Joint Czech-French Workshop on

Krylov methods for Inverse Problems

Survey of regularization by mollification

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MATHEMATICAL INSTITUTE OF TOULOUSE

Prague, July 19, 2010

Outline

- Introduction
- Approximate inverses
- Fourier synthesis
- Variational theorems
- Pseudo-commutants

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Mollifiers...

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Reminder Let $\varphi \in L^1(\mathbb{R}^n)$ be such that $\int \varphi(x)\,\mathrm{d}x = 1$.
For every $\varepsilon > 0$, let

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right).$$

Let $p \in [1, \infty)$. Then, for every $f \in L^p(\mathbb{R}^n)$,

$$\|\varphi_\varepsilon * f - f\|_p \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

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Example form imaging:

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- R is some integral operator

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A reasonable objective: the reconstruction of $\varphi_\beta * f$

Milestones...

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A LANNES, S ROQUES & M-J CASANOVE, *Stabilized reconstruction in signal and image processing*, J. Mod. Opt., 1987.

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A K LOUIS & P MAASS, A mollifier method for linear operator equations of the first kind, *Inverse Problems*, 1990.

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X. Bonnefond & P M, *A variational approach to the inversion of some compact operators*, Pacific Journal of Optimization, 2009.

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$$f_\beta(\mathbf{y}) := \int f(\mathbf{x}) \phi_\beta(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{x} \xrightarrow{a.e.} f(\mathbf{y}) \quad \text{as} \quad \beta \downarrow 0$$

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ψ_β is referred to as a *reconstruction kernel*

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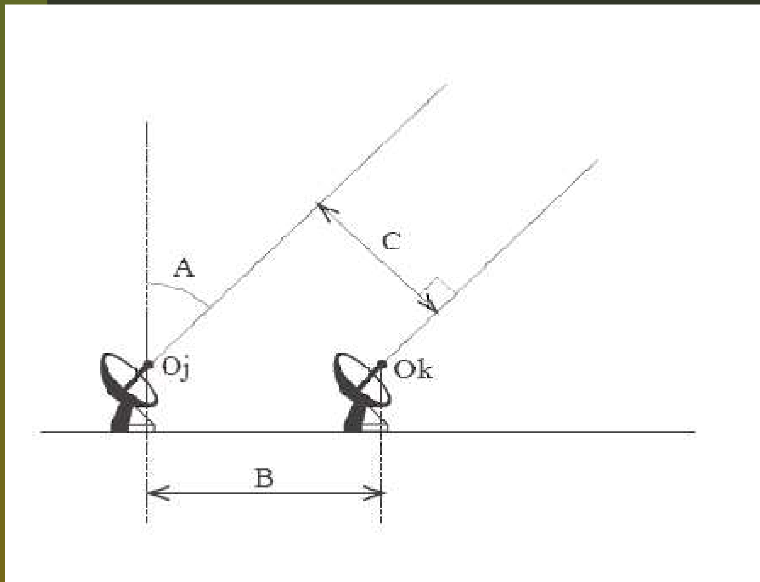
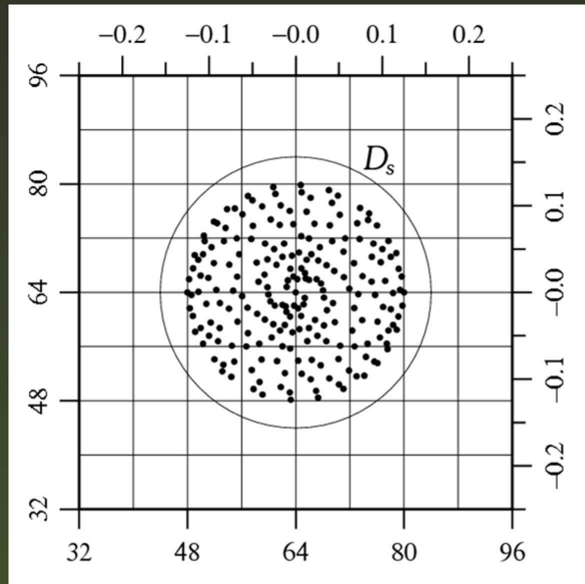
Fourier Synthesis

Recover a function from a partial and approximate knowledge of its Fourier transform.

Example 1: Aperture synthesis





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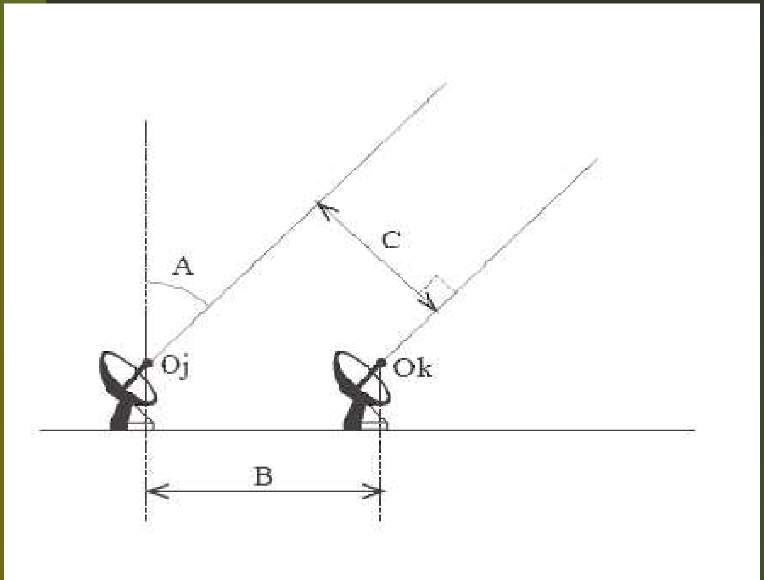
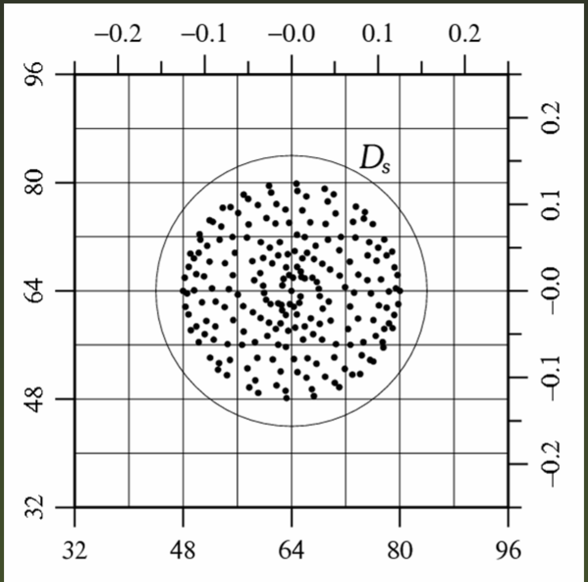
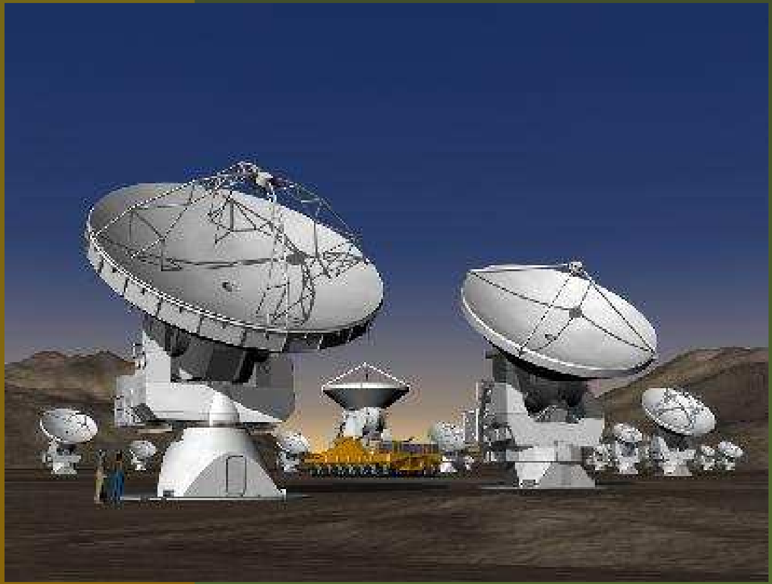
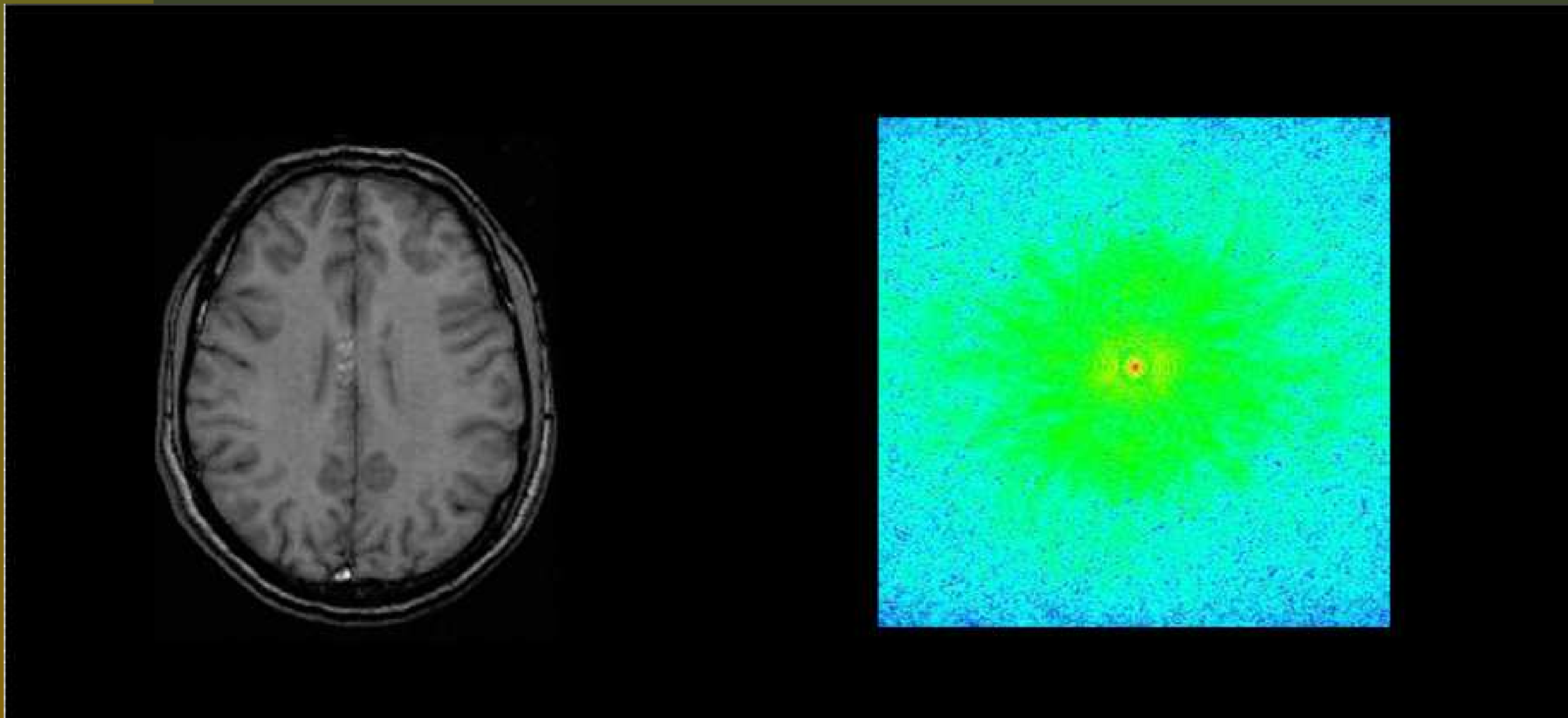






Example 2: MRI

Standard acquisitions:



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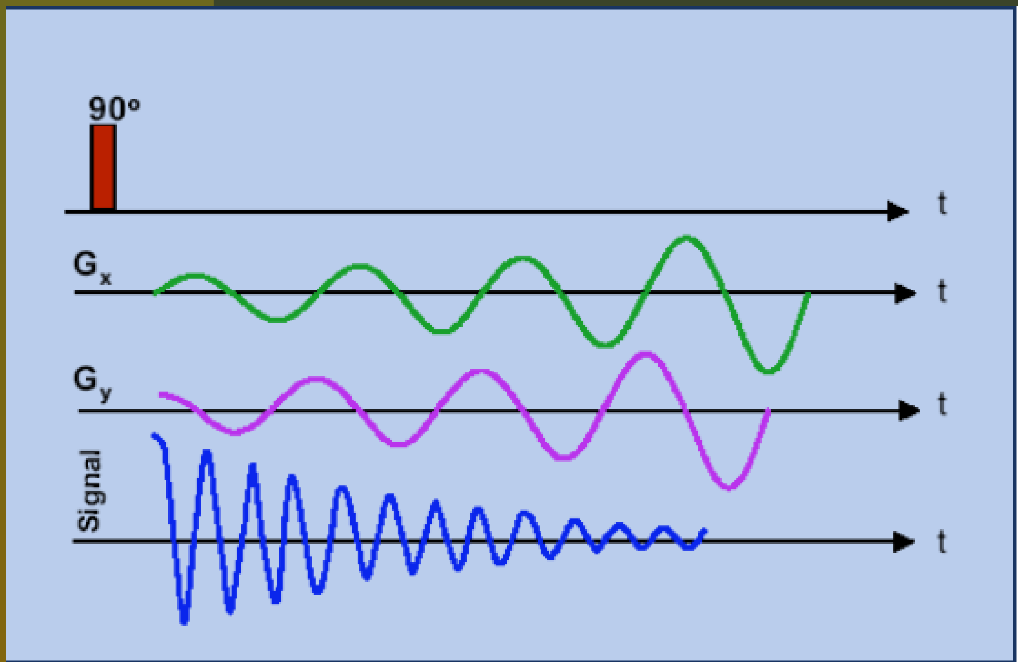
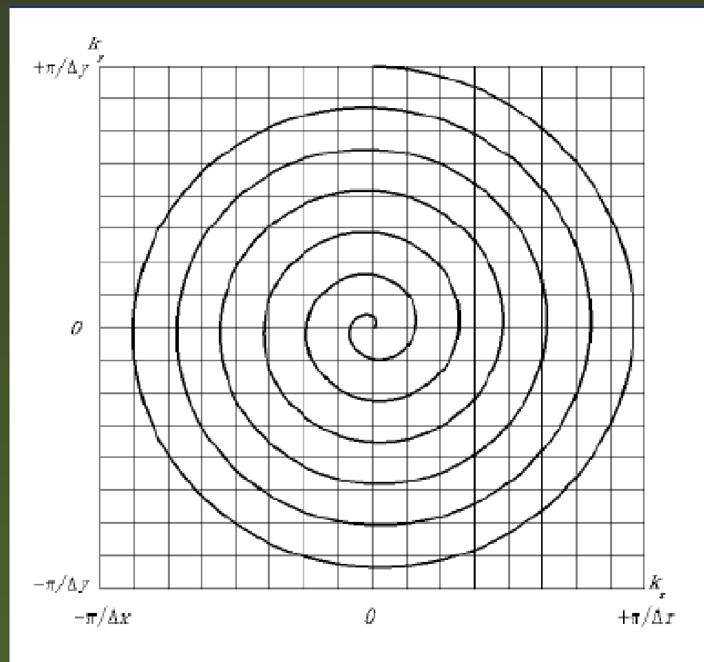
Non-Cartesian and sparse acquisitions:





Fourier extrapolation (Lannes *et al*)

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Let V and W be subsets of $\mathbb{R}^p$. Assume that V is bounded and that W has a non-empty interior. Recover $f_0 \in L^2(V)$ from the knowledge of its Fourier transform on W.

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Truncated Fourier operator:

$$\begin{array}{rccl} T_W : & L^2(V) & \longrightarrow & L^2(W) \\ & f & \longmapsto & T_W f := \mathbb{1}_W \hat{f} = \mathbb{1}_W U f. \end{array}$$

Properties of T_W

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$$(T_W f)(\xi) = \int_{\mathbb{R}^d} \underbrace{e^{-2i\pi\langle x,\xi\rangle} \mathbb{1}_V(x) \mathbb{1}_W(\xi)}_{\alpha(x,\xi) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)} f(x)\,\mathrm{d}x.$$

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Proposition Assume that $\Omega \subseteq \mathbb{R}^d$ is such that Ω^c is bounded. Then,

(i) T_Ω is bounded and injective;

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Reason

- $T_\Omega^\star T_\Omega = I - T_{\Omega^c}^\star T_{\Omega^c}$
- $T_\Omega^\star T_\Omega$ can be *diagonalized*, with eigenvalues $\mu_k := \lambda_k(T_\Omega^\star T_\Omega) = 1 - \lambda_k(T_{\Omega^c}^\star T_{\Omega^c})$

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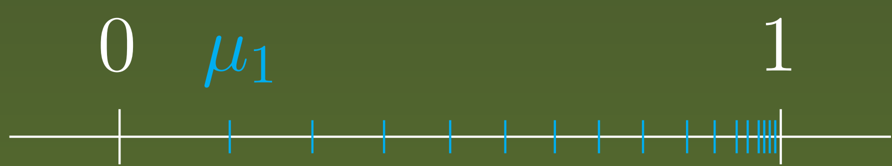

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$$\left|\begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\left\|\hat{\phi}_\beta g - T_W f\right\|^2_{L^2(W)} + \dfrac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}\right\|^2_{L^2(\mathbb{R}^d)} \\ \text{s.t.} & f \in L^2(V) \end{array}\right.$$

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Remark On denoting $C_\beta : f \mapsto \phi_\beta * f$ and $\Phi_\beta : g \mapsto \hat{\phi}_\beta g$, we have the *pseudo-commutation*:

$$T_W C_\beta = \Phi_\beta T_W$$

Well-posedness

$$(\mathcal{P}_{\alpha,\beta}) \quad \left| \begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\left\| \hat{\phi}_\beta g - T_W f \right\|^2 + \dfrac{\alpha}{2}\left\| (1-\hat{\phi}_\beta)\hat{f} \right\|^2 \\ \text{s.t.} & f \in L^2(V) \end{array} \right.$$

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Proposition Let $\alpha, \beta > 0$ be fixed. Then $(\mathcal{P}_{\alpha,\beta})$ has a unique solution $f_{\alpha,\beta}$, which depends continuously on $g \in L^2(W)$.

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$$H^s(\mathbb{R}^2) := \left\{ f \in L^2(\mathbb{R}^2) \;\middle|\; \int \left(1 + \|\xi\|^2\right)^s \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi < \infty \right\}$$

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- $\forall \xi \in \mathbb{R}^d \setminus \{0\}, \hat{\phi}(\xi) \neq 1$

If $g \in T_W(L^2(V) \cap H^s(\mathbb{R}^d))$, then $f_\beta \to T_W^+ g$ strongly as $\beta \downarrow 0$.

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Step 3: The convergence is in fact strong

Overview of the proof

Step 1: $(f_\beta)_{\beta\in(0,1]}$ is bounded

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$$\left.\begin{array}{r} (f_n) \text{ bounded} \\ \lim_{R\to\infty} \sup_n \int_{\|x\|>R} |f_n(x)|^2 \,\mathrm{d}x = 0 \\ \sup_n \|\mathcal{T}_h f_n - f_n\| \to 0 \text{ as } \|h\| \to 0 \end{array}\right\} \Rightarrow (f_n) \text{ precompact}$$

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$\hookrightarrow$ ϕ is positive, isotropic, radially decreasing, C^∞

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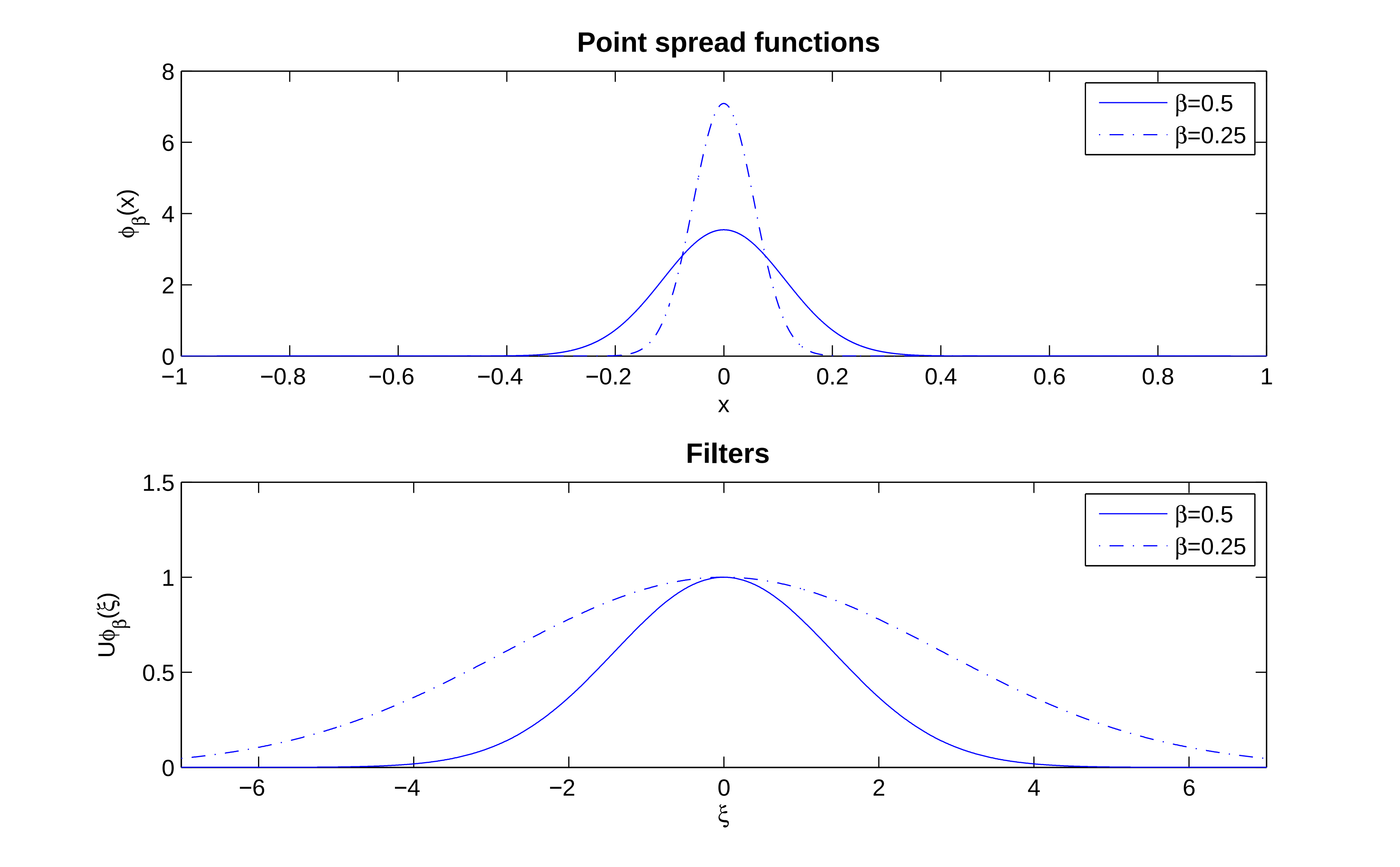

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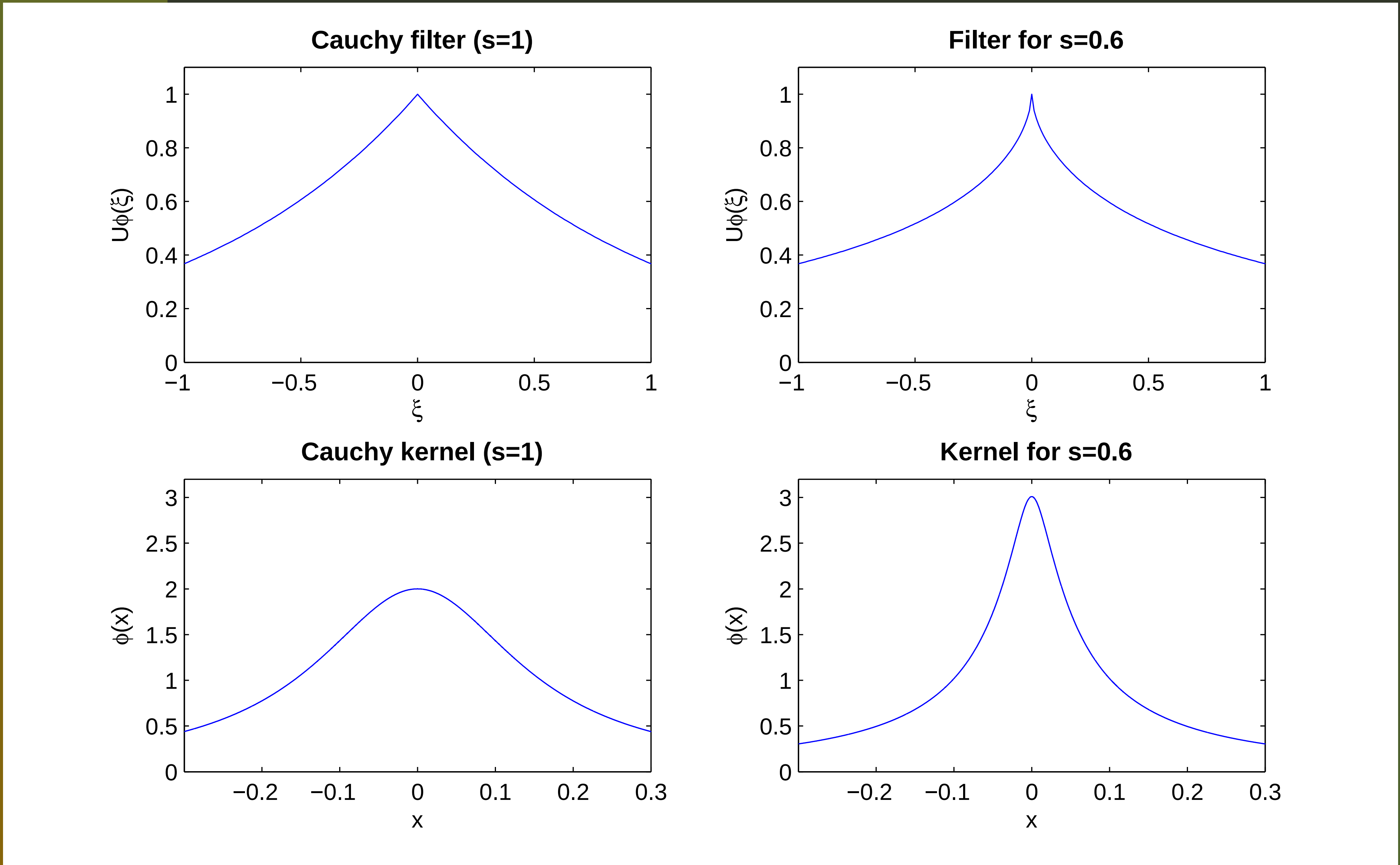

First extensions

$$RC_\beta = \Phi_\beta R \quad \text{with} \quad C_\beta := U^{-1}\hat{\phi}_\beta U$$

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$$R(f_1 * f_2) = Rf_1 \circledast Rf_2$$

$\circledast$ convolution selon s

$$\hookrightarrow \quad RC_\beta f = R(\phi_\beta * f) = R\phi_\beta \circledast Tf$$

$$\Phi_\beta = (g \mapsto R\phi_\beta \circledast g)$$

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$$(\mathcal{Q}_\beta) \quad \left| \begin{array}{l} \text{Minimize} \quad \dfrac{1}{2} \left\| RC_\beta - XR \right\|^2 \\ \text{s.t.} \quad X \in L(G), \;\; X = 0 \;\text{ on }\; (\operatorname{ran} R)^\perp \end{array} \right.$$

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$R\colon L^2(V) \to G, \quad G$ Hilbert space

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Theorem With the same assumptions on ϕ as before, assume that $g \in R(L^2(V) \cap H^s(\mathbb{R}^d))$. Then $f_\beta \to R^+ g$ strongly as $\beta \downarrow 0$.

Outline

- Introduction
- Approximate inverses
- Fourier synthesis
- Variational theorems
- Pseudo-commutants

Matrix formulation

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Definition We call *pseudo-commutant* of a matrix $C \in \mathbb{R}^{n\times n}$ w.r.t. a matrix $R \in \mathbb{R}^{m\times n}$ the unique solution $\Phi \in \mathbb{R}^{m\times m}$ of

$$(\mathcal{Q}) \quad \left| \begin{array}{rl} \text{Minimize} & \|XR - RC\|_F \\ \text{s.t.} & X \in \mathbb{R}^{m\times m},\ X(\ker R^\top) = \{\mathbf{0}\}. \end{array} \right.$$

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Proposition The matrix $\Phi = RCR^+$ is the unique solution to $(\mathcal{Q})$, and in the case where R is injective, then R^+R is the identity, so that Φ actually satisfies $\Phi R = RC$.

Spectral functions

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Remark The Frobenius norm satisfies

$$\| M \|_F^2 = \mathrm{tr}(M^\top M) = \sum_{j=1}^{m} \sigma_j^2(M),$$

where $\sigma_1(M) \geq \cdots \geq \sigma_m(M)$ are the singular values of M.

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We shall see that the solution RCR^+ remains unchanged if we replace $\| \cdot \|_F$ in Problem $(\mathcal{Q})$ by any convex spectral function.

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Definition A function $F\colon \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ is said to be orthogonally invariant if $F(UMV) = F(M)$ for all $M \in \mathbb{R}^{m\times n}$ and all $(U, V) \in \mathrm{O}(m) \times \mathrm{O}(n)$.

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Here, $\Pi(m)$ is the group of *signed* permutation matrices of size $m \times m$.

Back to spectral functions

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$$\boldsymbol{\sigma}: \quad \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^m$$
$$M \longmapsto \boldsymbol{\sigma}(M) := (\sigma_1(M), \ldots, \sigma_m(M))$$

Back to spectral functions

$$\begin{array}{rccl} \boldsymbol{\sigma}: & \mathbb{R}^{m\times n} & \longrightarrow & \mathbb{R}^m \\ & M & \longmapsto & \boldsymbol{\sigma}(M) := (\sigma_1(M), \ldots, \sigma_m(M)) \end{array}$$

Proposition A function $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is orthogonally invariant if and only if it satisfies

$$F = F \circ \operatorname{diag}_{m\times n} \circ \boldsymbol{\sigma}.$$

In such a case, $f := F \circ \operatorname{diag}_{m\times n}$ is the unique absolutely symmetric function such that $F = f \circ \boldsymbol{\sigma}$.

Remarkable facts

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Theorem [A. S. Lewis] Let F be orthogonally invariant, and let $f := F \circ \mathrm{diag}_{m \times n}$. Then, for all $M \in \mathbb{R}^{m \times n}$, the subdifferential of F at M is given by

$$\{U \, \mathrm{diag}_{m \times n}(\boldsymbol{\xi}) V \,|\, \boldsymbol{\xi} \in \partial f(\boldsymbol{\sigma}(M)), \, U \in \mathrm{O}(m), \, V \in \mathrm{O}(n)\}.$$

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Let $\mathcal{F}(X) = F(XR - RC)$, in which F is any convex orthogonaly invariant function.

Then RCR^{+} is a solution to

$$(\mathcal{Q}) \quad \left| \begin{array}{rl} \text{Minimize} & \mathcal{F}(X) \\ \text{s.t.} & X(\ker R^{\top}) = \{0\}. \end{array} \right.$$

Thank you for your attention !