# Sign Patterns of G-Matrices

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# Outline

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M. Fiedler/F. Hall, G-Matrices, LAA, 2012

All matrices in the paper are real.

We call a matrix $A$ a G-matrix if $A$ is nonsingular and there exist nonsingular diagonal matrices $D_1$ and $D_2$ such that

$$A^{-T} = D_1 A D_2,$$

where $A^{-T}$ denotes the transpose of the inverse of $A$.

The matrices $D_1$ and $D_2$ are not uniquely determined - we can multiply one by a nonzero number and divide the other by the same number.

## A rich class of matrices!

### Easy to see:

All orthogonal matrices are G-matrices.

All nonsingular diagonal matrices are G-matrices.

If $A$ is a G-matrix, then both $A^T$ and $A^{-1}$ are G-matrices.

If $A$ is an $n \times n$ G-matrix and $D$ is an $n \times n$ nonsingular diagonal matrix, then both $AD$ and $DA$ are G-matrices.

If $A$ is a G-matrix, then both matrices $A$ and $A^{-T}$ have the same zero, non-zero structure. Thus the zero, non-zero structures of $A$ and $A^{-1}$ are symmetric to each other.

The direct sum of G-matrices is again a G-matrix.

The product form of the definition leads to the following two facts:

Compound matrices of a G-matrix are also G-matrices.

The Kronecker product of G-matrices is a G-matrix.

Also:

A $2 \times 2$ matrix is G-matrix if and only if it is nonsingular and has four or two nonzero entries.

Cauchy matrices have form $C = [c_{ij}]$, where $c_{ij} = \frac{1}{x_i + y_j}$ for some numbers $x_i$ and $y_j$.

We shall restrict to square, say $n \times n$, Cauchy matrices - such matrices are defined only if $x_i + y_j \neq 0$ for all pairs of indices $i, j$, and it is well known that $C$ is nonsingular if and only if all the numbers $x_i$ are mutually distinct and all the numbers $y_j$ are mutually distinct.

From M. Fiedler, Notes on Hilbert and Cauchy matrices, LAA, 2010:

Every nonsingular Cauchy matrix is a G-matrix.

For generalized Cauchy matrices of order $n$, additional parameters $u_1, \ldots, u_n, v_1, \ldots, v_n$ are considered:

$$\hat{C} = \left[ \frac{u_i v_j}{x_i + y_j} \right].$$

Note that

$$\hat{C} = D_1 C D_2$$

where

$$D_1 = \mathrm{diag}(u_i),\ D_2 = \mathrm{diag}(v_j)$$

so that $\hat{C}$ is a G-matrix.

Just a couple of results re the diagonal matrices:

Theorem. Suppose $A$ is a G-matrix and $A^{-T} = D_1 A D_2$, where $D_1$ and $D_2$ are nonsingular diagonal matrices. Then the inertia of $D_1$ is equal to the inertia of $D_2$.

Proof. We have $A^T D_1 A D_2 = I$ and so $A^T D_1 A = D_2^{-1}$. Since $A$ is nonsingular, the result follows from Sylvester's Law of Inertia.

Remark. As the example of a diagonal matrix shows, the inertia of both matrices $D_1$ and $D_2$ need not be uniquely determined by the given G-matrix.

Theorem. Let $A$ be a symmetric irreducible G-matrix. Then the diagonal matrices $D_1$, $D_2$ in $A^{-T} = D_1 A D_2$ can be chosen to be either the same or the negative of each other.

If $A$ is positive definite then $D_1$ and $D_2$ can be chosen to be the same. If we denote the common matrix as $D$, then the matrix $AD$ is involutory. Denote the nonnegative square root of the modulus $|D|$ of $D$ as $D^{\frac{1}{2}}$. Then the matrix $B = D^{\frac{1}{2}} A D^{\frac{1}{2}}$ has the property that all its integral powers are G-matrices.

Finally, if $S = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is the matrix of the signs of $D$, ie. $S = D|D|^{-1}$, then the diagonal entries $a_{ii}$ of $A$ and the diagonal entries $\alpha_{ii}$ of $A^{-1}$ satisfy

$$\sum_i \varepsilon_i(\sqrt{a_{ii}\alpha_{ii}} - 1) = 0.$$

Two papers on Generalized G-Matrices

M. Matsuura, A note on generalized G-matrices, LAA, 2012

M. Fiedler/T. Markham, More on G-matrices, LAA, 2013

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$\mathcal{G}_n$ denotes the class of all $n \times n$ sign pattern matrices $A$ that allow a G-matrix, ie, there exists a nonsingular matrix $B \in Q(A)$ such that $B^{-T} = D_1 B D_2$ for some nonsingular diagonal matrices $D_1$ and $D_2$.

Theorem The class $\mathcal{G}_n$ is closed under
(i) multiplication (on either side) by a permutation pattern, and
(ii) multiplication (on either side) by a diagonal signature pattern.

The use of these operations in $\mathcal{G}_n$ then produces "equivalent" sign patterns.

Let $\mathcal{C}_n$ ($\mathcal{GC}_n$) be the class of all sign patterns of the $n \times n$ nonsingular Cauchy (generalized Cauchy) matrices. It should be clear that $\mathcal{C}_n$ ($\mathcal{GC}_n$) is closed under operation (i) (operations (i) and (ii)) above. The classes $\mathcal{C}_n$ and $\mathcal{GC}_n$ are two particular sub-classes of $\mathcal{G}_n$.

Since $x + y = (x + c) + (y - c)$ and when we permute the rows/columns, we still have a Cauchy matrix $C$, for a given such nonsingular matrix, we can assume that

$$x_1 > x_2 > \cdots > x_n > 0,$$

and

$$y_1 > y_2 > \cdots > y_n.$$

Further, the sign pattern of $C$ is the same as the sign pattern of the matrix $[x_i + y_j]$. So, the following result is clear.

**Theorem** The class $\mathcal{C}_n$ is the same as the class of $n \times n$ sign patterns permutation equivalent to a staircase sign pattern, where the part above (below) the staircase is all $+$ $(-)$.

In this form, whenever there is a minus, then to the right and below there are also minuses.

Note that this form includes the all $+$ and all $-$ patterns. However, for example, $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ is excluded as a pattern in $\mathcal{C}_2$. In fact:

Theorem An $n \times n$ sign pattern matrix $A$ belongs to $\mathcal{C}_n$ if and only if it does not contain either one of the patterns

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix} \text{ or } \begin{bmatrix} - & + \\ + & - \end{bmatrix}$$

as submatrices.

Recall the form of generalized Cauchy matrix: $\hat{C} = D_1 C D_2$.

Theorem The class $\mathcal{GC}_n$ is the same as the class of $n \times n$ sign patterns that are signature equivalent to a sign pattern in $\mathcal{C}_n$. Equivalently, $A \in \mathcal{GC}_n$ if and only if there exists a minimum rank one sign pattern matrix $M$ such that $A \circ M \in \mathcal{C}_n$, where $\circ$ denotes the Hadamard product.

Every $2 \times 2$ $(+, -)$ sign pattern is a matrix in $\mathcal{GC}_2$.
Every $3 \times 3$ $(+, \ -)$ sign pattern is a matrix in $\mathcal{GC}_3$.

But, in general $\mathcal{GC}_n$ does not contain all the $(+, -)$ $n \times n$ sign patterns, that is, there are forbidden sign patterns.

No GC-matrix can have the following pattern (or its equivalent or transpose) as submatrix:

$$S_3 = \begin{bmatrix} + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}.$$

By the way,

$$S_3 = \begin{bmatrix} + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}$$

is an L-matrix. Every $3 \times 4$ real matrix with the same sign pattern as $S_3$ has linearly independent rows.

R.A. Brualdi, K.L. Chavey, B.L. Shader, Rectangular L-matrices, LAA, 1994.

$\mathcal{PO}_n$ is the class of $n \times n$ sign patterns which allow an orthogonal matrix. The class $\mathcal{PO}_n$ is then a subclass of the class $\mathcal{G}_n$. In EHHL, CMJ, 1999, a number of nontrivial sign patterns in $\mathcal{PO}_n$ (and hence in $\mathcal{G}_n$) are constructed.

EG The patterns

$$\begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}, \quad \begin{bmatrix} - & - & - \\ - & + & - \\ - & - & + \end{bmatrix}$$

arising from Householder transformations, are in $\mathcal{PO}_3$. These two patterns are then in the intersection of $\mathcal{PO}_3$ and $\mathcal{GC}_3$.

The sign patterns

$$\begin{bmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{bmatrix}, \quad \begin{bmatrix} - & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{bmatrix}$$

are examples of $\mathcal{PO}_4$ patterns which are not in $\mathcal{GC}_4$, by a forbidden submatrix. The same holds for the corresponding Householder type patterns in $\mathcal{PO}_n, \quad n \geq 4$.

Remark Not every sign pattern in $\mathcal{G}_n$ is in $\mathcal{PO}_n$ or in $\mathcal{GC}_n$. For example, the Kronecker product

$$A = \begin{bmatrix} + & + & - \\ - & + & - \\ + & + & - \end{bmatrix} \otimes \begin{bmatrix} - & + & - \\ - & - & + \\ - & - & - \end{bmatrix}$$

is in $\mathcal{G}_9$ since the Kronecker product of G-matrices is a G-matrix.

Now,

$$A = \begin{bmatrix} - & + & - & - & + & - & + & - & + \\ - & - & + & - & - & + & + & + & - \\ - & - & - & - & - & - & + & + & + \\ + & - & + & - & + & - & + & - & + \\ + & + & - & - & - & + & + & + & - \\ + & + & + & - & - & - & + & + & + \\ - & + & - & - & + & - & + & - & + \\ - & - & + & - & - & + & + & + & - \\ - & - & - & - & - & - & + & + & + \end{bmatrix}.$$

By observing columns 4 and 7 of $A$, it is clear that $A \notin \mathcal{PO}_9$. Also, by considering the submatrix at the intersection of rows 1, 3, 5 and columns 2,4,5,6, we know that $A$ is equivalent to a pattern with a forbidden submatrix. So, $A \notin \mathcal{GC}_9$.

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Back to $\mathcal{C}_n$, the class of all sign patterns of the $n \times n$ nonsingular Cauchy matrices.

Recall: The class $\mathcal{C}_n$ is the same as the class of $n \times n$ sign patterns permutation equivalent to a staircase sign pattern, where the part above (below) the staircase is all $+$ $(-)$.

Let $C$ be an $n \times n$ nonsingular Cauchy matrix. Now, the sign pattern of $C$ is the same as the sign pattern of the reciprocal matrix $[x_i + y_j]$. The reciprocal matrix has rank at most 2, as any 3rd order subdet is 0. But, clearly, excluding the all $+$ and all $-$ patterns, any real matrix in the sign pattern class has rank $\geq 2$. Thus, excluding equivalences of the all $+$ and all $-$ patterns, if $A \in \mathcal{C}_n$, then the minimum rank of $A$ is 2.

Recall: The class $\mathcal{GC}_n$ is the same as the class of $n \times n$ sign patterns that are signature equivalent to a sign pattern in $\mathcal{C}_n$.

Thus, excluding equivalences of the all $+$ and all $-$ patterns, if $A \in \mathcal{GC}_n$, then the minimum rank of $A$ is 2.

Also, if $A \in \mathcal{GC}_n$, then there are signature sign patterns $D_1, D_2$ and permutation sign patterns $P_1, P_2$ such that $P_1D_1AD_2P_2$ is a staircase sign pattern, where the part above (below) the staircase is all $+$ $(-)$, that is, each row and each column of $P_1D_1AD_2P_2$ is nonincreasing. This then also holds for the condensed sign pattern $A_c$, where in general condensed means no zero line, no identical rows (or columns), and no two rows (or columns) which are negatives of each other.

This latter result is a special case of the following, from Z. Li et al, Sign patterns with minimum rank 2 and upper bounds on minimum ranks, LAMA, 2013:

Theorem An $m \times n$ sign pattern $A$ has minimum rank 2 iff its condensed sign pattern $A_c$ satisfies the conditions
(i) $A_c$ has at least two rows and two columns,
(ii) each row and each column of $A_c$ has at most one zero entry, and
(iii) there are signature sign patterns $D_1, D_2$ and permutation sign patterns $P_1, P_2$ such that each row and each column of $P_1 D_1 A_c D_2 P_2$ is nonincreasing.

Note that the conditions in the above theorem imply that $A_c$ is equivalent to a certain staircase pattern.

Next, suppose $A$ is an $n \times n$ $(+, -)$ sign pattern with minimum rank 2. From condition (iii) of the above theorem, there are signature sign patterns $D_1, D_2$ and permutation sign patterns $P_1, P_2$ such that each row and each column of $P_1 D_1 A_c D_2 P_2$ is nonincreasing. This then also holds for the sign pattern matrix $A$, so that $A \in \mathcal{GC}_n$.

We thus see that, excluding equivalences of the all $+$ and all $-$ patterns, $\mathcal{GC}_n$, the class of all sign patterns of the $n \times n$ nonsingular generalized Cauchy matrices, is the same as the class of all $n \times n$ $(+, -)$ sign patterns with minimum rank 2!

Recall: The patterns

$$\begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}, \quad \begin{bmatrix} - & - & - \\ - & + & - \\ - & - & + \end{bmatrix}$$

are in the intersection of $\mathcal{PO}_3$ and $\mathcal{GC}_3$. So, the minimum rank of each pattern is 2 (it's also easy to see that each pattern is not SNS).

Recall: The sign pattern

$$A = \begin{bmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{bmatrix}$$

is an example of a pattern which is not in $\mathcal{GC}_4$, by a forbidden submatrix.

From N. Alon, J.H. Spencer, The Probabilistic Method, 2000: If each row of a $(+,-)$ sign pattern $M$ has at most $k$ sign changes, then $\text{mr}(M) \leq k+1$.

So, $\text{mr}(A) \leq 3$. But, since $A \notin \mathcal{GC}_4$, $\text{mr}(A) \neq 2$, and clearly, $\text{mr}(A) \neq 1$. Thus, $\text{mr}(A) = 3$,

The same holds for the corresponding Householder type pattern in $\mathcal{PO}_n$, $\quad n \geq 4$.

The same can be said for the Householder pattern

$$\begin{bmatrix} - & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{bmatrix}.$$

Next, what about the 3rd type, up to signature and permutation similarity, irreducible Householder pattern (of course in $\mathcal{PO}_3$):

$$\begin{bmatrix} 0 & - & - \\ - & + & - \\ - & - & + \end{bmatrix}.$$

It's easy to see that this pattern is not SNS and its minimum rank is 2.

What about the corresponding Householder type pattern $A \in \mathcal{PO}_n, \quad n \geq 4$??

From the LAMA paper:
The number of polynomial sign changes of a sign pattern $M$, denoted $\mathrm{psc}(M)$, is the largest number of polynomial sign changes of the rows of $M$ (for a sign pattern with zero entries, this entails a more general definition).

For every sign pattern $M$, $\mathrm{mr}(M) \leq \mathrm{psc}(M_c) + 1$.

Now, for our pattern $A$, $\mathrm{psc}(A_c) = 2$, so that $\mathrm{mr}(A) \leq 3$.

Observe that the 2nd, 3rd, and 4th rows form an L-matrix.
So, $\mathrm{mr}(A) \geq 3$.

Thus, $\mathrm{mr}(A) = 3$.

A minimum rank 3 reference:
Z. Li et al, Sign patterns with minimum rank 3 and point-line configurations

What is the minimum rank of the $9 \times 9$ pattern $A$ on pages 20, 21??

$$A = \begin{bmatrix} - & + & - & - & + & - & + & - & + \\ - & - & + & - & - & + & + & + & - \\ - & - & - & - & - & - & + & + & + \\ + & - & + & - & + & - & + & - & + \\ + & + & - & - & - & + & + & + & - \\ + & + & + & - & - & - & + & + & + \\ - & + & - & - & + & - & + & - & + \\ - & - & + & - & - & + & + & + & - \\ - & - & - & - & - & - & + & + & + \end{bmatrix}.$$

$A = A_1 \otimes A_2$ where $\mathrm{mr}(A_1) = \mathrm{mr}(A_2) = 2$.

For real $B_1, B_2$, $\mathrm{rank}(B_1 \otimes B_2) = \mathrm{rank}(B_1)\mathrm{rank}(B_2)$.

So, $\mathrm{mr}(A) \leq \mathrm{mr}(A_1)\mathrm{mr}(A_2) = 4$.

Rows 3,4,5 of $A$ form an L-matrix. So, $\mathrm{mr}(A) \geq 3$.

Thus, $\mathrm{mr}(A) = 3$ or 4.

Note that $A$ does not contain an L-matrix with 4 rows.

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If $A \in \mathcal{G}_n$ then there exist a nonsingular matrix $B \in Q(A)$ and nonsingular diagonal matrices $D_1$ and $D_2$ such that $B^{-T} = D_1 B D_2$. Hence, there are signature patterns $S_1$ and $S_2$ such that

$$B^{-1} \in Q(S_1 A^T S_2). \tag{1}$$

In particular, we may study the sign patterns $A \in \mathcal{G}_n$ for which there exist diagonal matrices $D_1$, $D_2$ with positive diagonal entries in the equation $B^{-T} = D_1 B D_2$. Then, (1) becomes

$$B^{-1} \in Q(A^T). \tag{2}$$

In fact, in EHHL, CMJ, 1999, the class $\mathcal{T}_n$ of all $n \times n$ sign patterns $A$ for which there exists a nonsingular matrix $B \in Q(A)$ where (2) holds was studied. There it was asked if the class $\mathcal{T}_n$ is the same as the subclass $\mathcal{PO}_n$. In the same spirit, we may pose the following questions:

(i) Suppose $A$ is an $n \times n$ sign pattern matrix and there exists $B \in Q(A)$ such that (2) holds. Do there then exist diagonal matrices $D_1$, $D_2$ with positive diagonal entries such that $B^{-T} = D_1 B D_2$?

(ii) More generally, suppose that (1) holds for some $A$ and $B$. Is $B$ then necessarily a G-matrix, or, does there exist some $\hat{B} \in Q(A)$ that is a G-matrix?

Let $A$ be an $n \times n$ sign pattern. Suppose there exist a nonsingular matrix $B \in Q(A)$ and nonsingular diagonal matrices $D_1, D_2$ with $+$ diagonal entries such that $B^{-T} = D_1 B D_2$. Let $F = \sqrt{D_1}, G = \sqrt{D_2}$. Then

$$(FBG)^{-1} = (FBG)^T.$$

So, $FBG$ is an orthogonal matrix in $Q(A)$. Conversely, if $C$ is an orthogonal matrix in $Q(A)$, then $C^{-T} = C = I_n C I_n$.

Thus, $A$ allows a G-matrix with associated diagonal matrices with $+$ diagonal entries iff $A \in \mathcal{PO}_n$. For such a sign pattern $A$, we have that $A \in \mathcal{T}_n$. Does the reverse hold???

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Fiedler/Hall/Stroev, Discriminant matrices, in preparation.
Fiedler/Hall, More on generalized complementary basic matrices, in preparation.
Maybe hamburger matrices .....