Joint Czech-French Workshop on Krylov methods for Inverse Problems

Survey of regularization by mollification

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Outline

- Introduction
- Approximate inverses
- Fourier synthesis
- Variational theorems
- Pseudo-commutants
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- Approximate inverses
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Mollifiers...

... in approximation theory:
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**Reminder**  Let \( \varphi \in L^1(\mathbb{R}^n) \) be such that \( \int \varphi(x) \, dx = 1 \).

For every \( \varepsilon > 0 \), let

\[
\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n} \varphi \left( \frac{x}{\varepsilon} \right).
\]

Let \( p \in [1, \infty) \). Then, for every \( f \in L^p(\mathbb{R}^n) \),

\[
\| \varphi_\varepsilon \ast f - f \|_p \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.
\]
Mollifiers...

... in the theory of inverse problems:
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Ill-posed operator equation: $Rf = g_0$
Mollifiers...

... in the theory of inverse problems:

Ill-posed operator equation: \( Rf = g_0 \)

Example form imaging:

- \( f \in L^2(V) \), where \( V \subset \mathbb{R}^2 \) is the field
- \( R \) is some integral operator
Mollifiers...

... in the theory of inverse problems:

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Example form imaging:

- \( f \in L^2(V) \), where \( V \subset \mathbb{R}^2 \) is the field
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A reasonable objective: the reconstruction of \( \varphi_\beta \ast f \)
Milestones...

... from Fourier synthesis to approximate inverses:
Milestones...

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Milestones...

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Milestones...

... variational theorems:
... variational theorems:

Milestones...

... variational theorems:


Approximate inverses (Louis, Maass)
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Ill-posed equation: $Rf = g$
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Mollifier: $\phi$ continuous, nonnegative, with $\int \phi = 1$
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\phi_\beta(x, y) := \frac{1}{\beta^n} \phi \left( \frac{x - y}{\beta} \right)
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\[
f_\beta(y) := \int f(x) \phi_\beta(x, y) \, dx \xrightarrow{a.e.} f(y) \quad \text{as} \quad \beta \downarrow 0
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$$\langle f, \phi_\beta(\cdot, y) \rangle$$

$$R^* \psi_\beta(y)$$
Approximate inverses
Assumption: $\phi_\beta(\cdot, y) \in \text{ran } R^*$, and one can calculate explicitly

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\psi_\beta(y) = (R^*)^{-1}\phi_\beta(\cdot, y)
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\( \psi_\beta \) is referred to as a reconstruction kernel
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Fourier Synthesis

Recover a function from a partial and approximate knowledge of its Fourier transform.
Example 1: Aperture synthesis
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Example 2: MRI

Standard acquisitions:
Example 2: MRI

Non-Cartesian and sparse acquisitions:
Fourier extrapolation (Lannes et al)
Let $V$ and $W$ be subsets of $\mathbb{R}^p$. Assume that $V$ is bounded and that $W$ has a non-empty interior. Recover $f_0 \in L^2(V)$ from the knowledge of its Fourier transform on $W$. 
Fourier extrapolation (Lannes et al)

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**Truncated Fourier operator:**

$$T_W : L^2(V) \rightarrow L^2(W)$$

$$f \mapsto T_W f := 1_W \hat{f} = 1_W U f.$$
Properties of $T_W$
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$$(T_W f)(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi \langle x, \xi \rangle} \mathbb{1}_V(x) \mathbb{1}_W(\xi) f(x) \, dx.$$
Properties of $T_W$

$$(T_W f)(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi \langle x, \xi \rangle} 1_V(x) 1_W(\xi) f(x) \, dx.$$ 

$\alpha(x, \xi) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$

$\rightarrow$ $T_W$ is Hilbert-Schmidt
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Reminder  The Fourier transform of compactly supported functions are entire functions
Properties of $T_W$

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$\Rightarrow \quad T_W$ is Hilbert-Schmidt

**Reminder**  The Fourier transform of compactly supported functions are entire functions

$\Leftarrow \quad T_W$ is injective
Properties of $T_W$

Thus, $T_W^* T_W$ is compact, injective, Hermitian, positive.
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$\mapsto T_W^+ \text{ is unbounded and } \mathcal{D}(T_W^+) \subsetneq L^2(W)$
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$\mathcal{D}(T_W^+)$ is a dense subset of $L^2(W)$
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Proposition $\lambda_1(T_W^* T_W) < 1$. 
Properties of $T_W$

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Proposition $\lambda_1(T_W^* T_W) < 1.$

\[
\begin{pmatrix}
0 & \lambda_1 & 1
\end{pmatrix}
\]
Fourier interpolation

**Proposition** Assume that $\Omega \subseteq \mathbb{R}^d$ is such that $\Omega^c$ is bounded. Then,

(i) $T_\Omega$ is bounded and injective;

(ii) $\text{ran } T_\Omega$ is closed;

(iii) $T_\Omega^{-1} : \text{ran } T_\Omega \rightarrow L^2(V)$ is bounded.
Fourier interpolation

**Proposition**  Assume that $\Omega \subseteq \mathbb{R}^d$ is such that $\Omega^c$ is bounded. Then,

(i) $T_\Omega$ is bounded and injective;
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(iii) $T^{-1}_\Omega : \text{ran } T_\Omega \rightarrow L^2(V)$ is bounded.

**Reason**

- $T_\Omega^* T_\Omega = I - T^*_\Omega c T_{\Omega c}$
- $T_\Omega^* T_\Omega$ can be diagonalized, with eigenvalues $\mu_k := \lambda_k(T_\Omega^* T_\Omega) = 1 - \lambda_k(T_{\Omega c}^* T_{\Omega c})$
Fourier interpolation

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**Reason**

- $T^*_\Omega T_\Omega = I - T^*_{\Omega^c} T_{\Omega^c}$

- $T^*_\Omega T_\Omega$ can be diagonalized, with eigenvalues

\[
\begin{pmatrix}
0 & \mu_1 & 1
\end{pmatrix}
\]
Minimize \[ \frac{1}{2} \| g - T_wf \|^2_{L^2(W)} + \frac{\alpha}{2} \| 1_{W_\beta} U f \|^2_{L^2(W_\beta)} \]
Minimize \( \frac{1}{2} \| g - T_W f \|_{L^2(W)}^2 + \frac{\alpha}{2} \| 1_{W_\beta} U f \|_{L^2(W_\beta)}^2 \)

\( W_\beta \): complement of \( B_{1/\beta} \)
Minimize
\[ \frac{1}{2} \| g - T_{W} f \|_{L^2(W)}^2 + \frac{\alpha}{2} \| 1_{W_{\beta}} U f \|_{L^2(W_{\beta})}^2 \]

\( W_{\beta} \): complement of \( B_{1/\beta} \)

New object to be reconstructed: \( \phi_{\beta} \ast f_0 \)
Regularization

Minimize \( \frac{1}{2} \| g - Twf \|_{L^2(W)}^2 + \frac{\alpha}{2} \| 1_{W_\beta} Uf \|_{L^2(W_\beta)}^2 \)

\( W_\beta \): complement of \( B_{1/\beta} \)

New object to be reconstructed: \( \phi_\beta \ast f_0 \)

\( \phi_\beta := U^{-1} 1_{B_{1/\beta}} \)
Apodized version
Minimize \[ \frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|_{L^2(W)}^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_\beta) \hat{f} \right\|_{L^2(\mathbb{R}^d)}^2 \]

s.t. \( f \in L^2(V) \)
Minimize \[ \frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|^2_{L^2(W)} + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_\beta) \hat{f} \right\|^2_{L^2(\mathbb{R}^d)} \]

s.t. \( f \in L^2(V) \)

\[ \phi_\beta(x) = \frac{1}{\beta^d} \phi \left( \frac{x}{\beta} \right) \]
Apodized version

Minimize \[ \frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|_{L^2(W)}^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_\beta) \hat{f} \right\|_{L^2(\mathbb{R}^d)}^2 \]

s.t. \[ f \in L^2(V) \]

Regularized data: \[ T_W (\hat{\phi}_\beta * f_0) = \hat{\phi}_\beta T_W f_0 \approx \hat{\phi}_\beta g \]
Minimize \( \frac{1}{2} \left\| \hat{\phi}_g - T_W f \right\|_{L^2(W)}^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_g) \hat{f} \right\|_{L^2(\mathbb{R}^d)}^2 \)

s.t. \( f \in L^2(V) \)

Regularized data: \( T_W (\phi_* f_0) = \hat{\phi}_T W f_0 \approx \hat{\phi}_g \)

Remark: On denoting \( C_\beta : f \leftrightarrow \phi_* f \) and \( \Phi_\beta : g \leftrightarrow \hat{\phi}_g \), we have the pseudo-commutation:

\( T_W C_\beta = \Phi_\beta T_W \)
Well-posedness

\( (P_{\alpha,\beta}) \)

Minimize

\[ \frac{1}{2} \left\| \hat{\phi}_{\beta} g - T_{W} f \right\|^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_{\beta}) \hat{f} \right\|^2 \]

s.t.

\[ f \in L^2(V) \]
Well-posedness

**Definition** \[ \langle f_1, f_2 \rangle_\beta := \int_{\mathbb{R}^d} (1 - \hat{\phi}_\beta)^2 U f_1 U f_2 \]

\[ \minimize \frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_\beta) \hat{f} \right\|^2 \]

s.t. \[ f \in L^2(V) \]
Well-posedness

**Definition** \( \langle f_1, f_2 \rangle_\beta := \int_{\mathbb{R}^d} |1 - \hat{\phi}_\beta|^2 U f_1 \overline{U f_2} \)

**Lemma** \( \langle \cdot, \cdot \rangle_\beta \) is an inner product which turns \( L^2(V) \) into a Hilbert space. The corresponding norm \( \| \cdot \|_\beta \) is equivalent to \( \| \cdot \|_{L^2(V)} \).

\[
(P_{\alpha,\beta}) \quad \text{Minimize} \quad \frac{1}{2} \left\| \hat{\phi}_\beta g - TW f \right\|^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_\beta) \hat{f} \right\|^2 \\
\text{s.t.} \quad f \in L^2(V)
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Well-posedness

**Definition** \( \langle f_1, f_2 \rangle_\beta := \int_{\mathbb{R}^d} |1 - \hat{\phi}_\beta|^2 U f_1 U f_2 \)

**Lemma** \( \langle \cdot, \cdot \rangle_\beta \) is an inner product which turns \( L^2(V) \) into a Hilbert space. The corresponding norm \( \| \cdot \|_\beta \) is equivalent to \( \| \cdot \|_{L^2(V)} \).

**Proposition** Let \( \alpha, \beta > 0 \) be fixed. Then \((P_{\alpha,\beta})\) has a unique solution \( f_{\alpha,\beta} \), which depends continuously on \( g \in L^2(W) \).

\[
(P_{\alpha,\beta}) \quad \text{Minimize} \quad \frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_\beta) \hat{f} \right\|^2
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s.t. \( f \in L^2(V) \)
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For simplicity of notation: $T = T_W$, and $f_\beta = f_{\alpha,\beta}$
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\( f \in L^2(V) \)
For simplicity of notation: \( T = T_{W} \), and \( f_{\beta} = f_{\alpha,\beta} \)

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\frac{1}{2} \left\| \hat{\phi}_{\beta} g - T f \right\|^2 + \frac{\alpha}{2} \left\| (1 - \hat{\phi}_{\beta}) \hat{f} \right\|^2
\]

s.t. \( f \in L^2(V) \)

\( H^{s}(\mathbb{R}^2) := \left\{ f \in L^2(\mathbb{R}^2) \mid \int (1 + \| \xi \|^2)^{s} \left| \hat{f}(\xi) \right|^2 \, d\xi < \infty \right\} \)
Main Result
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Theorem  Assume that

- $\alpha > 0$ (fixed)
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- $\phi \in L^1(\mathbb{R}^d)$ with $\int \phi(x) \, dx = 1$ (i.e. $\hat{\phi}(0) = 1$)
Main Result

Theorem Assume that

- $\alpha > 0$ (fixed)
- $\phi \in L^1(\mathbb{R}^d)$ with $\int \phi(x) \, dx = 1$ (i.e. $\hat{\phi}(0) = 1$)
- $|1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} K\|\xi\|^s$ for some $K, s > 0$
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**Theorem** Assume that

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- $|1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} K\|\xi\|^s$ for some $K, s > 0$
- $\forall \xi \in \mathbb{R}^d \setminus \{0\}, \hat{\phi}(\xi) \neq 1$
Main Result

Theorem  Assume that

- \( \alpha > 0 \) (fixed)
- \( \phi \in L^1(\mathbb{R}^d) \) with \( \int \phi(x) \, dx = 1 \) (i.e. \( \hat{\phi}(0) = 1 \))
- \(|1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} K \|\xi\|^s \) for some \( K, s > 0 \)
- \( \forall \xi \in \mathbb{R}^d \setminus \{0\}, \hat{\phi}(\xi) \neq 1 \)

If \( g \in T_W(L^2(V) \cap H^s(\mathbb{R}^d)) \), then \( f_\beta \to T_W^+ g \) strongly as \( \beta \downarrow 0 \).
Overview of the proof
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Step 1: $(f_\beta)_{\beta \in (0,1]}$ is bounded
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Step 2: \((f_\beta)_{\beta \in (0,1]}\) converges weakly to \(T_W^+ g\)
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$\beta_n \downarrow 0$, $f_n := f_{\beta_n}$
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\[\exists (f_{n_k}) \rightharpoonup T_W^+ g\]
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Step 3: The convergence is in fact strong
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Step 1: \((f_\beta)_{\beta \in (0,1]}\) is bounded

Step 2: \((f_\beta)_{\beta \in (0,1]}\) converges weakly to \(T_W^+ g\)

\[ \beta_n \downarrow 0, f_n := f_{\beta_n} \]

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Step 3: The convergence is in fact strong

\((f_n)\) bounded

\[ \lim_{R \to \infty} \sup_n \int_{\|x\| > R} |f_n(x)|^2 \, dx = 0 \]

\[ \sup_n \| T_h f_n - f_n \| \to 0 \text{ as } \|h\| \to 0 \]

\[ \Rightarrow (f_n) \text{ precompact} \]
Overview of the proof

Step 1: \((f_{\beta})_{\beta \in (0,1]}\) is bounded

Step 2: \((f_{\beta})_{\beta \in (0,1]}\) converges weakly to \(T_{W}^{+}g\)

\[\beta_{n} \downarrow 0, \quad f_{n} \equiv f_{\beta_{n}}\]

\[\exists (f_{n_{k}}) \rightarrow T_{W}^{+}g\]

Step 3: The convergence is in fact strong

\[(f_{n})\) is bounded (Step 1)\]
Overview of the proof

Step 1: \((f_{\beta})_{\beta \in (0,1]}\) is bounded

Step 2: \((f_{\beta})_{\beta \in (0,1]}\) converges weakly to \(T_{W}^{+} g\)

\[ \beta_n \downarrow 0, \ f_n := f_{\beta_n} \]

\[ \exists (f_{n_k}) \rightharpoonup T_{W}^{+} g \]

Step 3: The convergence is in fact strong

\[ (f_n) \text{ is bounded (Step 1)} \]

\[ V \text{ bounded} \iff \lim_{R \to \infty} \sup_{n} \int_{\|x\| > R} |f_n(x)|^2 \, dx = 0 \]
Overview of the proof

Step 1: \((f_\beta)_{\beta \in (0,1]}\) is bounded

Step 2: \((f_\beta)_{\beta \in (0,1]}\) converges weakly to \(T^+_W g\)

\[\beta_n \downarrow 0, \ f_n := f_{\beta_n}\]

\[\exists (f_{n_k}) \rightarrow T^+_W g\]

Step 3: The convergence is in fact strong

\( (f_n) \) is bounded (Step 1)

\[V \text{ bounded} \quad \lim_{R \to \infty} \sup_n \int_{\|x\| > R} |f_n(x)|^2 \, dx = 0\]

\[\sup_n \|T_h f_n - f_n\| \to 0 \text{ as } \|h\| \to 0\]
Examples: Lévy kernels
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\[ |1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} \| \xi \|^s \]
Examples: Lévy kernels

\[ |1 - \hat{\phi}(\xi)| \sim_{\xi \to 0} |\xi|^s \]

\[ \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \hat{\phi}(\xi) \neq 1 \]
Examples: Lévy kernels

\[ \left| 1 - \hat{\phi}(\xi) \right| \sim_{\xi \to 0} \| \xi \|^s \]

\[ \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \hat{\phi}(\xi) \neq 1 \]

\[ \hat{\phi}: \xi \mapsto \exp(-\|\xi\|^s), \quad s \in (0, 2] \]
Examples: Lévy kernels

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\[ \phi: x \mapsto U^{-1} \exp(-\| \cdot \|^s)(x) \]
Examples: Lévy kernels

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\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \hat{\phi}(\xi) \neq 1

\hat{\phi}: \xi \mapsto \exp(-\|\xi\|^s), \quad s \in (0, 2]

\phi: x \mapsto U^{-1} \exp(-\|\cdot\|^s)(x)

\phi \text{ is positive, isotropic, radially decreasing, } C^\infty
Examples: Lévy kernels

Point spread functions

Filters

\[ \phi(x) \]

\[ U_{\phi}(\xi) \]
Examples: Lévy kernels

- Cauchy filter (s=1)
- Filter for s=0.6
- Cauchy kernel (s=1)
- Kernel for s=0.6
First extensions

\[ RC_\beta = \Phi_\beta R \quad \text{with} \quad C_\beta := U^{-1}\hat{\phi}_\beta U \]
First extensions

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\[ R = U^{-1} \hat{k} U, \quad \text{convolution by} \quad k \]

\[ \iff \quad RC_\beta = C_\beta R \]

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\( R \) Radon operator

\[
(Rf)(\theta, s) = \int f(x) \delta(s - \langle \theta, x \rangle) \, dx
\]

\[ R(f_1 \ast f_2) = Rf_1 \otimes Rf_2 \]

\( \otimes \) convolution selon \( s \)

\[ \leftarrow RC_\beta f = R(\phi_\beta \ast f) = R\phi_\beta \otimes Tf \]

\[ \Phi_\beta = (g \mapsto R\phi_\beta \otimes g) \]
Further extension

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Minimize \( \frac{1}{2} \| RC_\beta - XR \|_2^2 \)

s.t. \( X \in L(G), \ X = 0 \text{ on } (\text{ran } R)^\perp \)
Further extension

\[ R: L^2(V) \rightarrow G, \quad G \text{ Hilbert space} \]
\[ C_\beta \text{ convolution by } \phi_\beta \]
Assume \( R \) is still defined on \( \text{ran } C_\beta \)
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\[ (P_\beta) \quad \text{Minimize} \quad \frac{1}{2} \left\| \Phi_\beta g - Rf \right\|^2_G + \frac{\alpha}{2} \left\| (I - C_\beta)f \right\|^2_{L^2(\mathbb{R}^d)} \]
Further extension

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\[ (Q_\beta) \quad \text{Minimize} \quad X \mapsto \| RC_\beta - XR \| \\
\text{s.c.} \quad X \in L(G), \quad X = 0 \text{ on } (\text{ran } R)^\perp \]
**Proposition** If $RC_\beta R^+$ is bounded, then $RC_\beta R^+$ has a continuous extension on $G$ which is solution of $(Q_\beta)$. 
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**Remark** The operator $RCR^+$ is bounded if and only if there exists a positive constant $K$ telle que

$$\forall f \in (\ker R)^\perp, \quad \|RCf\|_F \leq K\|Rf\|_G.$$
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Theorem With the same assumptions on $\phi$ as before, assume that $g \in R(L^2(V) \cap H^s(\mathbb{R}^d))$. Then $f_{\beta} \to R^+ g$ strongly as $\beta \downarrow 0$. 
Outline

- Introduction
- Approximate inverses
- Fourier synthesis
- Variational theorems
- Pseudo-commutants
Matrix formulation
Matrix formulation

**Definition** We call *pseudo-commutant* of a matrix $C \in \mathbb{R}^{n \times n}$ w.r.t. a matrix $R \in \mathbb{R}^{m \times n}$ the unique solution $\Phi \in \mathbb{R}^{m \times m}$ of

$$
(Q) \quad \begin{aligned}
\text{Minimize} & \quad \| X R - R C \|_F \\
\text{s.t.} & \quad X \in \mathbb{R}^{m \times m}, \quad X (\ker R^\top) = \{0\}.
\end{aligned}
$$
Matrix formulation

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\[
\begin{align*}
\text{(Q)} & \quad \text{Minimize } \|XR - RC\|_F \\
& \quad \text{s.t. } X \in \mathbb{R}^{m \times m}, \ X(\ker R^\top) = \{0\}.
\end{align*}
\]

**Proposition** The matrix $\Phi = RCR^+$ is the unique solution to (Q), and in the case where $R$ is injective, then $R^+R$ is the identity, so that $\Phi$ actually satisfies $\Phi R = RC$. 
Remark The Frobenius norm satisfies

\[ \| M \|_F^2 = \text{tr}(M^\top M) = \sum_{j=1}^{m} \sigma_j^2(M), \]

where \( \sigma_1(M) \geq \cdots \geq \sigma_m(M) \) are the singular values of \( M \).
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We shall see that the solution \( RCR^+ \) remains unchanged if we replace \( \| \cdot \|_F \) in Problem (Q) by any convex spectral function.
Group invariance
**Definition** A function $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is said to be orthogonally invariant if $F(UMV) = F(M)$ for all $M \in \mathbb{R}^{m \times n}$ and all $(U, V) \in O(m) \times O(n)$. 
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Here, $\Pi(m)$ is the group of signed permutation matrices of size $m \times m$. 
Back to spectral functions
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\[ \sigma : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^m \]

\[ M \mapsto \sigma(M) := (\sigma_1(M), \ldots, \sigma_m(M)) \]
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\[ M \mapsto \sigma(M) := (\sigma_1(M), \ldots, \sigma_m(M)) \]

**Proposition** A function \( F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \) is orthogonally invariant if and only if it satisfies

\[ F = F \circ \text{diag}_{m \times n} \circ \sigma. \]

In such a case, \( f := F \circ \text{diag}_{m \times n} \) is the unique absolutely symmetric function such that \( F = f \circ \sigma \).
Remarkable facts
Theorem Let $F$ be orthogonally invariant, and let $f := F \circ \text{diag}_{m \times n}$. Then $F$ is convex if and only if $f$ is convex.
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Theorem [A. S. Lewis] Let $F$ be orthogonally invariant, and let $f := F \circ \text{diag}_{m \times n}$. Then, for all $M \in \mathbb{R}^{m \times n}$, the subdifferential of $F$ at $M$ is given by

$$\{ U \text{diag}_{m \times n}(\xi)V | \xi \in \partial f(\sigma(M)), U \in O(m), V \in O(n) \}.$$
Theorem [X Bonnefond and PM]
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Let $R \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{n \times n}$. 
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Let $R \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{n \times n}$.

Let $\mathcal{F}(X) = F(XR - RC)$, in which $F$ is any convex orthogonally invariant function.

Then $RCR^+$ is a solution to

$$
\begin{array}{c}
\text{(Q)} \\
\text{Minimize } \mathcal{F}(X) \\
\text{s.t. } X(\ker R^\top) = \{0\}.
\end{array}
$$
Thank you for your attention!