Algorithmic-algebraic canonicity for mu-calculi

WILLEM CONRADIE AND ANDREW CRAIG

Department of Mathematics
University of Johannesburg
South Africa

wconradie@uj.ac.za acraig@uj.ac.za

The modal mu-calculus was defined by Kozen [6] and is obtained by adding to the basic modal logic the least and greatest fixed point operators. (For an overview of mu-calculus see the chapter by Bradfield and Stirling [2].) The correspondence and completeness of logics with fixed point operators has been the subject of recent studies by Bezhanishvili and Hodkinson [1] and Conradie *et al.* [3]. Both of these works aim to develop a Sahlqvist-like theory for their respective fixed point settings.

Sahlqvist theory is one of the most important and powerful ideas in the study of modal and related logics. The theory consists of two parts: *canonicity* and *correspondence*. The Sahlqvist formulas are a recursively defined class of modal formulas with a particular syntactic shape. Any modal logic axiomatized by Sahlqvist formulas is strongly complete (via canonicity) with respect to its class of Kripke frames, and the latter is moreover guaranteed to be an elementary class. The work in [1] looks at both of these aspects, obtaining a syntactic class which allows for limited use of fixed point operators and for which a modified version of canonicity is proved. By contrast, [3] looks at only correspondence, and using an algorithmic approach obtains results for a much broader class of formulas. This algorithmic approach builds on work by Conradie and Palmigiano [4] on canonicity and correspondence for distributive modal logic.

In this work we prove that the members of a certain class of intuitionistic mu-formulas are *canonical*, in the sense of [1]; that is, they are preserved under certain modified canonical extensions. Our methods use a variation of the ALBA algorithm (Ackermann Lemma Based Algorithm) developed in [4]. We show that all mu-inequalities that can be successfully processed by our algorithm, μ^*-ALBA, are canonical. This is done via a "U-shaped argument" (see Figure 1), a generic version of which we now outline.

Let $\mathbf{A}$ be a bounded lattice with additional operations, and $\mathcal{L}$ a language interpretable in $\mathbf{A}$. We denote by $\mathbf{A}^\delta$ the canonical extension of $\mathbf{A}$; this is a dense and compact completion of $\mathbf{A}$ with the additional operations extended to the completion as defined by Gehrke and Harding [5]. Note that $\mathbf{A}^\delta$ is a perfect lattice, i.e., it is a complete lattice which is join-generated by its completely join-irreducible elements ($J^\infty(\mathbf{A}^\delta)$) and meet-generated by its completely meet-irreducible elements ($M^\infty(\mathbf{A}^\delta)$). In $\mathbf{A}^\delta$ we can interpret an extended language $\mathcal{L}^+$ which adds special variables called *nominals* and *co-nominals* which range over $J^\infty(\mathbf{A}^\delta)$ and $M^\infty(\mathbf{A}^\delta)$ respectively, and also possibly operations corresponding to the adjoints and residuals of those of $\mathcal{L}$. An $\mathcal{L}^+$ formula is *pure* if it contains no ordinary (propositional) variables but only, possibly, nominals and co-nominals.

An assignment on $\mathbf{A}^\delta$ sends propositional variables to elements of $\mathbf{A}^\delta$, nominals into $J^\infty(\mathbf{A}^\delta)$ and co-nominals into $M^\infty(\mathbf{A}^\delta)$. An *admissible assignment* on $\mathbf{A}^\delta$ is one which takes all propositional variables to elements of $\mathbf{A}$. An $\mathcal{L}^+$ inequality $\alpha \leq \beta$ is *admissibly valid* on $\mathbf{A}^\delta$, denoted $\mathbf{A}^\delta \models_{\mathbf{A}} \alpha \leq \beta$, if it holds under all admissible assignments.

The aim is to "purify" an inequality $\alpha \leq \beta$ by rewriting it as a (set of) pure (quasi-)inequalities, denoted $\mathtt{pure}(\alpha \leq \beta)$ in Figure 1. The fact that admissible and ordinary validity coincide for pure inequalities is the lynchpin for the transition from validity in $\mathbf{A}$ (simulated as admissible validity in $\mathbf{A}^\delta$) to validity in $\mathbf{A}^\delta$, i.e., canonicity.

1 Language and interpretation

A *bi-Heyting algebra* is an algebra $(A, \wedge, \vee, \to, -, \top, \bot)$ such that both $(A, \wedge, \vee, \to, \top, \bot)$ and $(A, \wedge, \vee, -, \top, \bot)^\partial$ are Heyting algebras. A *modal bi-Heyting algebra* is an algebra $(A, \wedge, \vee, \to, -, \top, \bot, \Box, \Diamond)$ such that $(A, \wedge, \vee, \to, -, \top, \bot)$ is bi-Heyting

$$
\begin{array}{ccc}
\mathbf{A} \models \alpha \leq \beta & & \mathbf{A}^\delta \models \alpha \leq \beta \\
\Updownarrow & & \\
\mathbf{A}^\delta \models_{\mathbf{A}} \alpha \leq \beta & & \Big\Updownarrow \\
\Updownarrow & & \\
\mathbf{A}^\delta \models_{\mathbf{A}} \mathtt{pure}(\alpha \leq \beta) & \Longleftrightarrow & \mathbf{A}^\delta \models \mathtt{pure}(\alpha \leq \beta)
\end{array}
$$

Figure 1: The U-shaped argument for canonicity of inequalities interpreted on a lattice-based algebra $\mathbf{A}$.

algebra and $\Box$ and $\Diamond$ preserve finite meets and joins, respectively. A *perfect bi-Heyting algebra* is a bi-Heyting algebra the lattice reduct of which is a perfect distributive lattice. A *perfect modal bi-Heyting algebra* is a modal bi-Heyting algebra the bi-Heyting reduct of which is a perfect bi-Heyting algebra, and moreover such that $\Box$ and $\Diamond$ preserve arbitrary meets and joins, respectively. The canonical extension $\mathbf{A}^\delta$ of any modal bi-Heyting algebra is perfect.

Formulas in the basic language $\mathcal{L}$ of modal bi-Heyting algebras are defined recursively by

$$\varphi ::= \bot \mid \top \mid p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \varphi - \psi \mid \Diamond\varphi \mid \Box\varphi$$

where $p \in \mathsf{PROP}$. Formulas in the extended language $\mathcal{L}^+$ are defined by

$$\varphi ::= \bot \mid \top \mid p \mid \mathbf{j} \mid \mathbf{m} \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \varphi - \psi \mid \Diamond\varphi \mid \Box\varphi \mid \blacksquare\varphi \mid \blacklozenge\varphi$$

where $p \in \mathsf{PROP}$, $j \in \mathsf{NOM}$ and $m \in \mathsf{CO\text{-}NOM}$. On perfect modal bi-Heyting algebras $\blacksquare$ and $\blacklozenge$ are interpreted as the right and left adjoints of $\Diamond$ and $\Box$, respectively. We will use the terms 'formula' and 'term' interchangeably, and also denote the set of all terms in $\mathcal{L}$ by $\mathtt{Term}$ and the terms of $\mathcal{L}^+$ by $\mathtt{Term}^+$. We now describe two extensions of $\mathtt{Term}$ obtained by adding fixed point operators. The distinction between the two extensions will become clear when we define their interpretations on distributive lattices with operators.

We define $\mathtt{Term}_1$ to be the set of terms which extends $\mathtt{Term}$ by allowing terms $\mu x.t(x)$ and $\nu x.t(x)$ where $t \in \mathtt{Term}_1$ and $t(x)$ is positive in x. The second extension is denoted $\mathtt{Term}_2$ and extends $\mathtt{Term}$ by allowing construction of the terms $\mathtt{LFP}_2 x.t(x)$ and $\mathtt{GFP}_2 x.t(x)$ where $t \in \mathtt{Term}_2$ and $t(x)$ is positive in x.

The terms of $\mathtt{Term}$, $\mathtt{Term}_1$ and $\mathtt{Term}_2$ are interpreted on modal bi-Heyting algebras as follows: Suppose $t(x_1, x_2, \ldots, x_n) \in \mathtt{Term}_1$ and $a_1, \ldots, a_n \in A$. Then $\mu x_1.t(x_1, a_2, \ldots, a_n) := \bigwedge\{a \in A \mid t(a, a_2, \ldots, a_n) \leq a\}$ if this meet exists, otherwise $\mu x_1.t(x_1, a_2, \ldots, a_n)$ is undefined. Similarly, $\nu x_1.t(x_1, a_2, \ldots, a_n) := \bigvee\{a \in A \mid a \leq t(a, a_2, \ldots, a_n)\}$ if this join exists, otherwise $\nu x_1.t(x_1, a_2, \ldots, a_n)$ is undefined. For each ordinal α we define $t^\alpha(\bot, a_2, \ldots, a_n)$ as follows:

$$
\begin{aligned}
&t^0(\bot, a_1, \ldots, a_n) = \bot, \qquad t^{\alpha+1}(\bot, a_1, \ldots, a_n) = t(t^\alpha(\bot, a_1, \ldots, a_n), a_2, \ldots, a_n),\\
&t^\lambda(\bot, a_1, \ldots, a_n) = \bigvee_{\alpha<\lambda} t^\alpha(\bot, a_2, \ldots, a_n) \quad \text{for limit ordinals } \lambda;\\
&t_0(\top, a_1, \ldots, a_n) = \top, \qquad t_{\alpha+1}(\top, a_1, \ldots, a_n) = t(t_\alpha(\top, a_1, \ldots, a_n), a_2, \ldots, a_n),\\
&t_\lambda(\top, a_1, \ldots, a_n) = \bigwedge_{\alpha<\lambda} t_\alpha(\top, a_2, \ldots, a_n) \quad \text{for limit ordinals } \lambda.
\end{aligned}
$$

We then define $\mathtt{LFP}_2 x.t(x, a_2, \ldots, a_n) := \bigvee_{\alpha\geq 0} t^\alpha(\bot, a_2, \ldots, a_n)$ and $\mathtt{GFP}_2 x.t(x, a_2, \ldots, a_n) := \bigwedge_{\alpha\geq 0} t_\alpha(\top, a_2, \ldots, a_n)$ if this join and this meet exist, and they are undefined otherwise.

A modal bi-Heyting algebra $\mathbf{A}$ is called a μ-algebra *of type 1* (of type 2) if $t^{\mathbf{A}}(a_1, \ldots, a_n)$ is defined for all $a_1, \ldots, a_n \in \mathbf{A}$ and all $t \in \mathtt{Term}_1$ ($t \in \mathtt{Term}_2$).

Lemma 1.1. *If* $\mathbf{A}$ *is a* μ*-algebra of type 2, then it is also a* μ*-algebra of type 1.*

The final sets of terms, $\mathtt{Term}_*$ (resp., $\mathtt{Term}_*^+$), are obtained as an extension of $\mathcal{L}$ (resp., $\mathcal{L}^+$) by allowing $\mu^*x.t(x)$ and $\nu^*x.t(x)$ whenever $t \in \mathtt{Term}_*$ (resp., $t \in \mathtt{Term}_*^+$). Terms in $\mathtt{Term}_*$ and $\mathtt{Term}_*^+$ are only interpreted in the canonical extensions $\mathbf{A}^\delta$ of modal bi-Heyting algebras $\mathbf{A}$. If $t(x_1, x_2, \ldots, x_n) \in \mathtt{Term}_* \cup \mathtt{Term}_*^+$ and $a_1, \ldots, a_n \in A^\delta$, then $\mu^*x_1.t(x_1, a_2, \ldots, a_n) := \bigwedge\{a \in A \mid t(a, a_2, \ldots, a_n) \leq a\}$ and $\nu^*x_1.t(x_1, a_2, \ldots, a_n) := \bigvee\{a \in A \mid a \leq t(a, a_2, \ldots, a_n)\}$. Given a term $\varphi \in \mathtt{Term}_1^+$ we write φ^* for the $\mathtt{Term}_*^+$ term obtained from φ by replacing all occurrences of μ and ν with μ^* and ν^*, respectively.

2 μ^*-ALBA

The restricted version of μ-ALBA, called μ^*-ALBA, is based on a calculus consisting of the following derivation rules:

First approximation rule.

$$\frac{\varphi \leq \psi}{\forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq \varphi \ \&\ \psi \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]}\ (\text{FA})$$

Approximation and adjunction rules for connectives.

$$\frac{\varphi \vee \chi \leq \psi}{\varphi \leq \psi \quad \chi \leq \psi}\ (\vee\text{LA}) \qquad \frac{\varphi \leq \chi \vee \psi}{\varphi - \chi \leq \psi}\ (\vee\text{RR}) \qquad \frac{\psi \leq \varphi \wedge \chi}{\psi \leq \varphi \quad \psi \leq \chi}\ (\wedge\text{RA}) \qquad \frac{\chi \wedge \psi \leq \varphi}{\chi \leq \psi \to \varphi}\ (\wedge\text{LR}) \qquad \frac{\Diamond\varphi \leq \psi}{\varphi \leq \blacksquare\psi}\ (\Diamond\text{LA})$$

$$\frac{\varphi \leq \Box\psi}{\blacklozenge\varphi \leq \psi}\ (\Box\text{RA}) \qquad \frac{\Box\psi \leq \mathbf{m}}{\exists \mathbf{n}(\Box\mathbf{n} \leq \mathbf{m} \ \&\ \psi \leq \mathbf{n})}\ (\Box\text{Appr}) \qquad \frac{\varphi \leq \chi \to \psi}{\varphi \wedge \chi \leq \psi}\ (\to\text{RR}) \qquad \frac{\chi - \psi \leq \varphi}{\chi \leq \psi \vee \varphi}\ (-\text{LR})$$

$$\frac{\mathbf{j} \leq \Diamond\psi}{\exists \mathbf{i}(\mathbf{j} \leq \Diamond\mathbf{i} \ \&\ \mathbf{i} \leq \psi)}\ (\Diamond\text{Appr}) \qquad \frac{\chi \to \varphi \leq \mathbf{m}}{\exists \mathbf{j} \exists \mathbf{n}(\mathbf{j} \to \mathbf{n} \leq \mathbf{m} \ \&\ \mathbf{j} \leq \chi \ \&\ \varphi \leq \mathbf{n})}\ (\to\text{Appr}) \qquad \frac{\mathbf{i} \leq \chi - \varphi}{\exists \mathbf{j} \exists \mathbf{n}(\mathbf{i} \leq \mathbf{j} - \mathbf{n} \ \&\ \mathbf{j} \leq \chi \& \ \varphi \leq \mathbf{n})}\ (-\text{Appr})$$

Restricted approximation rules for fixed point binders.

$$\frac{\mathbf{i} \leq \mu^*X.\psi(\overline{\varphi}/\overline{x}, X)}{\bigotimes_{i=1}^n (\exists \mathbf{j}^{\epsilon_i}[\mathbf{i} \leq \mu^*X.\psi(\overline{\mathbf{j}}_i^{\,\epsilon}/\overline{x}, X) \ \&\ \mathbf{j}^{\epsilon_i} \leq^{\epsilon_i} \varphi_i])}\ (\mu_*^\epsilon\text{-A-R}) \qquad \frac{\nu^*X.\varphi(\overline{\psi}/\overline{x}, X) \leq \mathbf{m}}{\bigotimes_{i=1}^n (\exists \mathbf{n}^{\epsilon_i}[\nu^*X.\varphi(\overline{\mathbf{n}}_i^{\,\epsilon}/\overline{x}, X) \leq \mathbf{m} \ \&\ \psi_i \leq^{\epsilon_i} \mathbf{n}^{\epsilon_i}])}\ (\nu_*^\epsilon\text{-A-R})$$

where

1. in each rule, the variables $\overline{x} \in Var$ do not occur in any formula in $\overline{\psi}$ or in $\overline{\varphi}$;
2. all propositional variables and free fixed point variables in $\psi(\overline{x}, X)$ and $\varphi(\overline{x}, X)$ are among $\overline{x}$ and X.
3. in (μ^ϵ-A) we have $\mu^*X.\psi(\overline{\varphi}/\overline{x}, X) \in \mathtt{Term}_*$ and the associated term function of $\psi(\overline{x}, X)$ is completely $\bigvee$-preserving in $(\overline{x}, X) \in \mathbf{A}^\epsilon \times \mathbf{A}$; in particular we require that $\psi(\overline{x}, X)$ is positive (negative) in x_i if $\epsilon_i = 1$ ($\epsilon_i = \partial$);
4. in (ν^ϵ-A) we have $\nu^*X.\varphi(\overline{\psi}/\overline{x}, X) \in \mathtt{Term}_*$ and the associated term function of $\varphi(\overline{x}, X)$ is completely $\bigwedge$-preserving in $(\overline{x}, X) \in \mathbf{A}^\epsilon \times A$; in particular we require that $\varphi(\overline{x}, X)$ is positive (negative) in x_i if $\epsilon_i = 1$ ($\epsilon_i = \partial$).

Recursive Ackermann rules. These are the key rules used to eliminate propositional variables form inequalities. The formulation of these rules require the notions of syntactically open and closed formulas. Informally, a $\mathtt{Term}_*^+$-formula is syntactically closed if, in it, all occurrences of nominals, $\blacklozenge$ and μ^* are positive, while all occurrences of co-nominals, $\blacksquare$ and ν^* are negative. Similarly, an $\mathtt{Term}_*^+$-formula is *syntactically open* if, in it all occurrences of nominals, $\blacklozenge$, and μ^* are negative, while all occurrences of co-nominals, $\blacksquare$ and ν^* are positive. We are now ready to formulate the right and left hand Ackermann rules, (RA$_{rec}$) and (LA$_{rec}$):

$$\frac{\exists p[\&_{i=1}^{n} \alpha_i(p) \leq p \ \& \ \&_{j=1}^{m} \beta_j(p) \leq \gamma_j(p)]}{\&_{j=1}^{m} \beta_j(\mu^* p.[\bigvee_{i=1}^{n} \alpha_i(p)]/p) \leq \gamma_j(\mu^* p.[\bigvee_{i=1}^{n} \alpha_i(p)]/p)} \ (\mathrm{RA}_{rec})$$

subject to the restrictions that the α_i and β_j are positive in p and syntactically closed, while the γ_j are negative in p and syntactically open.

$$\frac{\exists p[\&_{i=1}^{n} p \leq \alpha_i(p) \ \& \ \&_{j=1}^{m} \gamma_j(p) \leq \beta_j(p)]}{\&_{j=1}^{m} \gamma_j(\nu^* p.[\bigwedge_{i=1}^{n} \alpha_i(p)]/p) \leq \beta_j(\nu^* p.[\bigwedge_{i=1}^{n} \alpha_i(p)]/p)} \ (\mathrm{LA}_{rec})$$

subject to the restrictions that the α_i and β_j are positive in p and syntactically open, while the γ_j are negative in p and syntactically closed.

3 Canonicity

Theorem 3.1. *All μ^*-ALBA rules preserve admissible validity of $\mathtt{Term}_*^+$ (quasi-)inequalities on the canonical extensions $\mathbf{A}^\delta$ of modal bi-Heyting algebras $\mathbf{A}$ of type 2.*

This theorem allows us to instantiate the U-shaped argument of Figure 1 and derive the following corollary:

Corollary 3.2 (Canonicity)**.** *Let $\varphi \leq \psi$ be a $\mathtt{Term}_1$ inequality which can be reduced to (set of) pure (quasi-)inequalities by means of μ^*-ALBA rules. Then, for any modal bi-Heyting algebra $\mathbf{A}$ of type 2, it holds that*

$$\mathbf{A} \models \varphi \leq \psi \quad \textit{iff} \quad \mathbf{A}^\delta \models \varphi^* \leq \psi^*.$$

References

[1] N. Bezhanishvili and I. Hodkinson, *Sahlqvist theorem for modal fixed point logic*, Theor. Comput. Sci., 424 (2012), 1–19.

[2] J.C. Bradfield and C. Stirling, *Modal mu-calculi*, in Handbook of Modal Logic, P. Blackburn, J. van Benthem and F. Wolter, eds., Elsevier, 2006, pp. 721–756.

[3] W. Conradie, Y. Fomatati, A. Palmigiano, and S. Sourabh, *Algorithmic correspondence for intuitionistic mu-calculus*, Submitted.

[4] W. Conradie and A. Palmigiano, *Algorithmic correspondence and canonicity for distributive modal logic*, Ann. Pure Appl. Logic, 163 (2012), 338–376.

[5] M. Gehrke and J. Harding, *Bounded lattice expansions*, J. Algebra, 238 (2001), 345–371.

[6] D. Kozen, *Results on the propositional μ-calculus*, Theor. Comput. Sci., 27 (1983), 333–353.