The quest for the basic fuzzy logic*

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The quest for basic fuzzy logic was initiated by Petr Hájek when he proposed his basic fuzzy logic BL, complete with respect to the semantics given by all continuous t-norms. Later weaker systems, such as MTL, UL or psMTL^r, complete with respect to broader (but still meaningful for fuzzy logics) semantics, have been introduced and disputed the throne of the basic fuzzy logic. We contribute to the quest with our own proposal of a basic fuzzy logic. Indeed, we put forth a very weak logic called SL$^\ell$, introduced and studied in [4, 3], and propose it as a base of a new framework which allows to work in a uniform way with both propositional and first-order fuzzy logics.

1 T-norm based fuzzy logics and core fuzzy logics

Mathematical Fuzzy Logic (MFL) started as the study of logics based on particular continuous t-norms, most prominently Łukasiewicz logic Ł, Gödel–Dummett logic G and Product logic Π. These logics are rendered in a language with the truth-constant $\overline{0}$ (falsum) and binary connectives $\to$ (implication) and & (fusion, residuated/strong conjunction). They are complete with respect to the *standard semantics*, which has the real-unit interval $[0,1]$ as the set of truth degrees and interprets falsum $\perp$ by 0, fusion & by the corresponding t-norm, and the implication $\to$ by its residuum, which always exists for continuous t-norms. On the other hand, these systems are also complete with respect to an algebraic semantics and with respect to the subclass of their linearly ordered members. In this context, Petr Hájek introduced a natural question: is it possible to see Ł, G and Π (and, in general, any fuzzy logic with a continuous t-norm-based semantics) as extensions of the same fuzzy logic? In other words: is there *a basic fuzzy logic* underlying all (by then) known fuzzy logic systems? As an answer to this question, he introduced in his monograph [8] a system, weaker than Ł, G and Π, which he named BL (for *basic logic*). Nowadays the logic is called in his honor Hájek Logic HL.

Hájek proved completeness of HL with respect to the corresponding class of the so-called HL-algebras and even HL-chains and conjectured that it should be also complete with respect to the standard HL-algebras (i.e., algebras whose lattice reduct in the real unit interval with the usual order); the conjecture was later proved true in [2]. Therefore, HL could really be seen, at that time, as a *basic fuzzy logic*. Indeed, it was a genuine *fuzzy* logic because it retained what was then seen as the defining property of fuzzy logics: completeness with respect to a semantics based on continuous t-norms. And it was also *basic* in the following two senses:

1. *it could not be made weaker without losing essential properties* and
2. *it provided a base for the study of all fuzzy logics.*

The first item followed from the completeness of HL w.r.t. the semantics given by *all* continuous t-norms; thus, in a context of continuous t-norm based logics one could not possibly take a weaker system.

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The second meaning relied on the fact that the three main fuzzy logics (Ł, G, and Π) are all axiomatic extensions of HL and, in fact, the methods used by Hájek to introduce, algebraize, and study HL could be utilized for any other logic based on continuous t-norms. Actually, already in [8], Hájek developed a uniform mathematical theory for MFL. He considered all axiomatic extensions of HL (not just the three prominent ones) as fuzzy logics (he called them *schematic extensions*) and systematically studied their first-order extensions, extensions with modalities, complexity issues, etc.

However, the later development of MFL has shown that HL was actually *not basic enough*. That is, HL was indeed a good basic logic for the initial framework in which it was formulated, but the active research area that Hájek helped creating soon expanded its horizons to broader frameworks. Therefore, Hájek had not settled but only initiated the quest for the basic fuzzy logic. The first step towards a broader point of view was taken by Esteva and Godo, who noticed that the necessary and sufficient condition for a t-norm to have a residuum is not continuity, but left-continuity. Inspired by this fact they introduced in [5] the logic MTL. Similarly to the previous cases, Esteva and Godo proved that MTL is complete both w.r.t. the semantics given by all MTL-algebras and w.r.t. MTL-chains. Moreover, in [13], Jenei and Montagna proved MTL to be complete with respect to the *standard* semantics given by all left-continuous t-norms. Thus it was a better candidate than HL for a basic fuzzy logic: again it was a genuine fuzzy logic enjoying a standard completeness theorem, it could not be made weaker without losing this property, and all known fuzzy logics could be obtained as extensions of MTL, thus providing a good base for a new systematical study of MFL.

In fact, Petr Hájek saluted MTL as the new basic fuzzy logic and defined (in a joint work with Petr Cintula [11]) a precise general framework taking MTL as the basic system and not restricting to its axiomatic extensions (i.e. logics in the same language as MTL) but rather to its axiomatic expansions (by allowing new additional connectives). In particular they introduced two classes of logics large enough to cover the most of then studied fuzzy logics. The rough idea was to capture, by simple syntactic means, a class of logics which share many desirable properties with MTL.

Definition 1. *A logic* L *in a language* $\mathcal{L}$ *is a* core fuzzy logic *if it an axiomatic expansion of* MTL *such that for all* $\mathcal{L}$*-formulae* φ, ψ, χ *the following holds:*

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \chi', \tag{Cong}$$

where χ' *is a formula resulting from* χ *by replacing some occurrences of its subformula* φ *by the formula* ψ*.*

An important logic, which does not fall under the scope of the previous definition, is the logic $\mathrm{MTL}_{\triangle}$: the expansion of MTL with the Monteiro–Baaz projection connective $\triangle$. Taking $\mathrm{MTL}_{\triangle}$ as an alternative basic logic, Hájek and Cintula defined another class of fuzzy logics, the $\triangle$*-core fuzzy logics*.

Core and $\triangle$-core fuzzy logics are all finitary and well-behaved from several points of view. In particular, for every such logic L one can define in a natural way a corresponding class of algebraic structures, L-algebras, which provide a complete semantics as in the case of MTL or the previously mentioned logics and, more importantly, the completeness theorem is preserved if we restrict ourselves to *linearly ordered* L-algebras. However, there are logics expanding MTL studied in the literature which are not core or $\triangle$-core because they need some additional deduction rules, the prominent examples being the logic PL' (the extension of Łukasiewicz logic with an additional product-like conjunction which has no zero-divisors [12]) or logics with truth-hedges [6].

2 The quest goes on . . .

The quest for the basic fuzzy logic did not end with MTL (or $\mathrm{MTL}_{\triangle}$). Indeed, MTL has been further weakened in two different directions beyond the framework of core fuzzy logics:

(a) by dropping commutativity of conjunction Petr Hájek obtained a system, psMTL^r [9], which Jenei and Montagna proved to be complete with respect to the semantics on non-commutative residuated t-norms [14],

(b) by removing integrality (i.e. not requiring the neutral element of conjunction to be maximum of the order) Metcalfe and Montagna proposed the logic UL which is, in turn, complete with respect to left-continuous uninorms [15].

Petr Hájek liked to describe this process of successive weakening of fuzzy logics by telling the joke of the crazy scientist that studied fleas by removing their legs one by one and checking whether they could still jump [10].[1] Namely, if HL was the original flea, it lost the 'right-continuity leg' when it was substituted by MTL, and then psMTL^r and UL respectively lost the 'commutativity and the integrality leg' while retaining the ability to 'jump' (i.e., the completeness w.r.t. intended semantics based on reals).

These weaker fuzzy logics can be fruitfully studied in the context of substructural logics. Recall the bounded full Lambek logic FL, a basic substructural logic which does not satisfy any of the usual three structural rules: exchange, weakening, and contraction. This logic can be given an algebraic semantics of the variety of bounded pointed lattice-ordered residuated monoids (usually referred to as bounded pointed *residuated lattices* or FL-*algebras*). Intuitionistic logic together with logics FL_e, FL_w, and FL_{ew} are among the most prominent extensions of FL. Actually, many fuzzy logics have been shown to be axiomatic extensions of some of these prominent substructural logics by adding some axioms that enforce completeness with respect to some class of linearly ordered residuated lattices (or *chains*). For instance, Gödel–Dummett logic is the logic of linearly ordered Heyting algebras (FL_{ewc}-chains), MTL is the logic of FL_{ew}-chains, UL is the logic FL_e^ℓ of FL_e-chains, etc.

This common feature, completeness with respect to their corresponding linearly ordered algebraic structures, has motivated the methodological paper [1] where the authors postulate that *fuzzy logics are the logics of chains*, in the sense that they are logics complete with respect to a semantics of chains. However, all the fuzzy logics mentioned so far do enjoy a stronger property: the *standard completeness theorem*, i.e. completeness with respect to a semantics of algebras defined on the real unit interval $[0, 1]$ which Petr Hájek and many others have considered to be the intended semantics for fuzzy logics. Following Hájek's flea joke, we could say that those fleas are fuzzy logics that *jump well* provided that they satisfy standard completeness. Actually, many authors regard standard completeness as an essential requirement for fuzzy logics. It is, thus, reasonable to expect any candidate for the basic fuzzy logic to satisfy this stronger requirement. But, although they fulfill that, neither UL nor psMTL^r can be taken as basic because they are not comparable and hence do not satisfy our second meaning of *basic*. A reasonable candidate could be the logic of FL-chains. But, interestingly enough, this logic does not enjoy the standard completeness (as proved in [16]). Moreover, one can also argue that it is still *not basic enough* (in the first meaning) because it satisfies a remaining structural rule: associativity. Hence, in the context of substructural logics, it could still be made weaker by removing associativity.

The logic SL, a non-associative version of the bounded Full Lambek calculus, was introduced by Galatos and Ono in [7] with its algebraic semantics being the variety of bounded lattice-ordered residuated unital groupoids. The logic SL^ℓ of bounded linearly ordered residuated unital groupoids, was axiomatized in [3], where we have also show that it enjoys the standard completeness.

3 SL^ℓ and core semilinear logics

The main goal of our talk is to propose SL^ℓ as a new basic fuzzy logic and a framework analogous to (and encompassing) that of ($\triangle$-)core fuzzy logics.

Definition 2. *A logic* L *is a* core semilinear logic *if if satisfies the condition* Cong *and expands* SL^ℓ *by some sets of axioms Ax and rules R such that for each* $\langle \Gamma, \varphi \rangle \in R$ *and every formula* ψ *we have:*

$$\Gamma \vee \psi \vdash_{\mathrm{L}} \varphi \vee \psi,$$

where by $\Gamma \vee \psi$ *we denote the set* $\{\chi \vee \psi \mid \chi \in \Gamma\}$.

The class of core semilinear logics is a natural extension of ($\triangle$-)core fuzzy logics, and we show that it shares many of its nice properties, mainly that its logics are complete w.r.t. linearly ordered algebras and it provides a suitable base to study not only propositional but also first-order logics. Thus one could argue

[1] *A prominent biologist conducted a very important experiment. He trained a flea to jump upon giving her a verbal command ("Jump!"). In a first stage of the experiment he removed a flea's leg, told her to jump, and the flea jumped. So he wrote in his scientific notebook: "Upon removing one leg all flea organs function properly." So, he removed the second leg, asked the flea to jump, she obeyed, so he wrote again: "Upon removing the second leg all flea organs function properly." Thereafter he removed the last leg. Told flea to jump, and nothing happened. So he wrote the conclusion: "Upon removing the last leg the flea loses sense of hearing."*

that SL^ℓ is a good basic fuzzy logic in the second sense mentioned above. As regards the first sense, we have already seen that SL^ℓ enjoys the standard completeness theorem; therefore our flea still jumps (and jumps very well, even in the first-order case!). Moreover SL^ℓ is the weakest possible logic one could take in the context of substructural logics in a language with lattice connectives, a conjunction which is not required to satisfy any property corresponding to the usual structural rules and its left and right residua.

Thus we can arguably say that the quest for the basic fuzzy logic initiated by Petr Hájek so far seems to culminate with SL^ℓ. We do not know whether Mathematical Fuzzy Logic will require an even weaker system to serve as the basic fuzzy logic in the future. Only time will tell. What we can say is that, at the moment, we do not see any remaining legs to be pulled.

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