A completeness theorem for two-layer modal logics*

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Mathematical Fuzzy Logic (MFL) studies a family of non-classical logics with a semantics based on (linearly ordered) scales of degrees of truth. This is what makes these logics specially suited for the study of gradual aspects of vagueness and imprecision, found in sentences like 'it is heavily raining' or 'that man is tall'.

Moreover, a conceptually different issue, that of uncertainty, has also been addressed inside MFL. The main idea, introduced in [13] and later developed by Hájek in his monograph [11], is that one could use probability to determine the truth degree of statements such as 'tomorrow it will probably rain' or 'the probability that tomorrow it will rain is high'. Indeed, one takes classical logic and its formulae φ to describe crisp events, introduces a new modal operator P which can be applied on them to create atomic modal formulae $P\varphi$ which may be read as 'probably φ' (or better 'the probability of φ is high'), and finally these atomic modal formulae are combined by using the connectives of Łukasiewicz logic. What we obtain is a *two-layer modal fuzzy logic* built on atomic formulae $P\varphi$ whose truth values are given by a probability measure. Several works have followed this idea with variations. In [9] Godo, Esteva and Hájek replaced Łukasiewicz logic on the second layer by ŁΠ, but kept classical logic for non-modal formulae. The logic ŁΠ, with its expanded language, enabled them to deal with conditional probability. Flaminio and Montagna also considered conditional probability in [7], and Godo and Marchioni investigated coherent conditional probabilities in [10]. Marchioni also proposed a class of *logics of uncertainty* in [14] with different kinds of measures (besides probability) to quantify the uncertainty of events. In all of these works classical logic has been kept as the underlying logic for non-modal formulae.

However, if one wants to deal with uncertainty and vagueness at once, i.e. with the probability of vague events, as in 'tomorrow it will probably rain *heavily*', the two-layer paradigm can still be useful provided that the underlying classical logic is substituted by a fuzzy logic. This idea has been also investigated in some works, as [5] where finite Łukasiewicz systems $Ł_n$ are taken as the logics of vague events. Other recent works along these lines are surveyed in [6].

In this talk we provide a new general framework for two-layer modal fuzzy logics that encompasses all the mentioned system and paves the way for future development.[1] In fact, we go far beyond the landscape of fuzzy logics. Indeed, we show how one can construct a modal logic (for an arbitrary modality, not necessarily read as a probability) over an *arbitrary non-classical logic* (under certain technical requirements). Therefore, we need not assume that the starting logic is fuzzy, and we can develop a general theory of two-layer modal logics, showing how the methods used in the fuzzy literature can lead to completeness results using very few properties of the underlying logics. As a semantics, we propose particular kinds of *measured Kripke Frames* and prove corresponding completeness theorems.

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[1]There has already been an attempt at an abstract theory of two-layer modal fuzzy logics in Master thesis [15]; but it was rather restricted in its scope.

Convention 1. *Let $\mathcal{L}$ be a language containing at least a truth constant $\overline{1}$ and binary connectives $\to$ and $\vee$. In this paper a propositional logic* L *in $\mathcal{L}$ is a finitary lattice-disjunctive weakly implicative logic (as studied in [3]). In more details, this means that* L *is identified with the provability relation $\vdash_{\mathrm{L}}$ on $\boldsymbol{Fm}_{\mathcal{L}}$ given by a finitary Hilbert-style system such that:*[2]

$$\vdash_{\mathrm{L}} \varphi \to \varphi \qquad \varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi \qquad \varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi \qquad \varphi \dashv\vdash_{\mathrm{L}} \overline{1} \to \varphi$$

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \circ(\chi_1, \dots \chi_i, \varphi, \dots, \chi_n) \leftrightarrow \circ(\chi_1, \dots \chi_i, \psi, \dots, \chi_n) \quad \textit{for every } n\textit{-ary } \circ \in \mathcal{L} \textit{ and } i < n.$$

$$\vdash_{\mathrm{L}} \varphi \to \varphi \vee \psi \qquad \vdash_{\mathrm{L}} \psi \to \varphi \vee \psi \qquad \varphi \to \chi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \vee \psi \to \chi$$

$$\Gamma, \varphi \vdash_{\mathrm{L}} \chi \quad \textit{and} \quad \Gamma, \psi \vdash_{\mathrm{L}} \chi \quad \textit{imply} \quad \Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi$$

Note that our logics are algebraically implicative with a truth definition given by the single equation $x \vee \overline{1} \approx \overline{1}$. Let us fix a logic L in a language $\mathcal{L}$; then $\mathcal{L}$-algebras are algebras with signature $\mathcal{L}$ and homomorphisms from $\boldsymbol{Fm}_{\mathcal{L}}$ to an $\mathcal{L}$-algebra $\boldsymbol{A}$ are called $\boldsymbol{A}$*-evaluations*. For an $\mathcal{L}$-algebra $\boldsymbol{A}$ we define the set $F_{\boldsymbol{A}} = \{x \mid x \vee^{\boldsymbol{A}} \overline{1}^{\boldsymbol{A}} = x\}$.

Definition 2. *We say that $\boldsymbol{A}$ is an* L*-algebra, $\boldsymbol{A} \in \mathbb{L}$ in symbols, if*

- *for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ such that $\Gamma \vdash_{\mathrm{L}} \varphi$, we have that for each $\boldsymbol{A}$-evaluation e, if $e[\Gamma] \subseteq F_{\boldsymbol{A}}$, then $e(\varphi) \in F_{\boldsymbol{A}}$,*
- *for each $x, y \in A$, if $\{x \to^{\boldsymbol{A}} y, y \to^{\boldsymbol{A}} x\} \subseteq F_{\boldsymbol{A}}$, then $x = y$.*

$\mathbb{L}$ is in fact a quasivariety and it is the equivalent algebraic semantics of L in the sense of [2]. A non-trivial L-algebra $\boldsymbol{A}$ is *(finitely) subdirectly irreducible relative to* $\mathbb{L}$ if for every (finite non-empty) subdirect representation α of $\boldsymbol{A}$ with a family $\{\boldsymbol{A}_i \mid i \in I\} \subseteq \mathbb{L}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. $\mathbb{L}_{\mathrm{R(F)SI}}$ denotes the class of all (finitely) subdirectly irreducible algebras relative to $\mathbb{L}$. Of course $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{\mathrm{RFSI}}$.

Definition 3. *Let* L *be a logic and $\mathbb{K} \subseteq \mathbb{L}_{\mathrm{RFSI}}$. We say that* L *has* (finite) strong $\mathbb{K}$-completeness, S$\mathbb{K}$C *(or* FS$\mathbb{K}$C *resp.) whenever for each (finite) theory $\Gamma \cup \{\varphi\}$ holds that $\Gamma \vdash_{\mathrm{L}} \varphi$ iff for each $\boldsymbol{A} \in K$ and each $\boldsymbol{A}$-evaluation e we have $e(\varphi) \in F_{\boldsymbol{A}}$ whenever $e[\Gamma] \subseteq F_{\boldsymbol{A}}$.*

Algebraically we can say that L has FS$\mathbb{K}$C (or S$\mathbb{K}$C resp.) if $\mathbb{K}$ generates $\mathbb{L}$ as a (σ-)quasivariety. Note that every logic has S$\mathbb{L}_{\mathrm{RSI}}$C (and hence S$\mathbb{L}_{\mathrm{RFSI}}$C). If L is a fuzzy logic, then $\mathbb{L}_{\mathrm{RFSI}}$ is the class of L-chains.

Let us fix two logics L_1 and L_2 in disjoint languages $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\Box \notin \mathcal{L}_1 \cup \mathcal{L}_2$. Further we fix two classes of algebras $\mathbb{K}_i \subseteq (\mathbb{L}_i)_{\mathrm{RFSI}}$, $i \in \{1, 2\}$. We define three kinds of formulae of a two-level language $Fm^{Var}_{\mathcal{L}_2(\mathcal{L}_1)}$ over the set of variables Var:

- non-modal formulae from $Fm^{Var}_{\mathcal{L}_1}$,
- atomic modal formulae of the form $\Box\varphi$, for $\varphi \in Fm^{Var}_{\mathcal{L}_1}$,
- modal formulae resulting from atomic ones by connectives from $\mathcal{L}_2$.

Definition 4. *The* minimal L_2-modal logic over logic L_1 *(denoted by $\mathrm{L}_2(\mathrm{L}_1)$) has formulae $Fm^{Var}_{\mathcal{L}_2(\mathcal{L}_1)}$ and an axiomatic system consisting of*

- *the axioms and rules of L_1 for non-modal formulae,*
- *axioms and rules of L_2 for modal formulae,*
- *and the following congruence rule for each pair of non-modal formulae φ and ψ:*

$$\varphi \leftrightarrow \psi \vdash \Box\varphi \leftrightarrow \Box\psi \qquad \text{(CONGR)}$$

*An n-*ary modal rule *has n non-modal premises and a modal conclusion. An* L_2-modal logic over a logic L_1 *is an extension of $\mathrm{L}_2(\mathrm{L}_1)$ by some modal rules.*

[2] We write '$\varphi \leftrightarrow \psi$' for '$\{\varphi \to \psi, \psi \to \varphi\}$', '$T \vdash S$' for '$T \vdash \varphi$ for each $\varphi \in S$', and '$T \dashv\vdash S$' for '$T \vdash S$ and $S \vdash T$'.

We understand rules as schemata, i.e., for each substitution σ on $Fm^{Var}_{\mathcal{L}_1}$, if $\varphi_1, \ldots, \varphi_n \vdash \Psi$ is a modal rule then $\sigma\varphi_1, \ldots, \sigma\varphi_n \vdash \sigma\Psi$ is also a modal rule. We define the notion of proof in a modal logic in the usual way. One can imagine that the proof consists of three separate parts: proving non-modal formulae, application of the modal rules on proved non-modal formulae, and proving modal formulae.

Definition 5. *A* $\mathbb{K}_1$-based $\mathbb{K}_2$-measured Kripke frame *is a system* $\mathbf{F} = \langle W, (\boldsymbol{A}_w)_{w\in W}, \boldsymbol{B}, \mu\rangle$ *where* W *is a set (of possible worlds),* $\boldsymbol{A}_w \in \mathbb{K}_1$ *for each* $w \in W$, $\boldsymbol{B} \in \mathbb{K}_2$ *and* μ *is a* partial *mapping* $\mu\colon \prod_{w\in W} A_w \to B$.

Note the difference from the 'traditional' approach: in order to prove the completeness theorems in the full generality we cannot assume that all $\boldsymbol{A}_w$s are the same; we call such frames *uniform* and we will see later in which cases we can restrict ourselves to such frames.

Definition 6. *A* Kripke model $\mathbf{M}$ *over* $\mathbb{K}_1$*-based* $\mathbb{K}_2$*-measured Kripke frame* $\mathbf{F} = \langle W, (\boldsymbol{A}_w)_{w\in W}, \boldsymbol{B}, \mu\rangle$ *is a tuple* $\mathbf{M} = \langle \mathbf{F}, (e_w)_{w\in W}\rangle$ *where:*

- $e_w\colon Fm^{Var}_{\mathcal{L}_1} \to A_w$ *is an* $\boldsymbol{A}_w$*-evaluation,*
- *for each non-modal formula* φ*, the element* $\varphi_{\mathbf{M}} \in \prod_{w\in W} A_w$ *defined as* $\varphi_{\mathbf{M}}(w) = e_w(\varphi)$ *belongs to the domain of* μ*.*

The truth value of atomic modal formulae is defined (uniformly for all worlds) as:

$$||\Box\varphi||_{\mathbf{M}} = \mu(\varphi_{\mathbf{M}});$$

and the truth value of non-atomic modal formulae is (uniformly) computed by using operations from $\boldsymbol{B}$*.*

We say that $\mathbf{M}$ *is a* satisfies *the (non-)modal formula* Ψ *(*ψ *resp.) whenever* $||\Psi||_{\mathbf{M}} \in F_{\boldsymbol{B}}$ *(or* $e_w[\psi] \in F_{\boldsymbol{A}_w}$ *for each* $w \in W$ *respectively).*

Finally we say that $\mathbf{F}$ *is a frame for an* L_2*-modal logic over a logic* L_1 *if all its additional modal rules are valid in all Kripke models over* $\mathbf{F}$*, i.e. the conclusion of a modal rule is satisfied in all models of over* $\mathbf{F}$ *which satisfy all its premises.*

Next we state the main theorem, the completeness of an L_2-modal logic over a logic L_1. We will see that the form/strength of the completeness we obtain depends on the form/strength of the completeness of the logics L_2 and L_1. The proof has two main ingredients: Hájek's idea from [11] of a translation of formulae and proofs from an L_2-modal logic over a logic L_1 into the logic L_2 and the authors' characterization of completeness properties from [3].

Theorem 7. *Let* L *be an* L_2*-modal logic over a logic* L_1 *such that* L_i *has* $\mathrm{S}\mathbb{K}_i\mathrm{C}$*. Then the following are equivalent for each non-modal theory* T*, modal theory* Γ*, and a modal formula* Φ*:*

- $\Gamma, T \vdash_{\mathrm{L}} \Phi$
- *for each* $\mathbb{K}_1$*-based* $\mathbb{K}_2$*-measured Kripke frame* $\mathbf{F}$ *for* L *and each Kripke model* $\mathbf{M}$ *over* $\mathbf{F}$ *holds that* $\mathbf{M}$ *satisfies* Φ *whenever it satisfies all formulae from* Γ *and* T*.*

The same equivalence holds if L_2 *has* $\mathrm{FS}\mathbb{K}_2\mathrm{C}$ *only but at the price of restricting to finite* Γ *and* T *and additional assumptions that* $\mathbb{L}_1$ *is a locally finite and* L *has only finitely many additional modal rules.*

Note that any L_2-modal logic over a logic L_1 enjoys completeness w.r.t. its $(\mathbb{L}_1)_{\mathrm{RFSI}}$-based $(\mathbb{L}_2)_{\mathrm{RFSI}}$-measured Kripke frames and if L_1 enjoys completeness w.r.t. a single algebra, then we can restrict ourselves to uniform frames/models.

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