REPRESENTATIONS OF RAMSEY RELATION ALGEBRAS

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1. Introduction

Although the results we obtain have some significance for the theory of relation algebras, the method, and indeed the problem itself, can be presented without mentioning relation algebras at all. Consider the following Ramsey-like problem. For a given number n of colours, is there a complete graph K_m such that the edges of K_m can be coloured with the n colours in such a way that:

(1) there are no monochromatic triangles,
(2) every non-monochromatic triangle appears everywhere it can.

The first condition gives the upper bound for the possible size of K_m, via Ramsey theorem. The second condition is really a shorthand for the following two requirements: (i) every vertex of K_m has at least one outgoing edge of each colour, and (ii) given any edge (x, y) of colour c, and any colours a, b, such that $\{a, b, c\} \neq \{c\}$, there exist a vertex z such that (x, z) is coloured by a and (z, y) is coloured by b.

Clearly, to satisfy (2) the graph K_m cannot be too small, so (2) gives a lower bound for the possible size of K_m. For $n = 2$, the lower bound is 5, and so is the upper bound because $R(3, 3) = 6$. For $n = 3$, the upper bound is 16, the lower bound can be shown to be 13, and indeed K_{13} gives one possible answer. Curiously, no colourings satisfying (1) and (2) exist for K_{14} and K_{15}. But for K_{16} there exist two non-isomorphic ones. For $n = 4, 5$ suitable colourings were found by S. Comer [3]. For $n > 5$ the answer was not known. we will show that a required colouring can be obtained from a finite field satisfying certain conditions. Unfortunately, we have no general existence theorem for such fields, but computer searches have shown that these exist for all $2 \leq n \leq 120$, except $n = 8$, and $n = 13$.

2. Ramsey (or Monk, or Maddux) algebras

For a general introduction to Relation Algebras, the reader is referred to Hirsch, Hodkinson [1], and Maddux [2]. The relation algebras defined below are called *Ramsey algebras* here, because of the connection with Ramsey theorem, but they have been known under other names, for example, *Monk algebras* or *Maddux algebras* in [1]. They were also considered in [2]. Our choice of the name was influenced by a prominent algebraist, who remarked that it should at the very least point a non-expert in the right direction.

With the naming controversy thus avoided (or perhaps ignited?), we define the *Ramsey algebra* $\mathbf{R}_n$, for any $n \geq 2$, as a finite relation algebra on $n + 1$ atoms: $1', a_1, \ldots, a_n$, such that for each $i \in \{1, \ldots, n\}$ the triple $\langle a_i, a_i, a_i \rangle$ is forbidden. To prevent a notational confusion: what we call $\mathbf{R}_n$ here, is $\mathcal{M}_{n+1}$ in [1], and $\mathfrak{E}_{n+1}^{\{2,3\}}$ in [2].

By Ramsey Theorem, if $\mathbf{R}_n$ is representable then it is representable on a finite set. Representations of $\mathbf{R}_n$ were known to exists for $2 \leq n < 6$, where $\mathbf{R}_2$ is the pentagon algebra, so it has a unique representation on 5 points, $\mathbf{R}_3$ has precisely three non-isomprphic representations: one on 13 points and two on 16 points, representations for $\mathbf{R}_4$ and $\mathbf{R}_5$ were found by Comer [3].

We will now present a method of finding representations od Ramsey algebras, based on a rather simple observation about the representation of the pentagon algebra $\mathbf{R}_2$. Its unique representation can be described as follows. Consider $\mathbb{Z}_5$ as a Galois field, and let g be a generator of its multiplicative group. It happens that the order of the multiplicative group, $5 - 1 = 4$ is divisible by the number of colours, so we build a rectangular matrix

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \cong \begin{pmatrix} g & g^3 \\ g^2 & g^4 \end{pmatrix} \cong \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$$

where 2 and 3 are the only possible choices for g. The rows of this matrix give the representation over $\mathbb{Z}_5$, in the usual way, namely as a complex algebra of the partition $\{0 \mid 2, 3 \mid 4, 1\}$ of (the universe of) $\mathbb{Z}_5$. The representation is independent of the choice of the generator g, so we could write it as $\{0 \mid g, g^3 \mid g^2, g^4 = 1\}$.

Exactly the same procedure can be applied to $\mathbb{Z}_{13}$ and 3 colours (notice that, conveniently, $13 - 1$ is divisible by 3). We get the matrix

$$\begin{pmatrix} g & g^4 & g^7 & g^{10} \\ g^2 & g^5 & g^8 & g^{11} \\ g^3 & g^6 & g^9 & g^{12} \end{pmatrix} \cong \begin{pmatrix} 2 & 3 & 11 & 10 \\ 4 & 6 & 9 & 7 \\ 8 & 12 & 5 & 1 \end{pmatrix}$$

choosing $g = 2$. It is then not difficult to check that it produces a representation of $\mathbf{R}_3$ over $\mathbb{Z}_{13}$. Somewhat surprisingly, one of the two representations on 16 points, can be obtained similarly, this time taking $GF(16)$. Again, conveniently, $16 - 1$ is divisible by 3, so we get

$$\begin{pmatrix} g & g^4 & g^7 & g^{10} & g^{13} \\ g^2 & g^5 & g^8 & g^{11} & g^{14} \\ g^3 & g^6 & g^9 & g^{12} & g^{16} = 1 \end{pmatrix}$$

and this turns out to be a representation as well. The matrices that gave rise to the representations have the following crucial property that comes in four parts. Let a be an element of the bottom row. Then

- the additive inverse of a also belongs to the bottom row,
- $a + 1$ never belongs to the bottom row, and
- for any non-bottom row k, there is a choice of a such that $a + 1$ belongs to the row k.
- for any distinct rows k and ℓ, there is a b in row ℓ such that $b + 1$ belongs to row k.

The first part is a symmetry requirement, the second excludes monochromatic triangles, the third produces all isoceless triangles, the fourth all other non-monochromatic ones. The next section makes it precise.

3. A representability test

Consider $\mathbf{R}_n$. Let $GF(p^K)$, for a prime p, be such that n divides $p^K - 1$, say, $(p^K - 1)/n = m$. Let g be a generator of the multiplicative group of $GF(p^K)$, and

M be the $n \times m$ matrix

$$\begin{pmatrix} g & g^{n+1} & \dots & g^{(m-1)n+1} \\ g^2 & g^{n+2} & & g^{(m-1)n+2} \\ \vdots & \vdots & & \vdots \\ g^n & g^{n+n} & \dots & g^{(m-1)n+n} = g^{mn} = 1 \end{pmatrix}$$

and suppose that

(i) $-1 = g^{in}$ for some $i \in \{1, \dots, m\}$,
(ii) $g^{in} + 1 \neq g^{jn}$ for all $i, j \in \{1, \dots, m\}$,
(iii) for every $k \in \{1, \dots, n-1\}$ there are $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, m-1\}$, such that $g^{in} + 1 = g^{jn+k}$,
(iv) for every $k, \ell \in \{1, \dots, n-1\}$ with $k \neq \ell$, there are $i, j \in \{0, \dots, m-1\}$, such that $g^{in+\ell} + 1 = g^{jn+k}$.

We will write R_i for the i-th row of M, considered as a set. The complex operations on the rows have their usual meaning, so, for example

$$-R_i = \{-g^i, -g^{n+i}, \dots, -g^{(m-1)n+i}\}$$

and

$$R_i + R_j = \{a + b\colon a \in R_i,\ b \in R_j\}.$$

Lemma 1. *The matrix M above has the following properties:*

(1) $-R_i = R_i$,
(2) $R_i + R_i = \bigcup_{j \neq i} R_j$,
(3) $R_i + R_j = M$, *if* $i \neq j$,

for every $i, j \in \{1, \dots, m\}$.

Theorem 1. *Let $\mathbf{R}_n$ be a Ramsey algebra, and M be an $n \times m$ matrix over $GF(p^K)$ satisfying the properties stated before Lemma 1. Let $G(p^K)$ be the additive group of $GF(p^K)$. Then, $\mathbf{R}_n$ is representable over $G(p^K)$. More precisely, the representation of $\mathbf{R}_n$ is the subalgebra of the complex algebra of $G(p^K)$, whose atoms are the sets $\{0\}$ and R_i for $i \in \{1, \dots, n\}$.*

4. Emprirical results

As we said at the outset, we have no general existence theorem, only some empirical results. For $n = 2, \dots, 7$ our method produces representations, in fact, with the exception of the representation of $\mathbf{R}_3$ over $GF(2^4)$, they are all over prime fields. For $n = 8$ the method does *not* produce any representations. For $n = 9$ the only representation obtained is over $GF(19^2)$. For $n = 10, 11, 12$ representations over prime fields exist. For $n = 13$ we do not know whether the method produces any representations. For $n = 14, \dots, 120$ representations over prime fields exist. The sizes of the representations grow roughly as $n^{5/2}$ in the number of colours.

References

[1] R. Hirsch and I. Hodkinson, *Relation Algebras by Games*, Studies in Logic and the Foundations of Mathematics, vol.147, Elsevier 2002.
[2] R. Maddux, *Relation Algebras*, Studies in Logic and the Foundations of Mathematics, vol.150, Elsevier 2006.
[3] S. Comer, *Color schemes forbidding monochrome triangles*, Proceedings of the Fourteenth Southeastern Conference on Combinatorics, Graph Theory and Computing (Boca Raton, Fla., 1983), vol. 39, 1983, pp. 231-236.