ON THE ALGEBRAIZATION OF NON-FINITARY LOGICS

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In this talk, we exhibit a non-finitary sentential logic that is algebraized by a quasivariety—in fact by a finitely based variety of finite type. The algebraization process requires infinitely many defining equations. The existence of such a logic settles questions posed by J. Czelakowski [4] and by B. Herrmann [7]. This work has been published in [9].

For present purposes, a *deductive system* (or *logic*) is a substitution-invariant consequence relation on formulas in an algebraic signature. It is *sentential* if the formulas are just *terms* in the algebraist's sense (as opposed to equations, inequations, sequents, etc.) It is *finitary* if every consequence of a set Γ of formulas is already a consequence of a finite subset of Γ. Its *theories* are the sets of formulas closed under taking consequences, and they naturally form a complete lattice. For any class K of similar algebras, the *equational consequence relation* $\models_{\mathsf{K}}$ of K is the (non-sentential) logic in which $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$ means: for every homomorphism h from an absolutely free algebra into an algebra from K, if $h(\mu) = h(\nu)$ for all $\mu \approx \nu \in \Sigma$, then $h(\alpha) = h(\beta)$.

The notion of an algebraizable logic is due to Blok and Pigozzi [2]. Their original definition instantiates a more modern and general phenomenon of *equivalence* for arbitrary deductive systems (cf. [1, 5]). Two such systems are said to be *equivalent* if their respective lattices of theories are isomorphic under a map that also preserves the inverse image operators of all possible *substitutions* (of formulas for variables). Although it is not obvious, this amounts to the existence of a well-behaved pair of syntactic *translations* between the systems. A logic $\vdash$ is *algebraizable* if it is equivalent to $\models_{\mathsf{K}}$ for some class K of similar algebras—in which case K is said to *algebraize* $\vdash$. The translation from $\vdash$ to $\models_{\mathsf{K}}$ produces a set of 'defining equations,' while the reverse translation involves a set of 'equivalence formulas.' More exactly:

Theorem 1. (cf. [2]) *A (sentential) logic $\vdash$ and an equational consequence relation $\models_{\mathsf{K}}$ are equivalent iff there exist a family of unary equations $\delta_i(x) \approx \varepsilon_i(x)$, $i \in I$, and a family of binary terms $\Delta_j(x, y)$, $j \in J$, such that for any set of terms $\Gamma \cup \{\alpha\}$, we have*

$$\Gamma \vdash \alpha \quad \textit{iff} \quad \{\delta_i(\gamma) \approx \varepsilon_i(\gamma) : \gamma \in \Gamma,\, i \in I\} \models_{\mathsf{K}} \{\delta_i(\alpha) \approx \varepsilon_i(\alpha) : i \in I\},$$
$$\{\delta_i(\Delta_j(x, y)) \approx \varepsilon_i(\Delta_j(x, y)) : i \in I,\, j \in J\} \;\;=\!\mid\models_{\mathsf{K}}\;\; x \approx y.$$

In this case, for any set of equations $\Sigma \cup \{\alpha \approx \beta\}$, we have

$$\Sigma \models_{\mathsf{K}} \alpha \approx \beta \quad \text{iff} \quad \{\Delta_j(\mu, \nu) : \mu \approx \nu \in \Sigma,\, j \in J\} \vdash \{\Delta_j(\alpha, \beta) : j \in J\},$$
$$\{\Delta_j(\delta_i(x), \varepsilon_i(x)) : i \in I,\, j \in J\} \dashv\vdash x.$$

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Moreover, the set of *defining equations* $\{\delta_i(x) \approx \varepsilon_i(x) : i \in I\}$ and the set of *equivalence formulas* $\{\Delta_j(x,y) : j \in J\}$ are unique up to interderivability in $\models_{\mathsf{K}}$ and in $\vdash$, respectively (cf. [2, Thm. 2.15]).

An equational consequence relation $\models_{\mathsf{K}}$ is finitary iff it is equal to $\models_{\mathsf{M}}$ for some *quasivariety* M (which is then unique). A logic is said to be *elementarily algebraizable* if it is algebraized by a quasivariety.

Proposition 2. *Suppose the equivalent conditions of Theorem* 1 *hold for a logic* $\vdash$ *and a class of algebras* K.

(i) *When* $\vdash$ *is finitary, then* I *can be chosen finite. Dually*:
(ii) *When* $\models_{\mathsf{K}}$ *is finitary (e.g., when* K *is a quasivariety) then* J *can be chosen finite.*
(iii) *If* $\models_{\mathsf{K}}$ *is finitary and* I *can be chosen finite, then* $\vdash$ *is finitary.*

These facts can be extracted for instance from [1]. From (i) and (iii) we obtain:

Fact 3. *An elementarily algebraizable logic is finitary iff its algebraization is witnessed by some* finite *set of defining equations.*

B. Herrmann [6, 7, 8] exhibited a sentential logic $\vdash$ that is equivalent to an equational consequence relation $\models_{\mathsf{K}}$, yet $\vdash$ is finitary while $\models_{\mathsf{K}}$ is non-finitary. Thus, $\vdash$ is algebraizable but not elementarily algebraizable. Further examples of this kind appeared later. On the other hand, J. Czelakowski [4, Note 4.5.2(4), p. 314] asks whether there are algebraizable sentential systems whose finitarity properties are *dual* to those of Herrmann's $\vdash$. In other words, can a non-finitary sentential system be equivalent to the (finitary) equational consequence relation of a quasivariety?

We provide an example of such a system. Necessarily, the algebraization process requires infinitely many defining equations. This shows that, even when we translate into the most restrictive formalism (that of sentential systems), we may still lose finitarity in an equivalence.

Let V be the variety of all algebras $\boldsymbol{A} = \langle A; \leftrightarrow, \Diamond, \pi_1, \pi_2\rangle$ of type $\langle 2,1,1,1\rangle$ satisfying

$$\pi_1(x \leftrightarrow y) \approx x$$
$$\pi_2(x \leftrightarrow y) \approx y$$
$$\Diamond(x \leftrightarrow y) \approx x \leftrightarrow y.$$

For all terms α in the language of V, we define

$$\Diamond^0(\alpha) = \alpha\,; \quad \Diamond^{i+1}(\alpha) = \Diamond(\Diamond^i(\alpha))\,;$$
$$\delta_i(\alpha) = \pi_1(\Diamond^i(\alpha))\,; \quad \varepsilon_i(\alpha) = \pi_2(\Diamond^i(\alpha))$$

for all $i \in \omega$.

Definition 4. Let $\vdash^{\mathsf{V}}$ be the relation from sets of terms to single terms defined by

(1) $\Gamma \vdash^{\mathsf{V}} \alpha$ iff $\{\delta_i(\gamma) \approx \varepsilon_i(\gamma) : \gamma \in \Gamma,\, i \in \omega\} \models_{\mathsf{V}} \{\delta_i(\alpha) \approx \varepsilon_i(\alpha) : i \in \omega\}$.

Then $\vdash^{\mathsf{V}}$ is a sentential logic and we can easily show:

Fact 5. $\vdash^{\mathsf{V}}$ *is elementarily algebraizable, being equivalent to* $\models_{\mathsf{V}}$*. The defining equations are* $\delta_i(x) \approx \varepsilon_i(x)$*,* $i \in \omega$*, and* $\{x \leftrightarrow y\}$ *is the set of equivalence formulas.*

More significantly:

Fact 6. $\vdash^{\mathsf{V}}$ *is not finitary.*

To establish Fact 6, we need to construct an infinite matrix $\mathcal{A} = \langle \boldsymbol{A}, D\rangle$, where $\boldsymbol{A} \in \mathsf{V}$. This will be done in the talk. We choose $D \subseteq A$ so that $\vdash^{\mathsf{V}} \subseteq \vdash^{\mathcal{A}}$. Here, $\Gamma \vdash^{\mathcal{A}} \alpha$ means that, for every homomorphism h from an absolutely free algebra to $\boldsymbol{A}$, if $h[\Gamma] \subseteq D$ then $h(\alpha) \in D$.

It follows from (1) and Fact 5 that

$$\{\delta_i(x) \leftrightarrow \varepsilon_i(x) : i \in \omega\} \vdash^{\mathsf{V}} x,$$

so $\{\delta_i(x) \leftrightarrow \varepsilon_i(x) : i \in \omega\} \vdash^{\mathcal{A}} x$. Fact 6 may therefore be proved by showing that for every finite subset R of ω,

$$\{\delta_r(x) \leftrightarrow \varepsilon_r(x) : r \in R\} \nvdash^{\mathcal{A}} x.$$

Facts 5 and 6 settle the open questions mentioned above.

Although we defined $\vdash^{\mathsf{V}}$ semantically, a deductive base for an elementarily algebraizable system can always be constructed from an axiomatization of its equivalent quasivariety, provided the defining equations and equivalence formulas are known, cf. [3, Thm. 8.0.9]. In the present example we obtain:

Fact 7. $\vdash^{\mathsf{V}}$ *is the smallest logic containing the finitary postulates*

$$\begin{aligned}
&\vdash x \leftrightarrow x\\
&\vdash x \leftrightarrow \pi_1(x \leftrightarrow y)\\
&\vdash y \leftrightarrow \pi_2(x \leftrightarrow y)\\
&\vdash (x \leftrightarrow y) \leftrightarrow \Diamond(x \leftrightarrow y)\\
&x,\, x \leftrightarrow y \vdash y\\
&x \leftrightarrow y,\, z \leftrightarrow w \vdash (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow w)\\
&x \leftrightarrow y \vdash \alpha(x) \leftrightarrow \alpha(y), \ \textit{for } \alpha \in \{\pi_1, \pi_2, \Diamond\}\\
&x \vdash \pi_1(\Diamond^i(x)) \leftrightarrow \pi_2(\Diamond^i(x)), \ \textit{for all } i \in \omega
\end{aligned}$$

and the infinitary rule

$$\{\pi_1(\Diamond^i(x)) \leftrightarrow \pi_2(\Diamond^i(x)) : i \in \omega\} \vdash x.$$

In [7, p. 432], Herrmann disclaims knowledge of any algebraizable sentential system whose algebraization requires more than one defining equation. Observe that $\vdash^{\mathsf{V}}$ is an example of this kind, because Facts 3, 5 and 6 yield:

Fact 8. *The equivalence of* $\vdash^{\mathsf{V}}$ *and* $\models_{\mathsf{V}}$ *is not witnessed by any finite set of defining equations.*

It would be interesting to know whether there are also *finitary* systems of the kind that Herrmann sought.

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