An isomorphism criterion for colimits of sequences of finitely presented objects

A joint work with Vincenzo Marra (University of Milano)

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Abstract

We prove a general criterion reducing isomorphisms between colimits of sequences of finitely presented objects in a category to the notion of a *confluence* between their diagrams.

1. Introduction

Reducing problems regarding infinite objects to finite calculations is a central theme in applications of mathematics to computer science. Many successful attempts hinge on the fact that properties of the object one is interested in are of finite nature (mentioning only a finite number of elements and properties), so it might be sufficient to check them on a *finitely generated* object. In other less tame cases, it might still be the case that the checking can be broken down in an infinite number of tests on finite structures. The most popular technique of this last sort in mathematical logic goes under the name of *back&forth method* or *Ehrenfeucht-Fraïssé games* (see, e.g. [5]). Roughly, these methods allow to establish a relation (isomorphism, elementary equivalence) between two structures, by exhibiting finite partial correpondences between them.

This note abstracts this method to a categorical level, showing that, whenever two objects can be described as colimits of sequences of "finite" objects, isomorphisms between them are exactly induced by a correspondence between their finite parts, which we call *confluence*. As an immediate corollary of our result one obtains the main characterisation of [3], and, with some additional efforts, the main characterisation of [2].

Let C be a locally small category. Following [4], we say that an object C of C is **finitely presentable** if the covariant hom-functor $\mathsf{hom}\,(C,-)\colon \mathsf{C} \to \mathsf{Set}$ preserves directed colimits; that is, if whenever $D^\infty = \mathsf{colim}_{i\in I}\, D_i$ is the directed colimit of $\{D_i\}_{i\in I}$ in C, it follows that

$$\mathsf{hom}\,(C, \mathsf{colim}_{i\in I}\, D_i) = \mathsf{colim}_{i\in I}\, \mathsf{hom}\,(C, D_i)\,. \tag{*}$$

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Explicitly, this means that for every morphism $f\colon C \to D^\infty$ in C, and every directed diagram $(D_i, d_{ij})_{i,j\in I}$ such that $D = \mathsf{colim}_{i\in I} D_i$, and $d_i\colon D_i \to D^\infty$ are the colimit arrorws, the following two conditions are satisfied. (See Fig. 1.)

There is $g\colon C \to D_i$ such that $f = d_i \circ g$. (*Factorisation*)

If $g'\colon C \to D_i$ is such that $f = d_i \circ g'$, there is $j \geqslant i$ such that $d_{ij} \circ g = d_{ij} \circ g'$. (*Essential uniqueness*)

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[dr, "d_{ij}g = d_{ij}g'"] \arrow[ddr, bend right, "g"'] \arrow[ddr, bend right=20, "g'"] & D^\infty \\
& D_j \arrow[u, bend right, "d_j"'] \\
& D_i \arrow[u, "d_{ij}"] \arrow[uu, bend right=60, "d_i"']
\end{tikzcd}$$

Figure 1: Finite presentability according to Gabriel and Ulmer.

According to (the dual of) [1, Definition 2.11.1], a functor $\mathscr{G}\colon \mathsf{I} \to \mathsf{J}$ is called **cofinal** if for any category A and any functor $\mathscr{F}\colon \mathsf{J} \to \mathsf{A}$ the following conditions are satisfied:

If the colimit $(A^\infty, \{a_j\}_{j\in\mathsf{J}})$ of $\mathscr{F}$ exists, then $(A^\infty, \{a_{\mathscr{G}(i)}\}_{i\in\mathsf{I}})$ is the colimit of $\mathscr{F} \circ \mathscr{G}$;

If the colimit $(A^\infty, \{a_i\}_{i\in\mathsf{I}})$ of $\mathscr{F} \circ \mathscr{G}$ exists, then the colimit of $\mathscr{F}$ exists as well.

The idea being that a functor $\mathscr{G}\colon \mathsf{A} \to \mathsf{B}$ is cofinal if we can restrict diagrams on B to diagrams on A along $\mathscr{G}$ without changing their colimit.

$$\begin{tikzcd}
I \arrow[dr, "g"] & \\
& I'' \\
I' \arrow[ur, "g'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathscr{G}(I) \arrow[dr, "\mathscr{G}(g)"] & \\
J \arrow[ur, "f"] \arrow[dr, "f'"'] & & \mathscr{G}(I'') \\
& \mathscr{G}(I') \arrow[ur, "\mathscr{G}(g')"'] &
\end{tikzcd}$$

Figure 2: Condition (*filtering*) in the Proposition 1.1.

Proposition 1.1 ([1, Proposition 2.11.2]). *A functor $\mathscr{G}\colon \mathsf{I} \to \mathsf{J}$ is **cofinal** if:*

$$\forall J \in \mathsf{J} \quad \exists I \in \mathsf{I} \quad \exists f\colon \mathsf{J} \to G(I) \qquad (\textit{cofinality})$$

$$\forall I, I' \in \mathsf{I} \quad \forall J \in \mathsf{J} \quad \forall f\colon J \to \mathscr{G}(I),\ f'\colon J' \to \mathscr{G}(I')$$
$$\exists I'' \quad \exists g\colon I \to I'',\ g'\colon I' \to I'' \textit{ such that } \mathscr{G}(g) \circ f = \mathscr{G}(g') \circ f' \qquad (\textit{filtering})$$

See Figure 2.

When the order type of the diagram is ω, as will always be the case in this note, the condition (*filtering*) is automatically satisfied.

2. The isomorphism criterion

Definition 2.1. Let $D_A := (A_i, a_{ij})_{i,j \in I}$ and $D_B := (B_k, b_{kl})_{k,l \in K}$ be two diagrams in C. We say that the families of C-arrows $\hat{f} = \{f_i \colon A_i \to B_{k(i)} \mid i \in I, k(i) \in K\}$ and $\hat{g} = \{g_k \colon B_k \to A_{i(k)} \mid i(k) \in I, k \in K\}$ form a **confluence** if

1. The domains of the arrows in $\hat{f}$ are cofinal in D_A, i.e. for each $i \in I$ there exists $i_0 \in I$ such that $i \leqslant i_0$ and there exists a C-map $f_{i_0} \in \hat{f}$;
2. The domains of the arrows in $\hat{g}$ are cofinal in D_B, i.e. for each $k \in K$ there exists $k_0 \in K$ such that $k \leqslant k_0$ and there exists a C-map $g_{k_0} \in \hat{g}$;
3. For any arrow $f_i \in \hat{f}$ such that $f_i \colon A_i \to B_k$ there is an arrow $g_k \in \hat{g}$ such that $g_k \colon B_k \to A_j$ with $j \geqslant i$. Vice versa, for any arrow $g_k \in \hat{g}$ such that $g_k \colon B_k \to A_j$ there is an arrow $f_j \in \hat{f}$ such that $f_j \colon A_j \to B_l$ with $l \geqslant k$.
4. All the above triangles commute, i.e.
   (a) for all $i \in I$, if $f_i \in \hat{f}$, $\mathsf{cod}\, f_i = B_k$, $g_k \in \hat{g}$, and $\mathsf{cod}\, g_k = A_j$, then
   $$g_k \circ f_i = a_{ij}; \tag{1}$$
   (b) for all $k \in K$, if $g_k \in \hat{g}$, $\mathsf{cod}\, g_k = A_i$, $f_i \in \hat{f}$, and $\mathsf{cod}\, f_i = B_l$, then
   $$f_i \circ g_k = b_{kl}. \tag{2}$$

See Figure 3.

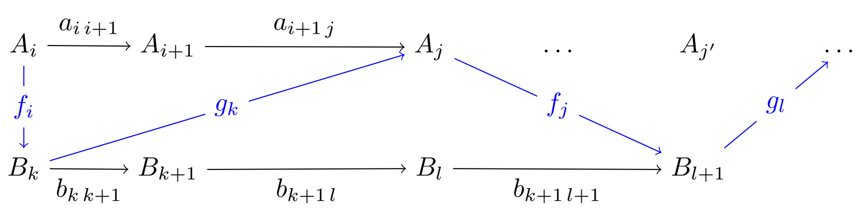

Figure 3: A sketch of a confluence between the diagrams D_A and D_B. (See Definition 2.1.)

The main result of this note is the following.

Theorem 2.2. *Let $D_A := (A_i, a_{ij})_{i,j \in I}$ and $D_B := (B_k, b_{kl})_{k,l \in K}$ be two sequences of finitely presented objects in C. Suppose $(A, \{a_i\}_{i \in I})$ and $(B, \{b_k\}_{k \in K})$ be their respective colimits. The objects A and B are isomorphic if, and only if, there exists a confluence between D_A and D_B.*

Easy counterexamples showing that the hypothesis of finite presentability cannot be dropped can be given.

Theorem 2.2 is particularly interesting in the case of varieties (or quasi-varieties) of algebras, as often are the algebraic semantics of logical systems. Indeed, in these cases every ω-generated algebra is the colimit of finitely presented algebras (in the sense of universal algebra, which happens to coincide in algebraic categories with the one by Gabriel-Ulmer). This is the principal reason we chose to present the result speaking of colimits, rather than presenting its dual statement about limits.

References

[1] Francis Borceaux. Handbook of categorical algebra 1. Basic category theory. *Encyclopedia of Mathematics and its Applications*, 50, 1994.

[2] Ola Bratteli. Inductive limits of finite dimensional C*-algebras. *Transactions of the American Mathematical Society*, 171:195–234, 1972.

[3] Manuela Busaniche, Leonardo Cabrer, and Daniele Mundici. Confluence and combinatorics in finitely generated unital lattice-ordered abelian groups. *Forum Mathematicum*, 24(2):253–271, 2012.

[4] Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien.* Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin, 1971.

[5] Bruno Poizat. *A course in Model Theory: an introduction to contemporary mathematical logic*. Springer, 2000.