Every deductive system (or logic) $S$ has an associated canonical class of algebras\textsuperscript{1}, denoted $\text{Alg} S$, and deductive systems are classified according to the relations they have with the algebras in $\text{Alg} S$. A deductive system $S$ has the congruence property if the interderivability relation $\vdash_S$ is a congruence of the algebra of formulas. Two formulas $\varphi$ and $\psi$ are related by $\vdash_S$ if and only if they belong to the same theories of $S$. When this property lifts to every algebra $A$ is said to be congruential\textsuperscript{2}, that is, when for every algebra $A$ in the language of $S$ the relation $\Lambda_A^S$ on $A$, defined by $\langle a, b \rangle \in \Lambda_A^S$ if and only if $a, b$ belong to the same $S$-filters\textsuperscript{3}, is a congruence of $A$. This is equivalent to saying that for every $A \in \text{Alg} S$, $\Lambda_A^S$ is the identity relation.

The results gathered in next theorem were proved in [3] and discussed and proved with different methods in [6, 7].

**Theorem 1.** Every finitary deductive system $S$ with the congruence property and the property of conjunction (with PC) or with a binary term $x \to y$ for which Modus Ponens and the deduction theorem hold (with DDT), is congruential and its canonical class of algebras $\text{Alg} S$ is a variety.

If $S$ is a finitary deductive system with the congruence property and (PC) for a binary term $x \land y$, then every algebra $A \in \text{Alg} S$ has an equationally definable order $\leq_A^S$, defined by the equation $x \land y \approx x$. Then the deductive system $S$ satisfies that $\Gamma \vdash_S \varphi$ if and only if

\begin{align}
(1) & \quad \forall A \in \text{Alg} S \forall \nu \in \text{Hom}(\text{Fm}, A) \forall a \in A((\forall \psi \in \Gamma, \ a \leq_A^S \nu(\psi)) \implies a \leq_A^S \nu(\varphi)), \\
(2) & \quad \forall A \in \text{Alg} S \forall \nu \in \text{Hom}(\text{Fm}, A) \nu(\varphi) = \nu(1) \text{ or} \\
& \quad \exists \varphi_0, \ldots, \varphi_n \in \Gamma \forall A \in \text{Alg} S \forall \nu \in \text{Hom}(\text{Fm}, A) \nu(\varphi_0 \to (\varphi_1 \to \ldots (\varphi_n \to \varphi) \ldots)) = \nu(1).
\end{align}

Condition (1) can be used to associate a finitary deductive system with every equationally-orderable (by a finite set of equations) quasivariety. Let $L$ be an algebraic language,

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\textsuperscript{1}The definitions of the concepts used, but not defined, in this abstract can be found in [4].
\textsuperscript{2}In [3] these deductive systems are called strongly selfextensional and in fully selfextensional.
\textsuperscript{3}A set $F \subseteq A$, is an $S$-filter of $A$ if for every set of formulas $\Gamma$, every formula $\varphi$ and every homomorphism $\nu$ from the algebra of formulas to $A$, if $\Gamma \vdash_S \varphi$ and $\nu(\Gamma) \subseteq F$, then $\nu(\varphi) \in F$. 

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\( \mu(x, y) \) a finite set of \( L \)-equations in two variables and \( Q \) a quasivariety of \( L \)-algebras. We say that \( Q \) is \( \mu \)-equationally orderable if for every algebra \( A \in Q \) the relation defined on \( A \) by the set of equations \( \mu \), that is,

\[
\leq^\mu_A := \{(a, b) \in A^2 : A \models \mu(x, y)[a, b]\},
\]

is a partial order of \( A \). The deductive system \( S^\mu_Q \) is then defined as follows:

\[
\Gamma \vdash_{S^\mu_Q} \varphi \iff \forall A \in Q \forall v \in \text{Hom}(\text{Fm}, A) \forall a \in A((\forall \psi \in \Gamma) a \leq^\mu_A v(\psi)) \implies a \leq^\mu_A v(\varphi).
\]

Since \( Q \) is a quasivariety and \( \mu \) is finite, this deductive system is finitary. Moreover it is easily seen to have the congruence property. We refer to \( S^\mu_Q \) as the deductive system of the \( \mu \)-order for \( Q \). When \( Q \) is a quasivariety of algebras with a binary term \( x \wedge y \) such that in every algebra in \( Q \) its interpretation gives a meet-semilattice, then \( Q \) is \( \{x \wedge y \approx x\} \)-equationally orderable and the deductive system \( S^\mu_Q \) (with \( \mu(x, y) = \{x \wedge y \approx x\} \)) has (PC). Thus, by Theorem 1, \( S^\mu_Q \) is congruential and from [6] follows that \( \text{Alg}S^\mu_Q \) is the variety generated by \( Q \). In particular, when \( Q \) is a variety of residuated lattices, the deductive system \( S^\mu_Q \) is the deductive system that in [1] is called the logic of \( Q \) that preserves degrees of truth and in this case we have \( \text{Alg}S^\mu_Q = Q \).

In [7] the quasivarieties \( Q \) with a binary term \( x \rightarrow y \) satisfying that 1) \( x \rightarrow x \) is a constant term, whose interpretation on every \( A \in Q \) we denote by \( 1^A \), and that 2) for every algebra \( A \in Q \), the algebra \( \langle A, \rightarrow^A, 1^A \rangle \) is a Hilbert algebra are called Hilbert-based. These quasivarieties are \( \{x \rightarrow y \approx 1\} \)-equationally orderable and therefore for each one of them \( Q \) the deductive system of the \( \{x \rightarrow y \approx 1\} \)-order for \( Q \). But for those quasivarieties condition (2) above can also be used to associate a finitary deductive system. Given a Hilbert-based quasivariety \( Q \) we denote the deductive system defined by condition (2) by \( S^\mu_Q \).

When a \( \mu \)-equationally orderable quasivariety \( Q \) has a constant term 1 such that for every \( A \in Q \), \( 1^A \) is the greatest element of the order \( \leq^\mu_A \), we can also consider the 1-assertional logic \( S^1_Q \) of \( Q \). If \( Q \) is 1-regular, then \( S^1_Q \) is algebraizable. This happens, for example, for all Hilbert-based quasivarieties. Recall that the 1-assertional logic of \( Q \) is defined by

\[
\Gamma \vdash_{S^1_Q} \varphi \iff \forall A \in Q \forall v \in \text{Hom}(\text{Fm}, A) \forall \psi \in \Gamma \forall a \in A((\forall v(\psi) = 1^A) \implies v(\varphi) = 1^A).
\]

If \( Q \) is a \( \mu \)-equationally orderable variety, we do not need that \( S^\mu_Q \) has (PC) to conclude that it is congruential and with \( \text{Alg}S^\mu_Q = Q \). In the talk we will present the following general result.

**Theorem 2.** Let \( Q \) be a \( \mu \)-equationally orderable variety. The deductive system \( S^\mu_Q \) is congruential and \( \text{Alg}S^\mu_Q = Q \).

We will also discuss other results on deductive systems associated with a \( \mu \)-equationally orderable quasivariety \( Q \) and the example of deductive systems associated with quasivarieties of BCK-algebras, BCK-meet-semilattices and BCK-join-semilattices, possibly with other operations apart form the implication. The quasivariety BCK of BCK-algebras is \( \{x \rightarrow y \approx 1\} \)-equationally orderable and so are the quasivarieties of algebras with a BCK
reduct. We will show that the 1-assertional logic of BCK differs from $S^{\leq}_{BCK}$ (where $\mu = \{ x \to y \approx 1 \}$). This also holds for the quasivariety of BCK-meet-semilattices, denoted $BCK^\wedge$, the quasivariety of BCK-join-semilattices, denoted $BCK^\vee$, and the quasivariety of BCK-lattices, denoted $BCK^\ell$. Since these quasivarieties are indeed a variety (see [5]), $S^{\leq}_{BCK^\wedge}$, $S^{\leq}_{BCK^\vee}$ and $S^{\leq}_{BCK^\ell}$ are congruential. Note that $S^{\leq}_{BCK^\wedge}$ is not encompassed by Theorem 1.

In the particular case of the variety $Hi$ of Hilbert algebras we find the same situation as for BCK-algebras: $S^{1}_{Hi}$ is different from $S^{\leq}_{BCK}$. In this case both logics are congruential. For the first it follows from Theorem 1 and for the second from Theorem 2, since $Hi$ is a variety. Moreover $S^{1}_{Hi} = S^{\leq}_{Hi}$. When we move to the variety $Hi^\wedge$ of Hilbert algebras with infimum (i.e. BCK-meet-semilattices whose BCK-reduct is a Hilbert algebra), the situation changes dramatically, the three deductive systems $S^{1}_{Hi^\wedge}$, $S^{\leq}_{Hi^\wedge}$ and $S^{\leq}_{Hi^\wedge}$ are different. Moreover it holds that for a subvariety $K \subseteq Hi^\wedge$, the three deductive systems $S^{1}_{K^\wedge}$, $S^{\leq}_{K^\wedge}$ and $S^{\leq}_{Hi^\wedge}$ are equal if and only if $K$ is a variety of implicational semilattices.

In [3] it is also shown that for every finitary deductive system $S$ with theorems, the congruence property and (PC) for a binary term $x \land y$, the Gentzen system $G_{S}$ defined by the structural rules (identity, weakening, exchange, contraction, cut), the congruence rules for every $n$-ary connective *$
abla$
,

$$\frac{\varphi_{1}, \ldots, \varphi_{n} \vdash \psi_{1}, \ldots, \psi_{n} : i < n}{\ast(\varphi_{0}, \ldots, \varphi_{n-1}) \vdash \ast(\psi_{0}, \ldots, \psi_{n-1})}$$

and the axioms rules $\Gamma \vdash \varphi$ for every finite set of formulas $\Gamma$ and every formula $\varphi$ such that $\Gamma \vdash_{S} \varphi$, and where $\Gamma$ is any finite sequence of the formulas in $\Gamma$, is algebraizable with equivalent algebraic semantics $Alg_{S}$, translation $s$ from equations to sequents given by $s(\varphi \approx \psi) := \{ \varphi \vdash \psi, \psi \vdash \varphi \}$ and translation $\rho$ from sequents to equations defined by

$$\rho(\varphi) := \{ \varphi \approx 1 \}, \quad \rho(\varphi_{0}, \ldots, \varphi_{n} \vdash \varphi) := \{ \varphi_{0} \land \ldots \land \varphi_{n} \land \varphi \approx \varphi_{0} \land \ldots \land \varphi_{n} \},$$

where 1 is a fixed theorem of $S$. Moreover it holds that $S$ is the internal deductive system of $G_{S}$.

Therefore, if $Q$ is a $\{ x \land y \approx x \}$-equationally orderable quasivariety, then the Gentzen system $G_{S^{\leq}_{Q}}$ is algebraizable with equivalent algebraic semantics the variety generated by $Q$ and $S^{\leq}_{Q}$ is the internal deductive system of $G_{S^{\leq}_{Q}}$.

In [3] it is also shown that every deductive system $S$ with the congruence property and a binary term $x \to y$ for which Modus Ponens and the deduction theorem hold, the Gentzen system $G_{S^{\leq}_{Q}}$ defined by the rules above together with the rule

$$\frac{\Pi_{i}, \varphi \vdash \psi}{\Pi \vdash \varphi \to \psi'}$$

is algebraizable with equivalent algebraic semantics $Alg_{S}$, the translation $s$ from equations to sequents defined as before and the translation $\eta$ from sequents to equations defined by

$$\eta(\varphi) := \varphi \approx 1, \quad \eta(\varphi_{0}, \ldots, \varphi_{n} \vdash \varphi) := \varphi_{0} \to (\varphi_{1} \to \ldots (\varphi_{n} \to \varphi) \ldots) \approx 1,$$

\[4\]Given sequent calculus $G$ with all the structural rules, its internal deductive system $S_{G}$ is defined by $\Gamma \vdash_{S_{G}} \varphi$ if and only if there is a finite set $\Delta \subseteq \Gamma$ such that for every finite sequence $\Delta$ of all the formulas in $\Delta$ the sequent $\Delta \vdash \varphi$ is derivable in $G_{S}$.
where again 1 is a fixed theorem, for example \( x \to x \), and \( \mathcal{S} \) is the internal deductive system of \( G_{\mathcal{S}} \).

If time permits, we will also discuss the relations of the deductive systems discussed for quasivarieties of BCK-algebras and Hilbert algebras (perhaps with additional operations) with the corresponding Gentzen systems associated to them according to [3].

References


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