Short Communication: Complete MV–algebra Valued Pavelka Logic

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Key words Mathematical fuzzy logic, MV–algebra.

Extended abstract

When Pavelka in 1970's introduced his fuzzy sentential logic [4], he had in mind to provide solid grounds to fuzzy logic, understood as a particular many–valued logic. This meant a generalization of classical logic in such a way that axioms, theories, theorems and tautologies can be not only fully true but also true to a degree and, therefore, giving rise to such concepts as fuzzy theories, fuzzy set of axioms, many–valued rules of inference, provability degree, truth degree, fuzzy consequence operation etc. Pavelka was inspired by a paper [2], where Goguen argued that the algebraic structure of fuzzy logic should be a (complete) residuated lattice in the same sense than Boolean algebras are the algebraic counterpart of Boolean logic. Pavelka defined his generalized concepts in a complete residuated lattice L and set a general research problem:

Do there exist a fuzzy set of logical axioms and a set of fuzzy rules of inference such that for any fuzzy theory $\mathcal{T}$ and any formula α the degree to which α follows from $\mathcal{T}$ equals exactly the degree to which α is provable from $\mathcal{T}$?

Pavelka limited to address the issue in the case L is a finite chain or the unit real interval $[0, 1]$ and proved that this question has an affirmative answer if, and only if L is equipped with Łukasiewicz structure i.e. the standard MV–algebra on the real unit interval. This result, called *Pavelka style completeness*, was proved by topological methods to cover propositional fuzzy logic. At the end of [4] Pavelka reported that the result also holds for first order predicate calculi, a detailed proof to this fact was given by Novák in [3].

Hájek showed in [1] that Pavelka's logic (both propositional and first order) can be significantly simplified while keeping the Pavelka style completeness results. Indeed, Hájek showed it is enough to extend the language only by a countable number of truth constants, one constant $\mathbf{r}$ for each rational $r \in [0, 1]$, and by adding to Łukasiewicz logic what are called book–keeping axioms. Some years earlier [5] Pavelka's ideas had been generalized to another direction by introducing a Pavelka style fuzzy sentential logic with truth values on an injective MV–algebra, thus generalizing $[0, 1]$–valued logic. Recall that injective MV–algebras L are complete (i.e. all supremas and infimas exist in L) and divisible (in a sense that every element $a \in L$ has n–divisors $b \in L$ for any natural n; $nb = a$ and $a^* \oplus (n - 1)b = b^*$.

In this paper we show that divisibility is redundant; Pavelka style completeness of a fuzzy sentential logic holds in a complete residuated lattice framework if, and only if the set of truth values is a complete MV–algebra. This is done by adding to Pavelka's original approach a new fuzzy rule of inference

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$$\frac{\alpha, \beta}{\alpha \ \texttt{or} \ \beta} \quad , \quad \frac{a, b}{a \oplus b}$$

called *Rule of Bold Disjunction*, Hájek's book–keeping axioms

$$\mathbf{a} \ \texttt{imp} \ \mathbf{b}, \quad a \rightarrow b$$

for all truth constants **a**, **b**, where $a \rightarrow b$ refers to the fact that these axioms are $a \rightarrow b$-tautologies, and a new axiom schema

$$[\alpha \ \texttt{or} \ (\texttt{not}\alpha \ \texttt{and} \ \beta)] \ \texttt{imp} \ [(\alpha \ \texttt{imp} \ \beta) \ \texttt{imp} \ \beta], \quad \mathbf{1}$$

where α, β are well formed formulas and **1** refers to the fact that this axiom is a **1**-tautology (for details see [5]). Then it is proved that the corresponding Lindenbaum algebra is isomorphic to the original set of truth values. Thus, if a formula α is provable at a degree a, then $|\alpha| = |\mathbf{a}|$ and there is a valuation v such that $v(\alpha) = a$. Consequently,

If a fuzzy theory $\mathcal{T}$ is consistent, then $\mathcal{C}^{sem}(\mathcal{T})(\alpha) = \mathcal{C}^{syn}(\mathcal{T})(\alpha)$ for any wff α.

Thus, we may talk about theorems of a degree $\mathcal{C}^{syn}(\mathcal{T})(\alpha) = a$ and tautologies of a degree $\mathcal{C}^{sem}(\mathcal{T})(\alpha) = b$ for $a, b \in L$, and these two values coincide for any formula α. In summary

Theorem 1 *A necessary and sufficient condition for Pavelka style completeness of fuzzy propositional logic is that the set of truth values is a complete MV–algebra.*

Open issues and future work

It is obvious that the above mentioned new axiom schema is provable from the original axioms and hence redundant. We also expect that set of rules of inference to obtain completeness can be reduced. An essential part of Pavelka's logic is that the propositional logic contains truth constants corresponding to each truth values; their counterpart in classical logic are the symbols $\perp$ and $\top$. Hajek's result allows us to have only countable many truth constant if the set of truth values L is the standard MV–algebra on the real unit interval $[0, 1]$, however, in the case L is an uncountable complete MV–algebra we might need uncountably many truth constants. These topics and their detailed proofs are left for a future work.

References

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