Discrete dualities for n-potent MTL and BL-algebras

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This talk is a contribution towards the project of developing discrete representability for the algebraic semantics of various non-classical logics. Discrete duality is a type of duality where a class of abstract relational systems is a dual counterpart to a class of algebras. These relational systems are referred to as 'frames' following the terminology of non-classical logics. There is no topology involved in the construction of these frames, so they may be thought of as having a discrete topology and hence the term: discrete duality. Having a discrete duality for an algebraic semantics for a logic often provides a Kripke-style semantics for the logic. In many cases it can also be used to develop filtration and tableau techniques for the logic. Another typical consequence of such a discrete duality in the case of lattice-ordered algebras is that we obtain a method of completing the algebras, i.e., an embedding of algebras into ones that are complete in the lattice sense.

Establishing discrete duality involves the following steps. Given a class of algebras *Alg* we define a class of frames *Fr*. Next, for any algebra A from *Alg* we define its 'canonical frame' $\mathcal{X}(A) \in Fr$ and for each frame X in *Fr* we define its 'complex algebra' $\mathcal{C}(X) \in Alg$. A duality between *Alg* and *Fr* holds provided that the following facts are provable:

- Every algebra $A \in Alg$ is embeddable into the complex algebra of its canonical frame.
- Every frame $X \in Fr$ is embeddable into the canonical frame of its complex algebra.

The logics of interest in this talk are *monoidal t-norm logic*, or MTL for short, (as defined in [EG]) and Hajek's *Basic Logic*, or BL for short (as defined in [H]). The algebraic semantics for these logics are, respectively, 'MTL-algebras' and 'BL-algebras', which are defined as follows.

- An MTL-*algebra* is an algebra $\langle A, \circ, \to, \wedge, \vee, 0, 1\rangle$ that is a bounded lattice with a commutative monoid operation $\circ$ with identity 1 that is
  residuated: $(\forall a, b, c \in A)(a \circ b \leq c \Leftrightarrow a \leq b \to c)$ and
  prelinear: $(\forall a, b \in A)((a \to b) \vee (b \to a) = 1)$.

- A *BL-algebra* is an MTL-algebra $\langle A, \circ, \rightarrow, \wedge, \vee, 0, 1\rangle$ that is *divisible:* $(\forall a, b \in A)(a \leq b \Rightarrow (\exists c \in A)(c \circ b = a))$.

Discrete dualities are developed for MTL-algebras in [ORe] building on the work of [CC]. The underlying order structure of MTL-algebras is a distributive lattice and hence the frames associated with these algebras are based on posets as is well known in the duality for distributive lattices [P]. To capture the properties of the operations of a residuated lattice an additional relation is required satisfying the appropriate conditions and hence the MTL-frames are structures of the form $\langle X, \leq, R\rangle$ where R is a ternary relation on X. The canonical frame of an MTL-algebra is the set of prime filters (in the lattice sense) together with the inclusion relation and a canonical form of R determined by the monoid product. The complex algebra of an MTL-frame is the family of upward closed subsets of X with the union and intersection of sets as the lattice operations. The operations of product and residuation are defined in terms of the relation R in such a way that they satisfy all the MTL axioms. The two discrete representation theorems for the MTL-algebras and MTL-frames hold.

For each positive integer n we define the class of *n-potent* MTL-algebras as the class of MTL-algebras satisfying: $x^{n+1} = x^n$, where x^n denotes the $\circ$-product, i.e., $x \circ x \circ \ldots \circ x$, of n x's. The n-potent classes of algebras are generalisations of the 1-potent case, in which $\circ$ corresponds to $\wedge$. In related classes of algebras in which 1 is not always the greatest element of the algebra, the one-sided 1-potence identity $x \leq x \circ x$ corresponds to the structural rule of contraction in the logic. Thus n-potence is a form of n-contraction (see, for example, [HNP]). The n-potent classes of algebras often have useful properties in terms of computability (see, for example, [V]).

The question we address here is: what are the additional frame conditions needed to characterize the frames of n-potent MTL-algebras? We give some positive results in this direction. Thereafter, we consider the n-potent BL-algebras, which are defined analogously. In the case of BL-algebras, there is no discrete duality; in fact, such a duality would provide a completion method for BL-algebras, contradicting a result from [KL]. Moreover, in [BC] it is shown that the only varieties of BL-algebras admitting completions are the n-potent ones. This observation, in part, motivated the current research direction. We present here a discrete duality for the variety of 2-potent BL-algebras (that is, satisfying $x^3 = x^2$) and indicate possible directions for obtaining discrete dualities for other n-potent classes.

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