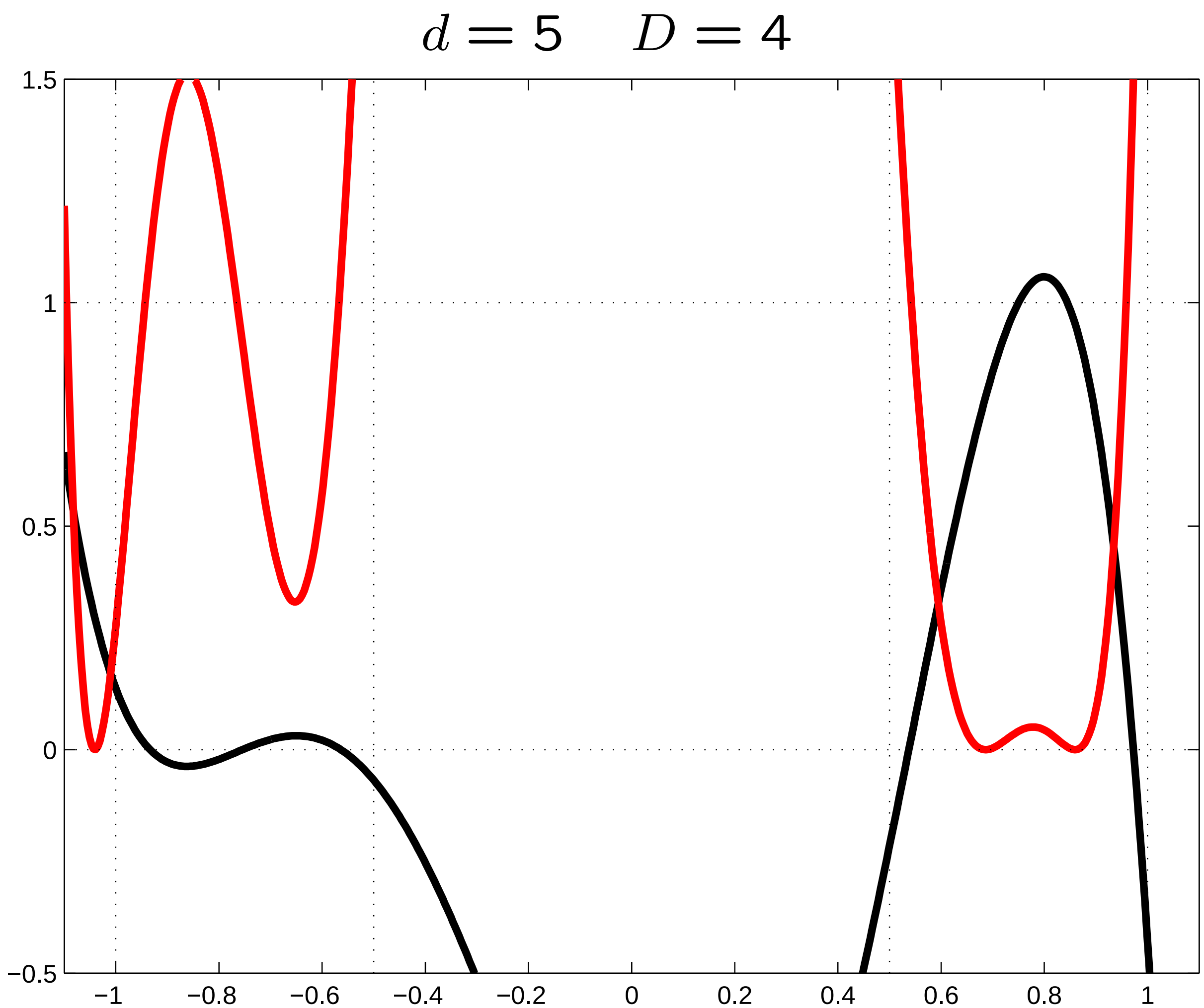


$d = 4 \quad D = 10$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

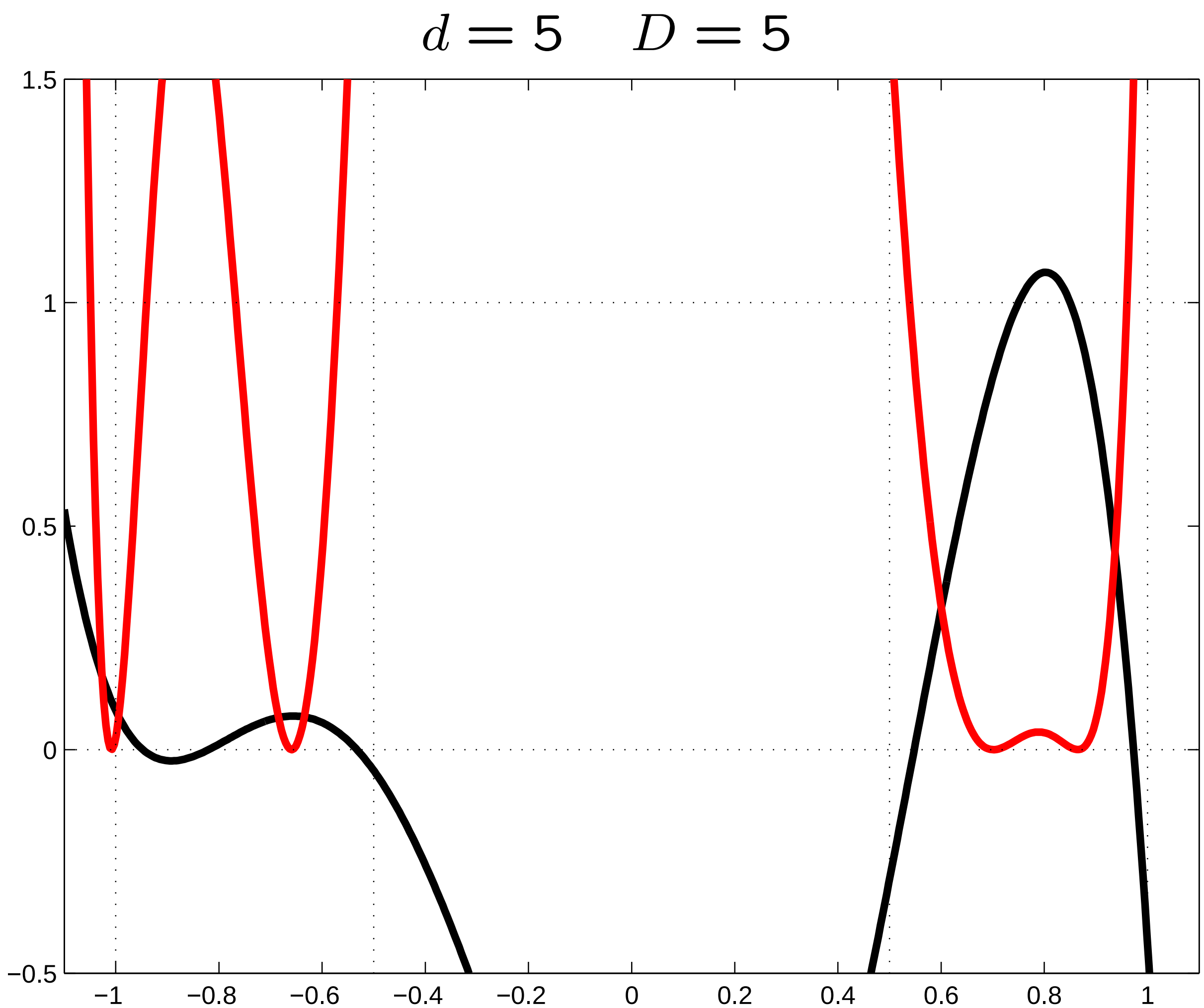
$$d = 5$$

$d = 5 \quad D = 3$

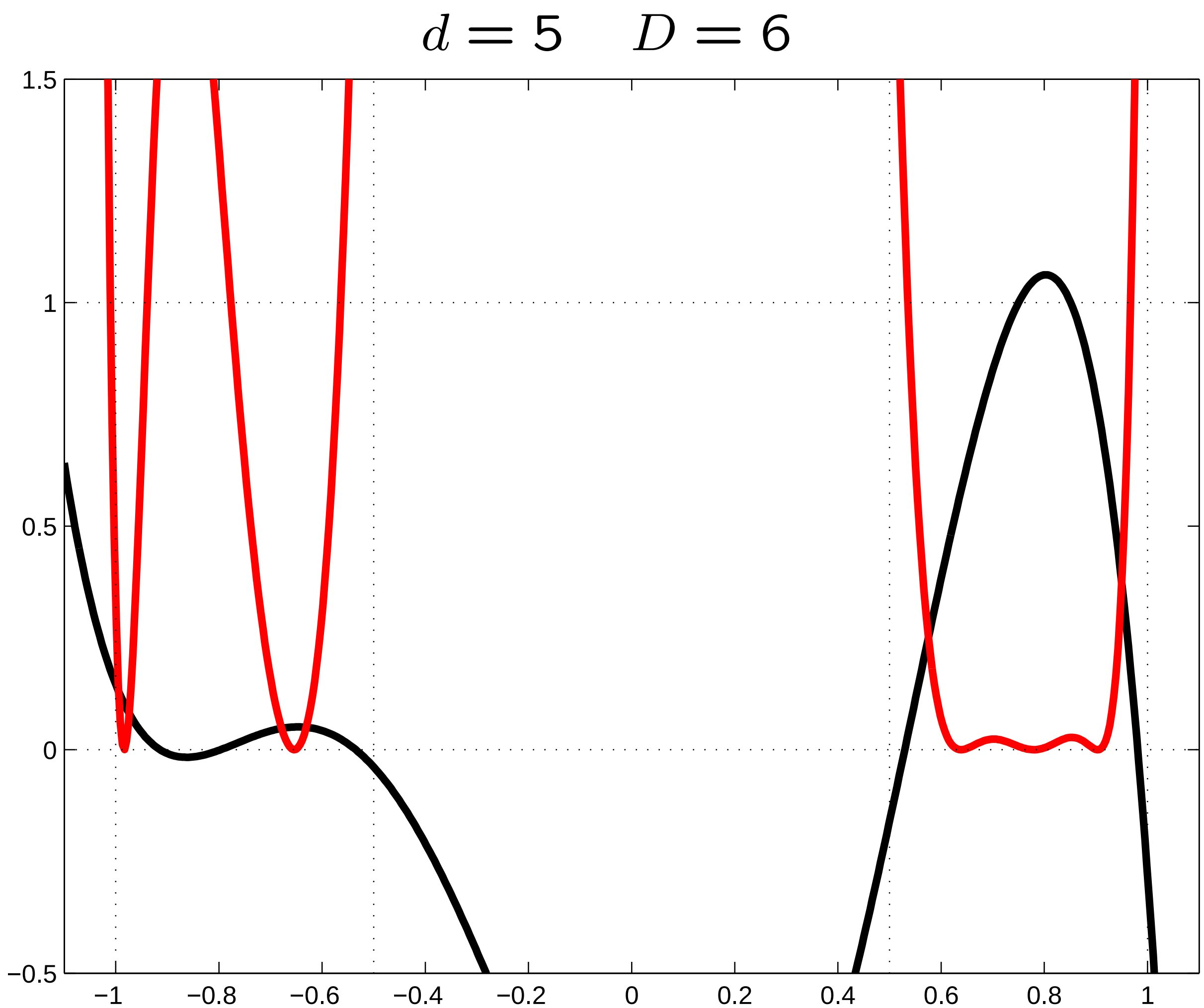




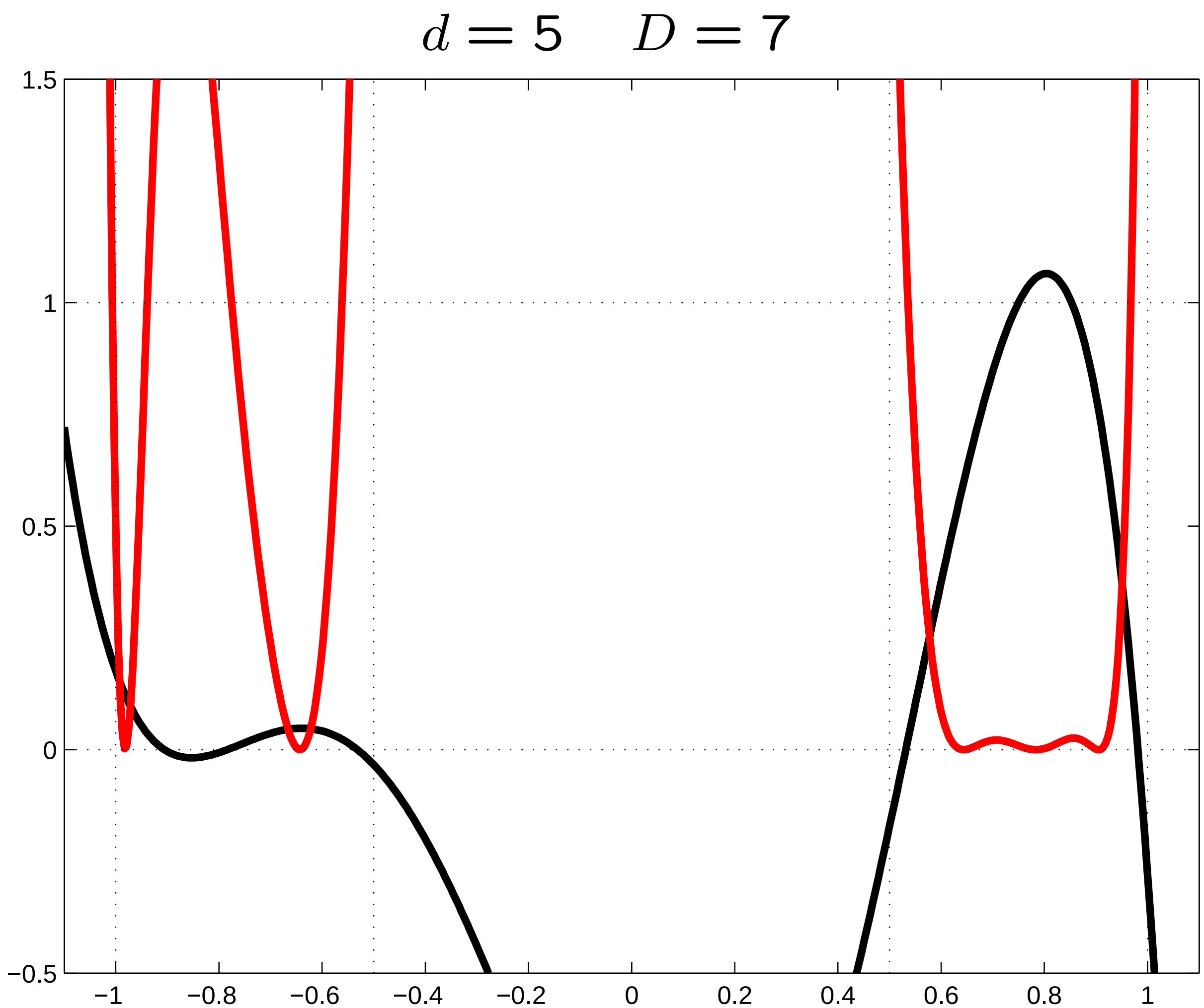
$d = 5 \quad D = 4$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

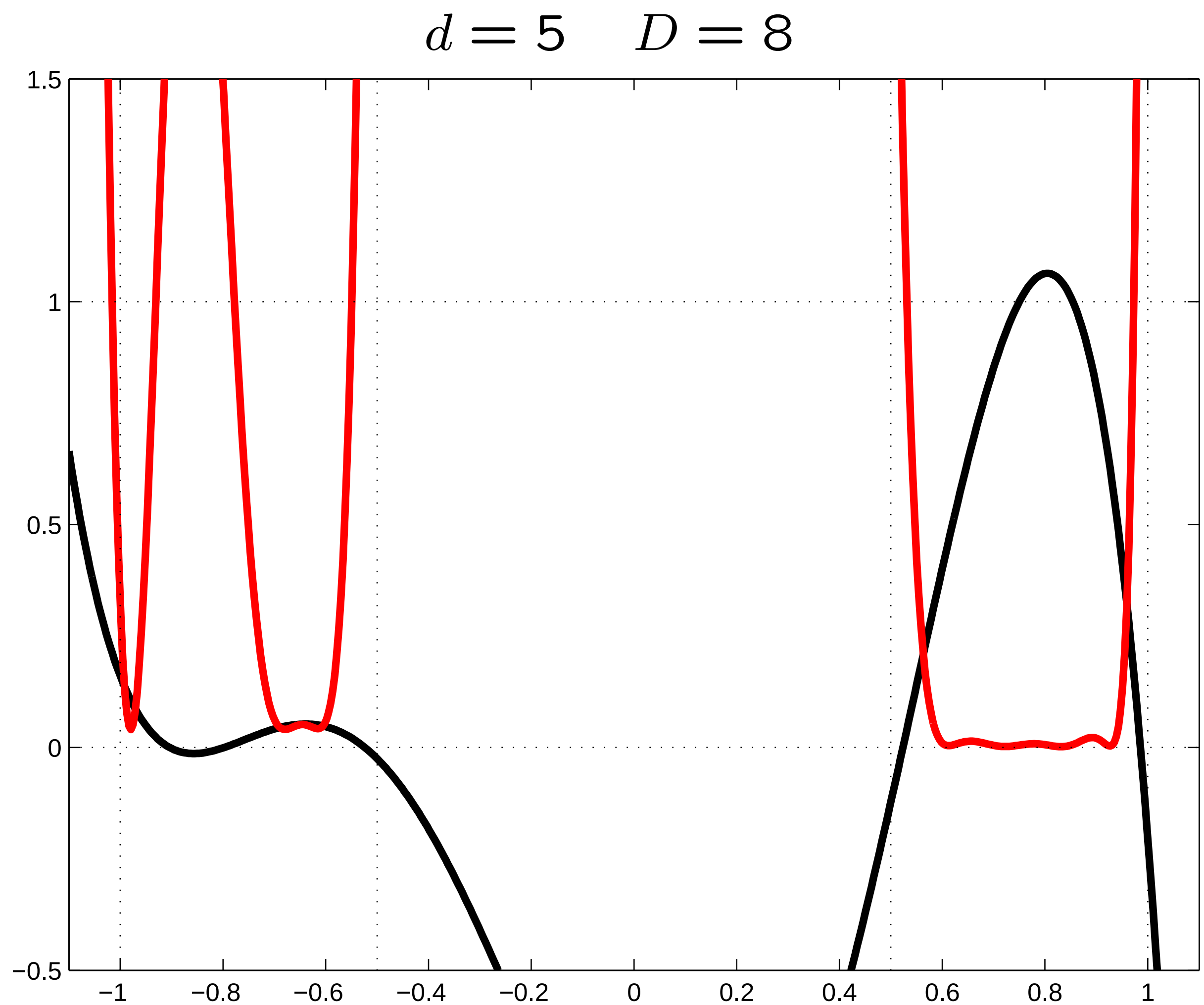


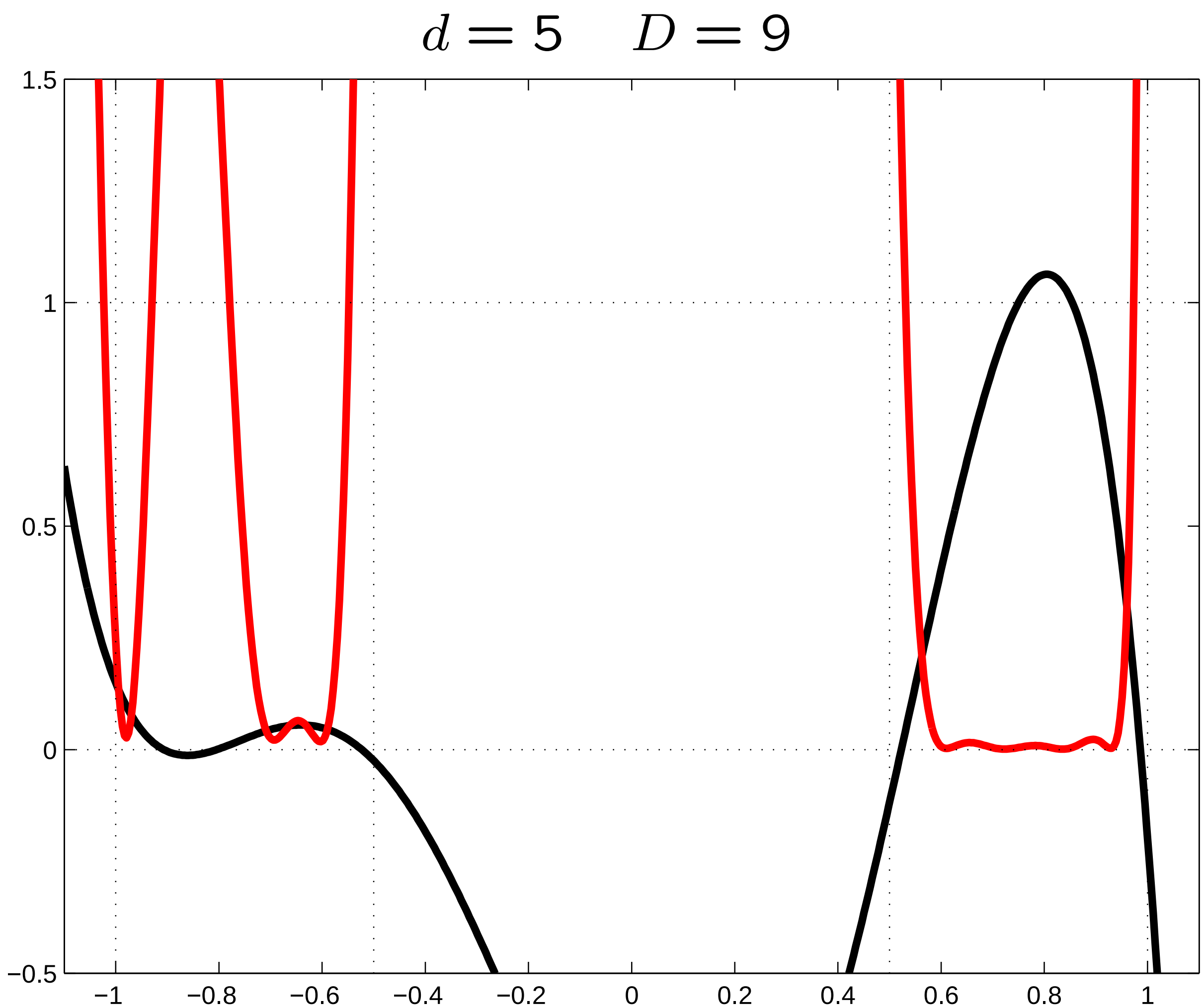
$d = 5 \quad D = 5$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



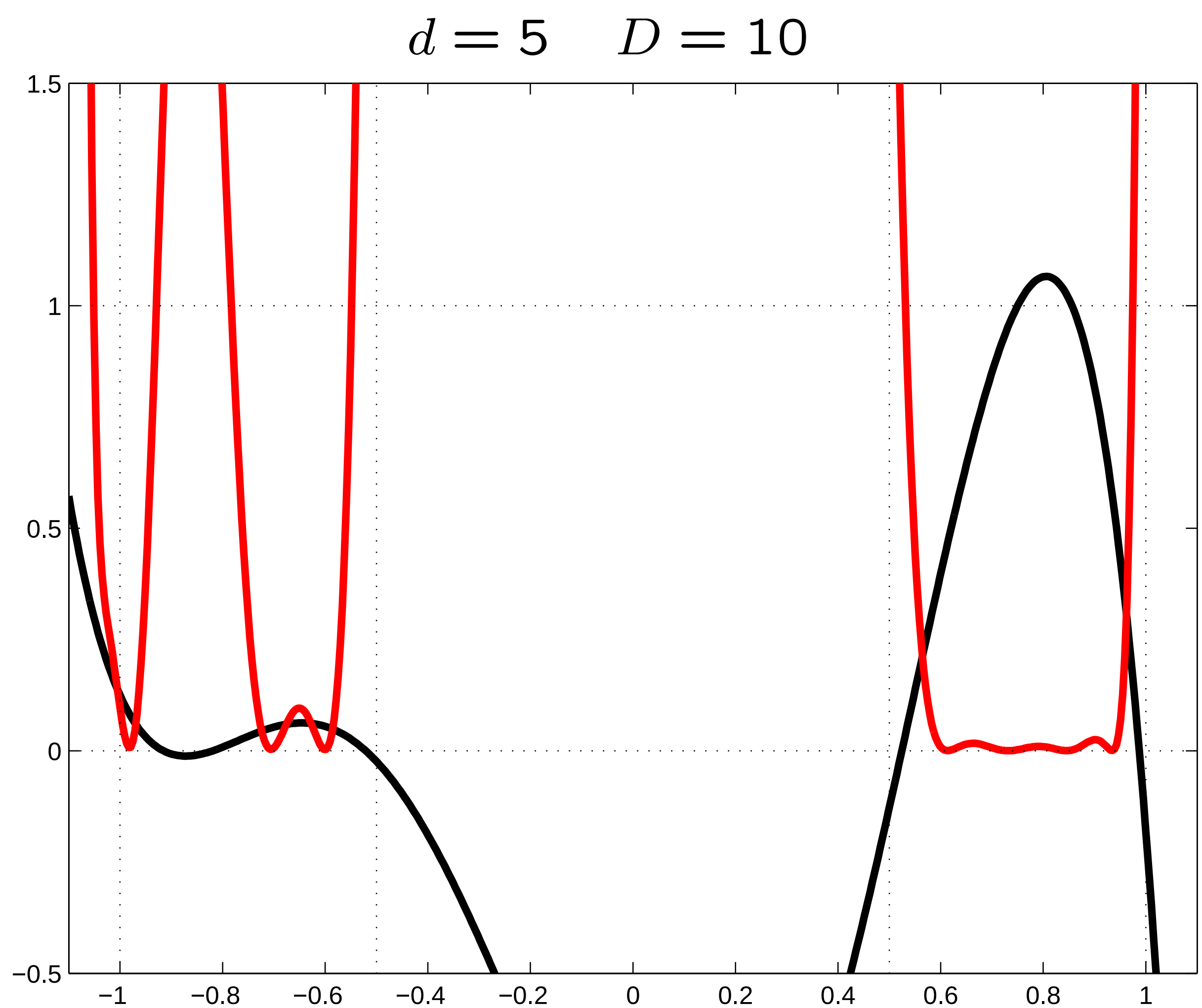
$d = 5 \quad D = 6$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


$d = 5 \quad D = 7$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


$d = 5 \quad D = 8$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



$d = 5 \quad D = 9$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



$d = 5 \quad D = 10$
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

Observations

Already for univariate problems severe oscillations occur when hierarchy order D increases

We should first look at lower degree models, i.e. d small

No significant improvement when $D \geq 10$ but prior experiments on volume computation indicate that this due to the choice of polynomial basis

For instance, Chebyshev basis on $[-1, 1]$ allows much more accurate models for $10 \leq D \leq 100$

Concluding remarks

In theory we have convergence to a valid defining polynomial

In practice numerical behavior is disappointing

Key tuning parameters:
- choice of objective function
- choice of dehomogenization constraint

Instead of minimizing $L_y(g) = \int_K g(x)d\mu(x)$ we tried $L_y(g^2) = \int_K g^2(x)d\mu(x)$ but it does not significantly reduce the oscillations

Should we try to minimize $\int_K \|\nabla g(x)\|^2 d\mu(x)$ inspired by PDE Dirichlet's problem ? Solution should be as smooth as possible

What is the connection with quadrature domains ?