ON DEDUCTIVE SYSTEMS ASSOCIATED WITH EQUATIONALLY ORDERABLE QUASIVARIETIES

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Every deductive system (or logic) $\mathcal{S}$ has an associated canonical class of algebras[1], denoted $\mathbf{Alg}\mathcal{S}$, and deductive systems are classified according to the relations they have with the algebras in $\mathbf{Alg}\mathcal{S}$. A deductive system $\mathcal{S}$ has the *congruence property* if the interderivability relation $\dashv_{\mathcal{S}}\vdash$ is a congruence of the algebra of formulas. Two formulas φ and ψ are related by $\dashv_{\mathcal{S}}\vdash$ if and only if they belong to the same theories of $\mathcal{S}$. When this property lifts to every algebra $\mathcal{S}$ is said to be *congruential*[2], that is, when for every algebra $\mathbf{A}$ in the language of $\mathcal{S}$ the relation $\Lambda^{\mathbf{A}}_{\mathcal{S}}$ on A, defined by $\langle a, b\rangle \in \Lambda^{\mathbf{A}}_{\mathcal{S}}$ if and only if a, b belong to the same $\mathcal{S}$-filters[3], is a congruence of $\mathbf{A}$. This is equivalent to saying that for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$, $\Lambda^{\mathbf{A}}_{\mathcal{S}}$ is the identity relation.

The results gathered in next theorem were proved in [3] and discussed and proved with different methods in [6, 7].

teor:starting **Theorem 1.** *Every finitary deductive system $\mathcal{S}$ with the congruence property and the property of conjunction (with PC) or with a binary term $x \to y$ for which Modus Ponens and the deduction theorem hold (with DDT), is congruential and its canonical class of algebras $\mathbf{Alg}\mathcal{S}$ is a variety.*

If $\mathcal{S}$ is a finitary deductive system with the congruence property and (PC) for a binary term $x \wedge y$, then every algebra $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ has an equationally definable order $\leq^{\wedge}_{\mathbf{A}}$, defined by the equation $x \wedge y \approx x$. Then the deductive system $\mathcal{S}$ satisfies that $\Gamma \vdash_{\mathcal{S}} \varphi$ if and only if

(1)
$$\forall \mathbf{A} \in \mathbf{Alg}\mathcal{S}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})\ \forall a \in A((\forall \psi \in \Gamma,\ a \leq^{\wedge}_{\mathbf{A}} v(\psi)) \Longrightarrow a \leq^{\wedge}_{\mathbf{A}} v(\varphi)), \tag{1}$$

for every set of formulas Γ and every formula φ. The algebras $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ of finitary deductive systems $\mathcal{S}$ with the congruence property and (DDT) for a binary term $x \to y$ also have an equationally definable order $\leq^{\to}_{\mathbf{A}}$, now defined by the equation $x \to y \approx 1$. In this case $\mathcal{S}$ may not satisfy the condition above, but it satisfies that for every set of formulas Γ and every formula φ, $\Gamma \vdash_{\mathcal{S}} \varphi$ if and only if

(2)
$$\forall \mathbf{A} \in \mathbf{Alg}\mathcal{S}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}) v(\varphi) = v(1) \text{ or} \tag{2}$$
$$\exists \varphi_0, \ldots, \varphi_n \in \Gamma\ \forall \mathbf{A} \in \mathbf{Alg}\mathcal{S}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})\ v(\varphi_0 \to (\varphi_1 \to \ldots (\varphi_n \to \varphi)\ldots)) = v(1).$$

Condition (1) can be used to associate a finitary deductive system with every equationally orderable (by a finite set of equations) quasivariety. Let L be an algebraic language,

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[1]The definitions of the concepts used, but not defined, in this abstract can be found in [4].

[2]In [3] these deductive systems are called strongly selfextensional and in fully selfextensional.

[3]A set $F \subseteq A$, is an $\mathcal{S}$-filter of $\mathbf{A}$ if for every set of formulas Γ, every formula φ and every homomorphism v from the algebra of formulas to $\mathbf{A}$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $v[\Gamma] \subseteq F$, then $v(\varphi) \in F$.

$\mu(x,y)$ a finite set of L-equations in two variables and Q a quasivariety of L-algebras. We say that Q is *μ-equationally orderable* if for every algebra $\mathbf{A} \in \mathsf{Q}$ the relation defined on A by the set of equations μ, that is,

$$\leq^{\mu}_{\mathbf{A}} := \{\langle a,b\rangle \in A^2 : \mathbf{A} \models \mu(x,y)[a,b]\},$$

is a partial order of A. The deductive system $\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}}$ is then defined as follows:

$$\Gamma \vdash_{\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}}} \varphi \;\text{ iff }\; \forall \mathbf{A} \in \mathsf{Q}\ \forall v \in \mathrm{Hom}(\mathbf{Fm},\mathbf{A})\ \forall a \in A((\forall \psi \in \Gamma)\ a \leq^{\mu}_{\mathbf{A}} v(\psi)) \Longrightarrow a \leq^{\mu}_{\mathbf{A}} v(\varphi)).$$

Since Q is a quasivariety and μ is finite, this deductive system is finitary. Moreover it is easily seen to have the congruence property. We refer to $\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}}$ as *the deductive system of the μ-order* for Q. When Q is a quasivariety of algebras with a binary term $x \wedge y$ such that in every algebra in Q its interpretation gives a meet-semilattice, then Q is $\{x \wedge y \approx x\}$-equationally orderable and the deductive system $\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}}$ (with $\mu(x,y) = \{x \wedge y \approx x\}$) has (PC). Thus, by Theorem 1, $\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}}$ is congruential and from [6] follows that $\mathbf{Alg}\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}}$ is the variety generated by Q. In particular, when Q is a variety of residuated lattices, the deductive system $\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}}$ is the deductive system that in [1] is called the logic of Q that preserves degrees of truth and in this case we have $\mathbf{Alg}\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}} = \mathsf{Q}$.

In [7] the quasivarieties Q with a binary term $x \to y$ satisfying that 1) $x \to x$ is a constant term, whose interpretation on every $\mathbf{A} \in \mathsf{Q}$ we denote by $1^{\mathbf{A}}$, and that 2) for every algebra $\mathbf{A} \in \mathsf{Q}$, the algebra $\langle A, \to^{\mathbf{A}}, 1^{\mathbf{A}}\rangle$ is a Hilbert algebra are called Hilbert-based. These quasivarieties are $\{x \to y \approx 1\}$-equationally orderable and therefore for each one of them Q the deductive system of the $\{x \to y \approx 1\}$-order for Q. But for those quasivarieties condition (2) above can also be used to associate a finitary deductive system. Given a Hilbert-based quasivariety Q we denote the deductive system defined by condition (2) by $\mathcal{S}^{\to}_{\mathsf{Q}}$.

When a μ-equationally orderable quasivariety Q has a constant term 1 such that for every $\mathbf{A} \in \mathsf{Q}$, $1^{\mathbf{A}}$ is the greatest element of the order $\leq^{\mu}_{\mathbf{A}}$, we can also consider the 1-assertional logic $\mathcal{S}^{1}_{\mathsf{Q}}$ of Q. If Q is 1-regular, then $\mathcal{S}^{1}_{\mathsf{Q}}$ is algebraizable. This happens, for example, for all Hilbert-based quasivarieties. Recall that the 1-assertional logic of Q is defined by

$$\Gamma \vdash_{\mathcal{S}^{1}_{\mathsf{Q}}} \varphi \;\text{ iff }\; \forall \mathbf{A} \in \mathsf{Q}\ \forall v \in \mathrm{Hom}(\mathbf{Fm},\mathbf{A})\ (\forall \psi \in \Gamma)\ v(\psi) = 1^{\mathbf{A}}) \Longrightarrow v(\varphi) = 1^{\mathbf{A}}).$$

If Q is a μ-equationally orderable variety, we do not need that $\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}}$ has (PC) to conclude that it is congruential and with $\mathbf{Alg}\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}} = \mathsf{Q}$. In the talk we will present the following general result.

teor:2 **Theorem 2.** *Let Q be a μ-equationally orderable variety. The deductive system $\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}}$ is congruential and $\mathbf{Alg}\mathcal{S}^{\leq^{\mu}}_{\mathsf{Q}} = \mathsf{Q}$.*

We will also discuss other results on deductive systems associated with a μ-equationally orderable quasivariety Q and the example of deductive systems associated with quasivarieties of BCK-algebras, BCK-meet-semilattices and BCK-join-semilattices, possibly with other operations apart form the implication. The quasivariety BCK of BCK-algebras is $\{x \to y \approx 1\}$-equationally orderable and so are the quasivarieties of algebras with a BCK reduct. We will show that the 1-assertional logic of BCK is different from $\mathcal{S}^{\leq^\mu}_{\mathsf{BCK}}$ (where $\mu = \{x \to y \approx 1\}$). This also holds for the quasivariety of BCK-meet-semilattices, denoted $\mathsf{BCK}^\wedge$, the quasivariety of BCK-join-semilattices, denoted $\mathsf{BCK}^\vee$, and the quasivariety of BCK-lattices, denoted BCK^L. Since these quasivarieties are indeed a variety (see [5]), $\mathcal{S}^{\leq^\mu}_{\mathsf{BCK}^\wedge}$, $\mathcal{S}^{\leq^\mu}_{\mathsf{BCK}^\vee}$ and $\mathcal{S}^{\leq^\mu}_{\mathsf{BCK}^L}$ are congruential. Note that $\mathcal{S}^{\leq^\mu}_{\mathsf{BCK}^\vee}$ is not encompassed by Theorem 1.

In the particular case of the variety Hi of Hilbert algebras we find the same situation as for BCK-algebras: $\mathcal{S}^1_{\mathsf{Hi}}$ is different from $\mathcal{S}^{\leq^\mu}_{\mathsf{BCK}}$. In this case both logics are congruential. For the first it follows form Theorem 1 and for the second from Theorem 2, since Hi is a variety. Moreover $\mathcal{S}^1_{\mathsf{Hi}} = \mathcal{S}^{\to}_{\mathsf{Hi}}$. When we move to the variety $\mathsf{Hi}^\wedge$ of Hilbert algebras with infimum (i.e. BCK-meet-semilattices whose BCK-reduct is a Hilbert algebra), the situation changes dramatically, the three deductive systems $\mathcal{S}^1_{\mathsf{Hi}}$, $\mathcal{S}^{\to}_{\mathsf{Hi}}$ and $\mathcal{S}^{\leq^\mu}_{\mathsf{Hi}}$ are different. Moreover it holds that for a subvariety $\mathsf{K} \subseteq \mathsf{Hi}^\wedge$, the three deductive systems $\mathcal{S}^1_{\mathsf{K}}$, $\mathcal{S}^{\to}_{\mathsf{K}}$ and $\mathcal{S}^{\leq^\mu}_{\mathsf{Hi}}$ are equal if and only if K is a variety of implicative semilattices.

In [3] it is also shown that for every finitary deductive system $\mathcal{S}$ with theorems, the congruence property and (PC) for a binary term $x \wedge y$, the Gentzen system $\mathbf{G}_{\mathcal{S}}$ defined by the structural rules (identity, weakening, exchange, contraction, cut), the congruence rules for every n-ary connective $\star$,

$$\frac{\varphi_i \rhd \psi_i,\ \ \psi_i \rhd \varphi_i : i < n}{\star(\varphi_0 \dots \varphi_{n-1}) \rhd \star(\psi_0 \dots \psi_{n-1})}, \quad \text{and the axiom rules} \quad \frac{}{\underline{\Gamma} \rhd \varphi}$$

for every finite set of formulas Γ and every formula φ such that $\Gamma \vdash_{\mathcal{S}} \varphi$, and where $\underline{\Gamma}$ is any finite sequence of all the formulas in Γ, is algebraizable with equivalent algebraic semantics $\mathbf{Alg}\mathcal{S}$, translation s from equations to sequents given by $s(\varphi \approx \psi) := \{\varphi \rhd \psi, \psi \rhd \varphi\}$ and translation ρ from sequents to equations defined by

$$\rho(\rhd \varphi) := \{\varphi \approx 1\}, \qquad \rho(\varphi_0, \dots, \varphi_n \rhd \varphi) := \{\varphi_0 \wedge \dots \wedge \varphi_n \wedge \varphi \approx \varphi_0 \wedge \dots \wedge \varphi_n\},$$

where 1 is a fixed theorem of $\mathcal{S}$. Moreover it holds that $\mathcal{S}$ is the internal deductive system of $\mathbf{G}_{\mathcal{S}}$[4]. Therefore, if Q is a $\{x \wedge y \approx x\}$-equationally orderable quasivariety, then the Gentzen system $\mathbf{G}_{\mathcal{S}^{\leq}_{\mathsf{Q}}}$ is algebraizable with equivalent algebraic semantics the variety generated by Q and $\mathcal{S}^{\leq}_{\mathsf{Q}}$ is the internal deductive system of $\mathbf{G}_{\mathcal{S}^{\leq}_{\mathsf{Q}}}$.

In [3] it is also shown that every deductive system $\mathcal{S}$ with the congruence property and a binary term $x \to y$ for which Modus Ponens and the deduction theorem hold, the Gentzen system $\mathbf{G}^{\to}_{\mathcal{S}}$ defined by the rules above together with the rule

$$\frac{\Pi, \varphi \rhd \psi}{\Pi \rhd \varphi \to \psi},$$

is algebraizable with equivalent algebraic semantics $\mathbf{Alg}\mathcal{S}$, the translation s from equations to sequents defined as before and the translation η from sequents to equations defined by

$$\eta(\rhd \varphi) := \varphi \approx 1, \qquad \eta(\varphi_0, \dots, \varphi_n \rhd \varphi) := \varphi_0 \to (\varphi_1 \to \dots (\varphi_n \to \varphi) \dots) \approx 1,$$

[4]Given sequent calculus $\mathbf{G}$ with all the structural rules, its internal deductive system $\mathcal{S}_{\mathbf{G}}$ is defined by $\Gamma \vdash_{\mathcal{S}_{\mathbf{G}}} \varphi$ if and only if there is a finite set $\Delta \subseteq \Gamma$ such that for every finite sequence $\underline{\Delta}$ of all the formulas in Δ the sequent $\underline{\Delta} \rhd \varphi$ is derivable in $\mathbf{G}_{\mathcal{S}}$.

where again 1 is a fixed theorem, for example $x \to x$, and $\mathcal{S}$ is the internal deductive system of $\mathbf{G}_{\mathcal{S}}^{\to}$.

If time permits, we will also discuss the relations of the deductive systems discussed for quasivarieties of BCK-algebras and Hilbert algebras (perhaps with additional operations) with the corresponding Gentzen systems associated to them according to [3].

References

EsFoGiGoToVe08-m [1] Bou, F. & Esteva, F. & Font, J. M. & Gil, A. & Godo, L. & Torrens, A. & Verdú, V. (2009). Logics preserving degrees of truth from varieties of residuated lattices. *Journal of Logic and Computation, 19*, pp. 1031–1069.

FiRaSa06 [2] Figallo Jr., A. & Ramón, G. & Saad, S. (2006). iH-propositional calculus. *Bulletin of the Section of Logic, 35*, pp. 157 – 162.

FJ [3] Font, J. M. & Jansana, R. (1996). *A General Algebraic Semantics for Sentential Logics*. vol. 7 of *Lecture Notes in Logic*, Springer-Verlag, 1996. Second revised edition by ASL 2009, available at:

FJP-Su [4] Font, J. M. & R. Jansana, R. & Pigozzi, D. (2003). A Survey of Abstract Algebraic Logic. *Studia Logica, Special Issue on Algebraic Logic II, 74*, pp. 13–97.

Id84 [5] Idziak, P. M. (1984). Lattice operations in BCK-algebras *Mathematica Japonica*, 29, pp. 839–846.

JaSelf [6] Jansana, R. (2006). Selfextensional logics with a conjunction *Studia Logica*, 84, pp. 63–104.

JaImp [7] Jansana, R. (2007). Selfextensional logics with implication. In Béziau, J-Y., ed. *Universal Logic*, Birkhäuser.

Pig-04 [8] Pigozzi, D. (2004). Partially ordered varieties and quasivarieties. Manuscript, available at: `http://orion.math.iastate.edu/dpigozzi/`

Raf-JSL [9] Raftery, J.G. (2006). Correspondence between Gentzen and Hilbert systems. *The Journal of Symbolic Logic, 71*, pp. 903–957.

Raf-09 [10] Raftery, J. G. (2013). Order algebraizable logics. *Annals of Pure and Applied Logic, 164(3)*, pp. 251–283.

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