Co-rotation, co-rotation-annihilation, and involutive ordinal sum constructions of residuated semigroups [*]

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Keywords: Residuated lattice, constructions

AMS Classification: 20M14 - commutative semigroups, 06F05 - ordered semigroups and monoids

1 Introduction

Residuated lattices form the algebraic counterpart of substructural logics [4]. *Rotation construction* for t-norms has been introduced in [9] and has been generalized to arbitrary residuated posets in [7]. The reader is referred also to [13, 6, 5]. The *rotation-annihilation construction* has been introduced in [10, 7]. The rotation construction has proved fundamental in the structural description of perfect and bipartite IMTL-algebras [12], of free nilpotent minimum algebras [1], of free Glivenko algebras [2], and has proved useful in many other mathematical applications.

De Morgan dualization can be defined in any algebra which has a binary multiplication $\circledast$ and an order-reversing involution $'$ on its universe by $x \cdot y = (x \circledast y)'$. However, de Morgan dualization does not fit well to the class of residuated lattices, since de Morgan dual of the monoidal operation of a residuated lattice, in general, is not residuated with respect to the same ordering relation. As an attempt to overcome this, and to describe the structure of a certain class of residuated chains, skew dualization has been introduced in [8]. However, skew-dualization is a notion hard to work with.

Two *co*-rotation constructions and two *co*-rotation-annihilation constructions (in both cases a disjoint one and a connected one) will be introduced. In some sense these can be considered as 'skew dual' constructions to the (disjoint and connected) rotation constructions. Just as the rotation(-annihilation) of FL_e-algebras results in positive rank involutive FL_e-algebras, the co-rotation(-annihilation) of FL_e-algebras results in non-positive (zero or negative) rank involutive FL_e-algebras; thus providing a wide spectrum of examples for the latter algebra. The fact that co-rotation and co-rotation-annihilation constructions are not simply dual of their rotation and rotation-annihilation counterparts is also reflected by the fact that the class of algebras which can be used in them are quite different.

Also, a construction, called *involutive ordinal sums* will be introduced, and its use will be demonstrated in the structural description of a certain class of involutive FL_e-

[*] Supported by the SROP-4.2.2.C-11/1/KONV-2012-0005 project.

algebras. This construction generalizes the generalized ordinal sums of Galatos [3] in the group-like case.

2 Involutive Ordinal Sums

Definition 1 Let $(\kappa, \leq)$ be a nonempty totally ordered set. For $i \in \kappa$, let $\mathcal{U}_i = \langle X_i, \circledast_i, \leq_i, t_i, t_i \rangle$ be a group-like FL$_e$-algebra, its involution, its negative and positive cone are being denoted by $'^i$, $(X_i^-, \otimes_i)$, and $(X_i^+, \oplus_i)$, respectively, and denote $\mathcal{F}$ the family $\{\mathcal{U}_i \mid i \in \kappa\}$. Define the involutive ordinal sum of the $\mathcal{U}_i$'s as follows:

Let $X = \bigcup_{i \in \kappa} X_i$ be the disjoint union of X_i's equipped by the following total-order:

1. If $x, y \in X_i$ for some $i \in \kappa$ then $x \leq y$ iff $x \leq_i y$.
2. if $x \in X_i^+$ and $y \in X_j^+$ for some $i, j \in \kappa$ and $i < j$ then $x \leq y$,
3. if $x \in X_i^-$ and $y \in X_j^-$ for some $i, j \in \kappa$ and $i < j$ then $y \leq x$.
4. If $x \in X_i^+$ and $y \in X_j^-$ for some $i, j \in \kappa$, $i \neq j$ then $x \geq y$.

We will identify $\mathcal{U}_i$ ($i \in \kappa$) with its embedding into X. Notice that 4. implies $t_i = t_j$ for any $i, j \in \kappa$; denote this element by t. Then the mapping $' : X \to X$, $x \mapsto x'^i$ if $x \in X_i$, is clearly an order-reversing involution of X with t being its unique fixed point. Define a binary operation $\circledcirc$ on X as follows:

$$x \circledcirc y = \begin{cases} x \circledast_i y & \text{if } x, y \in X_i \\ \min(x, y) & \text{if } x \in X_i, y \in X_j, i \neq j, \text{and } x \leq y' \\ \max(x, y) & \text{if } x \in X_i, y \in X_j, i \neq j, \text{and } x > y' \end{cases}, \tag{1}$$

and call

$$ios_{i \in \kappa}(\mathcal{U}_i) = \langle X, \circledcirc, \leq, t, t \rangle$$

the *involutive ordinal sum* of the family $\mathcal{F} = \{\mathcal{U}_i \mid i \in \kappa\}$.
Further, denote the (max-)ordinal sum of the positive cones of $\mathcal{U}_i$'s with respect to κ by $(X^+, \oplus)$, that is, let $(X^+, \oplus) = os_\kappa(X_i^+, \circledast_i)$. In addition, denote the (min-)ordinal sum of the negative cones of $\mathcal{U}_i$'s with respect to the dual of κ by $(X^-, \otimes)$, that is, let $(X^-, \otimes) = os_{\kappa^{co}}(X_i^-, \circledast_i)$.

Theorem 1 *The involutive ordinal sum of an arbitrary family of group-like FL$_e$-algebras is a group-like FL$_e$-algebra.*

Corollary 1 *The twin-rotation of the (Clifford-style) ordinal sum of any family of negative cones of group-like FL$_e$-algebras is a group-like FL$_e$-algebra.*

3 Co-rotations and co-rotation-annihilations

Lack of available space, instead of presenting the (lengthy) definitions, we shall show some example operations that can be constructed with the co-rotation and the co-rotation-annihilation constructions.

Example 1 Figure 1 shows the connected co-rotations of skewed duals ([8]) of the minimum, product, and Łukasiewicz t-norms, respectively, rescaled to $[0, 1]$. These operations coincide with the skewed duals of connected rotations of the minimum, product, and Łukasiewicz t-norms, respectively. For more on skew duals and skew symmetrization, see [8].

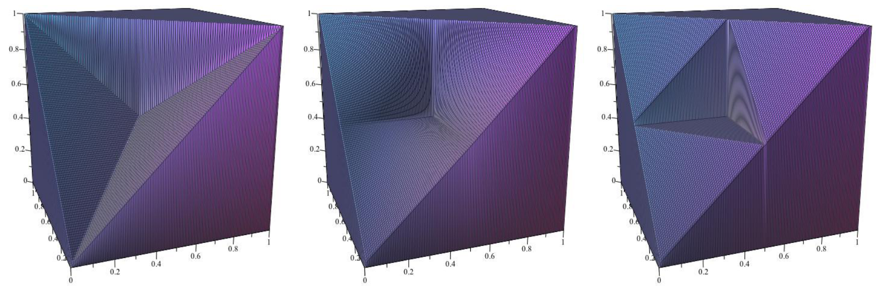

Figure 1: Connected co-rotations of skew duals of minimum, product, and Łukasiewicz t-norms, respectively, rescaled to $[0, 1]$, see Example 1

Example 2 Both the disconnected and the connected co-rotations can be iterated. That is, the resulted operation can serve as the starting operation of both co-rotation constructions. An example in in Figures 2. In Figure 2 the leftmost operation is a group-like FL_e-algebra, for a related classification theorem, see [11].

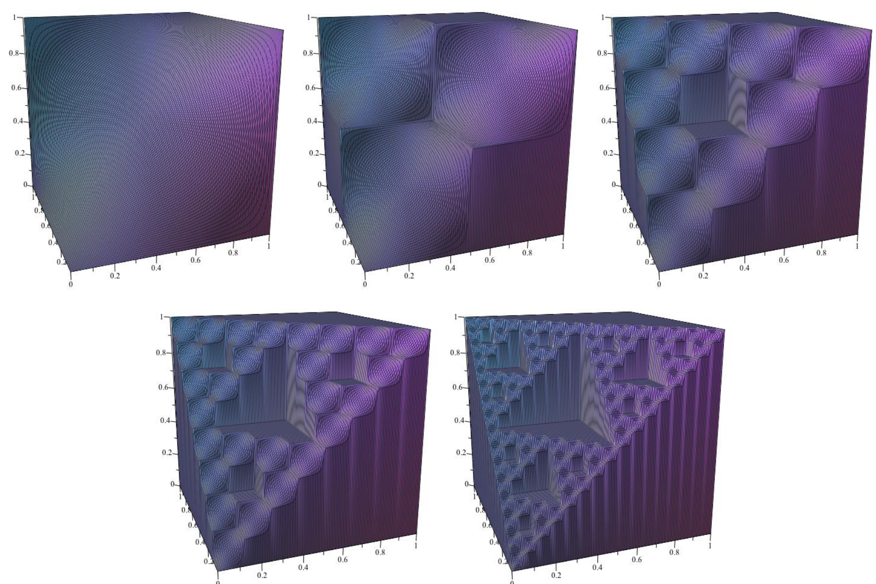

Figure 2: The (skew) symmetrization of the product t-norm and its iterated connected co-rotations, rescaled to $[0, 1]$, see Example 2

References

[1] M. Busaniche, Free nilpotent minimum algebras, Mathematical Logic Quartely 52 (3), (2006) 219–236.

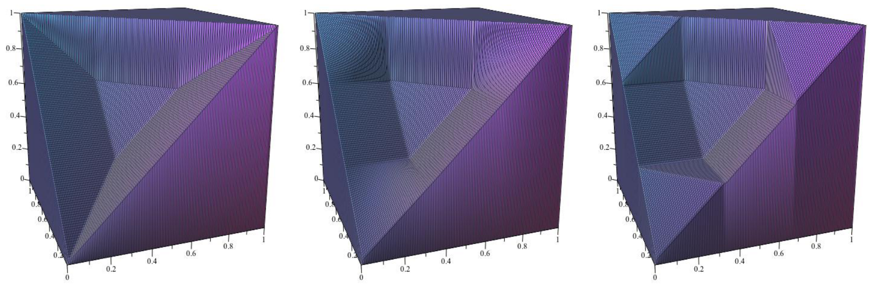

Figure 3: Connected co-rotation-annihilations of skew duals of minimum, product t-norms, and Łukasiewicz t-norms, respectively, with the skew duals of Łukasiewicz t-norm, rescaled to $[0, 1]$

[2] R. Cignoli, A. Torrens, Free algebras in varieties of Glivenko MTL-algebras satisfying the equation $2(x^2) = (2x)^2$, Studia Logica 83 (1-3), (2006), 157–181.

[3] N. Galatos, Generalized Ordinals Sums and Translations, Logic Journal of the IGPL, 19 (3), (2011)

[4] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated Lattices – An Algebraic Glimpse at Substructural Logics, Elsevier, 2007, 532 pp.

[5] N. Galatos, J. Raftery, Adding involution to residuated structures, Studia Logica, 77 (2), (2004), 181–207.

[6] S. Jenei, On the relationship between the rotation construction and ordered Abelian groups, Fuzzy Sets and Systems 161 (2) pp. 277-284. (2010)

[7] S. Jenei, On the structure of rotation-invariant semigroups, Archive for Mathematical Logic, 42(5), (2003), 489–514

[8] S. Jenei, Structural Description of a Class of Involutive Uninorms via Skew Symmetrization, Journal of Logic and Computation, 21(5), (2011), 729–737.

[9] S. Jenei, Structure of left-continuous triangular norms with strong induced negations – (I) Rotation construction, Journal of Applied Non-Classical Logics 10 (1) pp. 83-92. (2000)

[10] S. Jenei, Structure of left-continuous triangular norms with strong induced negations – (II) Rotation-annihilation construction, Journal of Applied Non-Classical Logics 11 (3-4) pp. 351-366. (2001)

[11] S. Jenei, F. Montagna, A classification of certain group-like FL_e-chains, (submitted)

[12] C. Noguera, F. Esteva, J. Gispert, Perfect and bipartite IMTL-algebras are disconnected rotations of prelinear semihoops, Archive for Mathematical Logic 44 (7): 869–886 2005

[13] A. Wronski, Reflections and distensions of BCK-algebras, Math. Japonica, 28 (1983), 215–225.