Abstract Algebraic Logic – 3rd lesson

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Completeness theorem for classical logic

- Suppose that $T \in \mathrm{Th}(\mathrm{CPC})$ and $\varphi \notin T$ ($T \nvdash_{\mathrm{CPC}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta\rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\boldsymbol{Fm}_{\mathcal{L}}$ compatible with T: if $\alpha \in T$ and $\langle \alpha, \beta\rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum-Tarski algebra: $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T)\rangle} \varphi$. 2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \mathrm{Th}(\mathrm{CPC})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T') \cong \mathbf{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \mathbf{2}, \{1\}\rangle} \varphi$. 3rd completeness theorem

Recall from Lesson 1:

A congruence θ is logical in the matrix $\mathbf{A} = \langle A, F\rangle$ if for each $x, y \in A$: $x \in F$ and $\langle x, y\rangle \in \theta$ imply $y \in F$

Let $\mathbf{A} = \langle A, F\rangle$ be an L-matrix for a weakly implicative logic L. Then:

1. the Leibniz congruence $\Omega_{\mathbf{A}}(F)$ of $\mathbf{A}$ is defined as

$$\langle a, b\rangle \in \Omega_{\mathbf{A}}(F) \qquad \text{iff} \qquad \{a \to^{\mathbf{A}} b, b \to^{\mathbf{A}} a\} \subseteq F.$$

2. $\Omega_{\mathbf{A}}(F)$ is the largest logical congruence of $\mathbf{A}$.
3. $\langle a, b\rangle \in \Omega_{\mathbf{A}}(F)$ if, and only if, for each formula χ and each $\mathbf{A}$-evaluation e:

$$e[p{\to}a](\chi) \in F \qquad \text{iff} \qquad e[p{\to}b](\chi) \in F.$$

Leibniz congruence in general

Definition 3.1

Let L be an arbitrary logic and $\mathbf{A} = \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$. We define the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$ on $\mathbf{A}$ as: $\langle a, b \rangle \in \Omega_{\boldsymbol{A}}(F)$ iff for each formula χ and each $\boldsymbol{A}$-evaluation e it is the case that

$$e[p \to a](\chi) \in F \quad \text{iff} \quad e[p \to b](\chi) \in F$$

Theorem 3.2

$\Omega_{\boldsymbol{A}}(F)$ is the largest logical congruence of $\boldsymbol{A}$.

Proof.

$\Omega_{\boldsymbol{A}}(F)$ is obviously a congruence. Logicity: $\langle a, b \rangle \in \theta$ consider $\chi = p$ then we get $a \in F$ iff $b \in F$.

Leibniz congruence in general

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$$e[p{\to}a](\chi) \in F \quad \text{iff} \quad e[p{\to}b](\chi) \in F$$

Theorem 3.2

$\Omega_{\boldsymbol{A}}(F)$ is the largest logical congruence of $\boldsymbol{A}$.

Proof.

Take a logical congruence θ st. $\langle a, b\rangle \in \theta$, a formula χ, and an $\mathbf{A}$-evaluation e. Clearly $\langle e[p{\to}a](\chi), e[p{\to}b](\chi)\rangle \in \theta$; logicity of θ yields $e[p{\to}a](\chi) \in F$ iff $e[p{\to}b](\chi) \in F$, i.e. $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$. □

Algebraic counterpart

Definition 3.3

An L-matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ is reduced, $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ in symbols, if $\Omega_{\boldsymbol{A}}(F)$ is the identity relation $\mathrm{Id}_{\boldsymbol{A}}$.

An algebra $\boldsymbol{A}$ is an L-algebra, $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ in symbols, if there is a set $F \subseteq A$ s.t. $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$.

Recall that $\Omega_{\boldsymbol{A}}(A) = A^2$ and $\mathcal{F}i_{\mathrm{Inc}}(\boldsymbol{A}) = \{A\}$ thus:

$$\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{Inc}) \qquad \text{iff} \qquad A \text{ is a singleton}$$

Note that $\Omega_{\boldsymbol{A}}(\emptyset) = A^2$ and $\mathcal{F}i_{\mathrm{AInc}}(\boldsymbol{A}) = \{\emptyset, A\}$ thus also:

$$\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{AInc}) \qquad \text{iff} \qquad A \text{ is a singleton}$$

$$\text{i.e., } \mathbf{ALG}^*(\mathrm{AInc}) = \mathbf{ALG}^*(\mathrm{Inc}).$$

Properties of the Leibniz operator

Proposition 3.4

Given a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *and a set* $G \subseteq B$, *we have:* $h^{-1}[\Omega_{\boldsymbol{B}}(G)] \subseteq \Omega_{\boldsymbol{A}}(h^{-1}[G])$.
If h *is surjective, then* $h^{-1}[\Omega_{\boldsymbol{B}}(G)] = \Omega_{\boldsymbol{A}}(h^{-1}[G])$.

Proposition 3.5

Let $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ *be a strict and surjective matrix homomorphism. Then:*

- $F = h^{-1}[G]$
- $G = h[F]$
- $F = h^{-1}[h[F]]$
- $\Omega_{\boldsymbol{A}}(F) = h^{-1}[\Omega_{\boldsymbol{B}}(G)]$
- $\Omega_{\boldsymbol{B}}(G) = h[\Omega_{\boldsymbol{A}}(F)]$

Properties of the Leibniz operator

Lemma 3.6

Given a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *and a set* $F \subseteq A$*, we have:*
$F = h^{-1}[h[F]]$ *iff* $Ker(h) \subseteq \Omega_{\boldsymbol{A}}(F)$*.*

Proposition 3.7

1. *Given a homomorphism* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *and a set* $F \subseteq A$*, we have:*
   h *is strict between* $\langle \boldsymbol{A}, F\rangle$ *and* $\langle \boldsymbol{B}, h[F]\rangle$ *iff* $Ker(h) \subseteq \Omega_{\boldsymbol{A}}(F)$*.*
2. *Given* $\theta \in Co(\boldsymbol{A})$*, the projection* $\pi\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{A}/\theta, F/\theta\rangle$ *is strict iff* $\theta \subseteq \Omega_{\boldsymbol{A}}(F)$*.*

Properties of the Leibniz operator

Corollary 3.8

If $\theta \subseteq \Omega_{\boldsymbol{A}}(F)$*, then* $\Omega_{\boldsymbol{A}/\theta}(F/\theta) = \Omega_{\boldsymbol{A}}(F)/\theta$*.*

Corollary 3.9

Given an L*-matrix* $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$*, we define its reduction as* $\mathbf{A}^* = \langle \boldsymbol{A}/\Omega_{\boldsymbol{A}}(F), F/\Omega_{\boldsymbol{A}}(F) \rangle$*.*
Then: $\mathbf{A}^* \in \mathbf{MOD}^*(\mathrm{L})$*.*

More things that work in general . . .

Theorem 3.10

Given a logic L *and* $\mathbf{A} = \langle A, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$, *we have:*

1. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{R(F)SI}}$ *iff* F *is (finitely)* $\cap$*-irreducible in* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.
2. $\Omega_{\boldsymbol{A}}[\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})] = \boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$.

Furthermore, for finitary logics: $\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P}_{\mathrm{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}})$.

Theorem 3.11

Let L *be a logic and* $\Gamma \cup \{\varphi\}$ *a set of formulae. TFAE:*

1. $\Gamma \vdash_{\mathrm{L}} \varphi$
2. $\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi$.

Furthermore if L *is finitary we can add:*

3. $\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}} \varphi$.

Not everything is as nice: monotonicity of Ω_A

Proposition 3.12

Let $\mathrm{L} \neq \mathrm{AInc}$ *be a logic without theorems, then* $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}$ *is not monotone.*

Proof.

Clearly: $\emptyset \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}})$ and $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}(\emptyset) = {Fm_{\mathcal{L}}}^2$. As $\mathrm{L} \neq \mathrm{AInc}$ then $\psi \nvdash_{\mathrm{L}} \varphi$ for some φ and ψ, i.e., there is a theory T st. $\psi \in T$ and $\varphi \notin T$. Then clearly ${Fm_{\mathcal{L}}}^2$ is not a logical congruence on $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle$ and so $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}(T) \neq {Fm_{\mathcal{L}}}^2$. □

In lesson 4 we will see less trivial examples . . .

Recall that in weakly implicative logics, the monotonicity of Leibniz operator was a trivial consequence of its definability via a pair of formulae.

From implications to generalized equivalences

Let $\mathbf{A} = \langle A, F \rangle$ be an L-matrix for a weakly implicative logic L.
We define the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$:

$$\langle a, b \rangle \in \Omega_{\boldsymbol{A}}(F) \quad \text{iff} \quad \{a \to^{\boldsymbol{A}} b, b \to^{\boldsymbol{A}} a\} \subseteq F$$

From implications to generalized equivalences

Let $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ be an arbitrary matrix.
We define a relation $\Omega^{E}_{\boldsymbol{A}}(F)$:

$$\langle a, b \rangle \in \Omega^{E}_{\boldsymbol{A}}(F) \quad \text{iff} \quad E^{\boldsymbol{A}}(a, b) \subseteq F$$

for some set E of formulae in two variables

From implications to generalized equivalences

Let $\mathbf{A} = \langle A, F\rangle$ be an arbitrary matrix.
We define a relation $\Omega^E_{\boldsymbol{A}}(F)$:

$$\langle a, b\rangle \in \Omega^E_{\boldsymbol{A}}(F) \quad \text{iff} \quad \{t^{\boldsymbol{A}}(a, b, \vec{x}) \mid t(p, q, \vec{v}) \in E \text{ and } \vec{x} \in A^{|\vec{v}|}\} \subseteq F$$

for some set E of formulae

Conventions: we write $\Leftrightarrow$ instead of E and we set:

$$a \Leftrightarrow^{\boldsymbol{A}} b = \{t^{\boldsymbol{A}}(a, b, \vec{x}) \mid t(p, q, \vec{v}) \in \Leftrightarrow \text{ and } \vec{x} \in A^{|\vec{v}|}\}$$

When is $\Omega^{\Leftrightarrow}$ the Leibniz congruence?

Theorem 3.13

Let L *be a logic and* $\Leftrightarrow$ *a set of formulae. TFAE:*

1. $\Omega^{\Leftrightarrow}_{\boldsymbol{A}}(F)$ *is the Leibniz congruence of each* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$
2. $\Omega^{\Leftrightarrow}_{\boldsymbol{A}}(F)$ *is the identity for all* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$
3. $\Omega^{\Leftrightarrow}_{\boldsymbol{Fm}_{\mathcal{L}}}(T)$ *is the Leibniz congruence for each theory* T
4. L *satisfies:*

$$\begin{array}{ll}
(\mathrm{R}) & \vdash_{\mathrm{L}} \varphi \Leftrightarrow \varphi \\
(\mathrm{T}) & \varphi \Leftrightarrow \psi, \psi \Leftrightarrow \chi \vdash_{\mathrm{L}} \varphi \Leftrightarrow \chi \\
(\mathrm{MP}) & \varphi, \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \psi \\
(\mathrm{Cng}) & \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \Leftrightarrow c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n) \\
 & \textit{for each } \langle c, n\rangle \in \mathcal{L} \textit{ and each } 0 \leq i < n.
\end{array}$$

Protoalgebraic logics

Theorem 3.14

A logic L *is protoalgebraic if it satisfies any of the following equivalent conditions:*

1. *There is a set* $\Leftrightarrow(p, q, \overrightarrow{r})$ *of formulae st.*

$$\begin{array}{ll} \text{(R)} & \vdash_{\mathrm{L}} \varphi \Leftrightarrow \varphi \\ \text{(MP)} & \varphi, \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \psi \\ \text{(Cng)} & \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots) \Leftrightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots) \\ & \textit{for each } \langle c, n \rangle \in \mathcal{L} \textit{ and each } 0 \leq i < n. \end{array}$$

2. *There exists a set* $\Leftrightarrow(p, q, \overrightarrow{r})$ *of formulae st.* $\Omega^{\Leftrightarrow}_{\boldsymbol{A}}(F)$ *is the Leibniz congruence of each* $\langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$.
3. *For every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$, $\Omega_{\boldsymbol{A}}$ *is monotone on* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.
4. $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}$ *is monotone on* $\mathrm{Th}(\mathrm{L})$.

Where is $\Leftrightarrow$ coming from?

Consider $\sigma(q) = p$ and $\sigma(r) = r$ for $r \neq q$. Define:

$$E_{\mathrm{L}} = \{\varphi(p, q, \vec{v}) \mid \ \vdash_{\mathrm{L}} \varphi(p, p, \vec{v})\} = \{\varphi \mid \ \vdash_{\mathrm{L}} \sigma\varphi\} = \sigma^{-1}[\mathrm{Thm}]$$

It can be shown that E_{L} satisfies (Cng) and (R) for any logic.

Note that E_{L} is a theory; we show that $\langle p, q\rangle \in \Omega_{\boldsymbol{Fm}_{\mathcal{L}}}(E_{\mathrm{L}})$: for any formula χ and substitution ρ we have:

$$\rho[p{\to}p](\chi) \in E_{\mathrm{L}} \text{ iff } \vdash_{\mathrm{L}} \sigma(\rho[p{\to}p](\chi)) \text{ iff } \vdash_{\mathrm{L}} \sigma(\rho[p{\to}q](\chi)) \text{ iff } \rho[p{\to}q](\chi) \in E_{\mathrm{L}}$$

Due to the monotonicity we get: $\langle p, q\rangle \in \Omega_{\boldsymbol{Fm}_{\mathcal{L}}}(\mathrm{Th}_{\mathrm{L}}(E_{\mathrm{L}} \cup \{p\}))$ and so by logicity of Ω:

$$p, E_{\mathrm{L}} \vdash_{\mathrm{L}} q$$

Note that for any generalized equivalence $\Leftrightarrow$ we have: $\Leftrightarrow \subseteq E_{\mathrm{L}}$.

Equivalential logics

Theorem 3.15

A logic L *is equivalential if it satisfies any of the following equivalent conditions:*

1. *There is a set* $\Leftrightarrow(p, q)$ *of formulae such that*

$$\begin{array}{ll} \text{(R)} & \vdash_{\mathrm{L}} \varphi \Leftrightarrow \varphi \\ \text{(MP)} & \varphi, \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \psi \\ \text{(Cng)} & \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots) \Leftrightarrow c(\chi_1, \dots, \chi_i, \psi, \dots) \\ & \textit{for each } \langle c, n\rangle \in \mathcal{L} \textit{ and each } 0 \leq i < n. \end{array}$$

2. *There exists a set* $\Leftrightarrow(p, q)$ *of formulae st.* $\Omega^{\Leftrightarrow}_{\boldsymbol{A}}(F)$ *is the Leibniz congruence of each* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$.
3. $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}$ *is monotone and commutes with preimages of substitutions on* $\mathrm{Th}(\mathrm{L})$, *i.e.* $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}(\sigma^{-1}[T]) = \sigma^{-1}[\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}(T)]$.
4. *For every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$, $\Omega_{\boldsymbol{A}}$ *is monotone and commutes with preimages of homomorphisms.*

Continuous Leibniz operators

Definition 3.16

The Leibniz operator on $\boldsymbol{A}$ is continuous if:

$$\Omega_{\boldsymbol{A}}\left(\bigcup_{F\in\mathcal{F}} F\right) = \bigcup_{F\in\mathcal{F}} \Omega_{\boldsymbol{A}} F$$

for every directed family $\mathcal{F} \subseteq \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ for which $\bigcup_{F\in\mathcal{F}} F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.

Finitely equivalential logics

Theorem 3.17

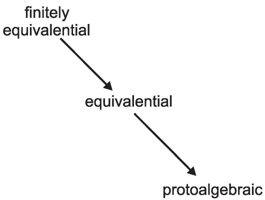
A logic L *is finitely equivalential if it satisfies any of the following equivalent conditions:*

1. *There is a finite set* $\Leftrightarrow(p,q)$ *of formulae st.*

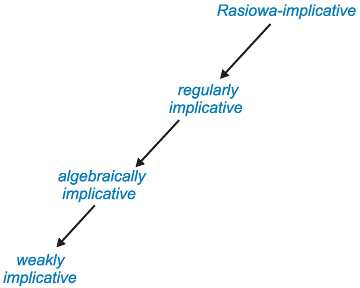
   $$\begin{array}{ll} \text{(R)} & \vdash_{\mathrm{L}} \varphi \Leftrightarrow \varphi \\ \text{(MP)} & \varphi, \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \psi \\ \text{(Cng)} & \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots) \Leftrightarrow c(\chi_1, \dots, \chi_i, \psi, \dots) \\ & \text{for each } \langle c, n\rangle \in \mathcal{L} \text{ and each } 0 \leq i < n. \end{array}$$

2. *There exists a finite set* $\Leftrightarrow(p,q)$ *of formulae st.* $\Omega^{\Leftrightarrow}_{\boldsymbol{A}}(F)$ *is the Leibniz congruence of each* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$.
3. $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}$ *is continuous on* $\mathrm{Th}(\mathrm{L})$.
4. *For every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$, $\Omega_{\boldsymbol{A}}$ *is continuous on* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.

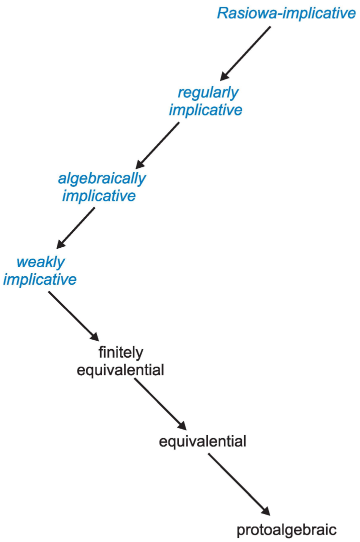
What do we have so far?



Hierarchy of weakly implicative logics



A part of (extended) Leibniz hierarchy



Injectivity of Ω_A

Recall that for the algebra $\boldsymbol{M} \in \mathbf{ALG}^*(\mathrm{BCI})$ defined via:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

we have

$\Omega_{\boldsymbol{M}}(\{t, \top\}) = \Omega_{\boldsymbol{M}}(\{t, f, \top\}) = \mathrm{Id}_{\boldsymbol{M}}$ i.e., $\Omega_{\boldsymbol{M}}$ is not injective

Relation to filter definability in reduced matrices

Recall

A weakly implicative logic L *is algebraically implicative if it satisfies any of the following equivalent conditions:*

1. *There is a set of equations* $\mathcal{T}$ *in one variable such that for each* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ *and each* $a \in A$ *holds:* $a \in F$ *if, and only if,* $\mu^{\boldsymbol{A}}(a) = \nu^{\boldsymbol{A}}(a)$ *for every* $\mu \approx \nu \in \mathcal{T}$.
2. *There is a set of equations* $\mathcal{T}$ *in one variable such that*
   $$(\mathrm{Alg}) \quad p \dashv\vdash_{\mathrm{L}} \{\mu(p) \leftrightarrow \nu(p) \mid \mu \approx \nu \in \mathcal{T}\}.$$
3. $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}$ *is injective on* $\mathrm{Th}(\mathrm{L})$.
4. *For every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$, $\Omega_{\boldsymbol{A}}$ *is injective on* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.

In the first two items the sets $\mathcal{T}$ *can be taken the same.*

Relation to filter definability in reduced matrices

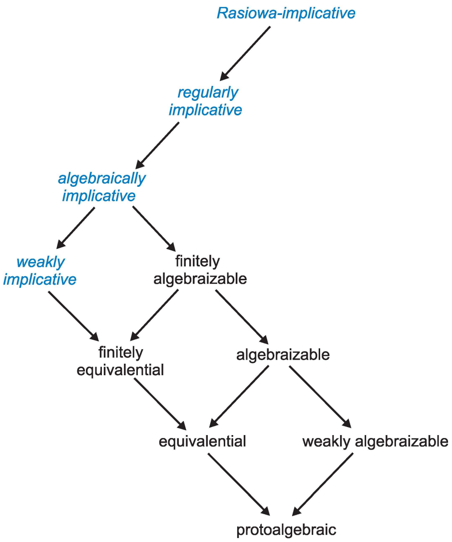
Theorem 3.18

A protoalgebraic logic L *is weakly algebraizable if it satisfies any of the following equivalent conditions:*

1. *There is a set of equations* $\mathcal{T}$ *in one variable such that for each* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ *and each* $a \in A$ *holds:* $a \in F$ *if, and only if,* $\mu^{\boldsymbol{A}}(a) = \nu^{\boldsymbol{A}}(a)$ *for every* $\mu \approx \nu \in \mathcal{T}$.
2. *There is a set of equations* $\mathcal{T}$ *in one variable such that*
   $$(\mathrm{Alg}) \quad p \dashv\vdash_{\mathrm{L}} \{\mu(p) \Leftrightarrow \nu(p) \mid \mu \approx \nu \in \mathcal{T}\}.$$
3. $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}$ *is injective on* $\mathrm{Th}(\mathrm{L})$.
4. *For every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$, $\Omega_{\boldsymbol{A}}$ *is injective on* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.

In the first two items the sets $\mathcal{T}$ *can be taken the same.*

A part of (extended) Leibniz hierarchy



The last few classes . . .

Recall

A weakly implicative logic L *is regularly implicative if it satisfies one of the equivalent conditions:*

1. *For each* $\mathbf{A} = \langle A, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$ *there is* $a \in A$ *st.* $F = \{a\}$.
2. L *satisfies:*

$$(\mathrm{Reg}) \quad p, q \vdash_{\mathrm{L}} p \leftrightarrow q.$$

The last few classes . . .

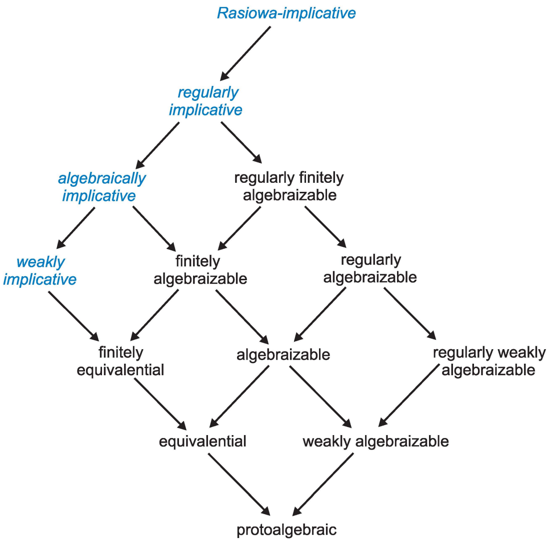
Theorem 3.19

A protoalgebraic logic L *is regularly weakly algebraizable if it satisfies one of the equivalent conditions:*

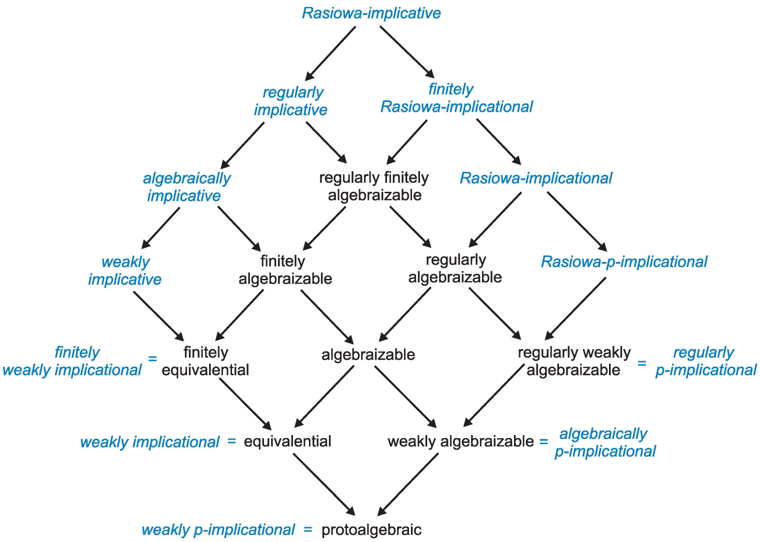
1. *For each* $\mathbf{A} = \langle A, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$ *there is* $a \in A$ *st.* $F = \{a\}$*.*
2. L *satisfies:*

$$(\mathrm{Reg}) \quad p, q \vdash_{\mathrm{L}} p \Leftrightarrow q.$$

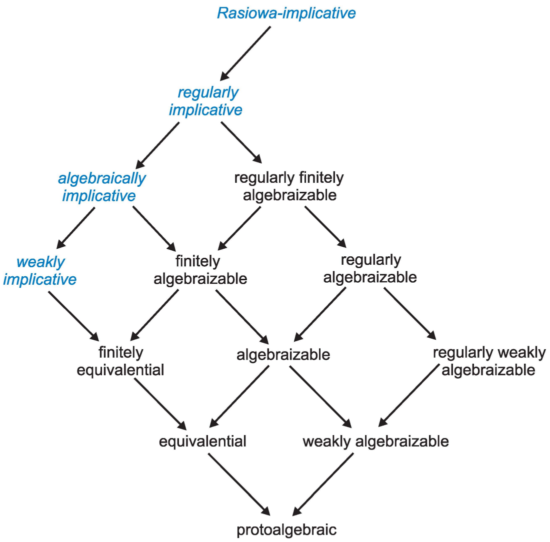
(Extended) Leibniz hierarchy



Hierarchy of implicational logics



(Extended) Leibniz hierarchy



Separation of classes in the Leibniz hierarchy

0. The logic with one binary connective $\to$, axiomatized by axiom $\varphi \to \varphi$ and modus ponens is protoalgebraic but neither weakly algebraizable nor equivalential.

A. BCI is weakly implicative but not weakly algebraizable.

B. Linear logic is algebraically implicative but not regularly weakly algebraizable.
(analogously for any substructural logic without weakening)

C. Equivalence fragment of classical logic regularly implicative but not Rasiowa-implicative.

Separation of classes in the Leibniz hierarchy

D. The logic $\models_{\langle \boldsymbol{D},\{1\}\rangle}$ is regularly finitely algebraizable but not weakly implicative:

$\to_1^{\boldsymbol{D}}$	0	a	1
0	1	1	1
a	0	1	1
1	1	0	1

$\to_2^{\boldsymbol{D}}$	0	a	1
0	1	1	1
a	0	1	1
1	0	1	1

E. The logic $\models_{\langle \boldsymbol{E},\{\omega\}\rangle}$ is regularly algebraizable but not finitely equivalential.

$\to_0^{\boldsymbol{E}}$	0	1	2	$\cdots$	$i-1$	i	$i+1$	$\cdots$	ω
0	ω	ω	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
1	ω	ω	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
2	ω	0	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
$\vdots$									
$i-1$	ω	0	0	$\cdots$	ω	ω	ω	$\cdots$	ω
i	ω	0	0	$\cdots$	0	ω	ω	$\cdots$	ω
$i+1$	ω	0	0	$\cdots$	0	0	ω	$\cdots$	ω
$\vdots$									
ω	0	ω	ω	$\cdots$	ω	ω	ω	$\cdots$	ω

$\to_i^{\boldsymbol{E}}$	0	1	2	$\cdots$	$i-1$	i	$i+1$	$\cdots$	ω
0	ω	ω	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
1	ω	ω	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
2	ω	0	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
$\vdots$									
$i-1$	ω	0	0	$\cdots$	ω	ω	ω	$\cdots$	ω
i	0	0	0	$\cdots$	0	ω	ω	$\cdots$	ω
$i+1$	ω	0	0	$\cdots$	0	0	ω	$\cdots$	ω
$\vdots$									
ω	ω	ω	ω	$\cdots$	ω	0	ω	$\cdots$	ω

F. The logic of ortholattices is weakly regularly algebraizable but not equivalential.

D. The logic $\models_{\langle \boldsymbol{D},\{1\}\rangle}$ is regularly finitely algebraizable but not weakly implicative:

$\to_1^{\boldsymbol{D}}$	0	a	1
0	1	1	1
a	0	1	1
1	1	0	1

$\to_2^{\boldsymbol{D}}$	0	a	1
0	1	1	1
a	0	1	1
1	0	1	1

E′. Dellunde's logic is finitary regularly algebraizable but not finitely equivalential. It has the language $\{\Box, \leftrightarrow\}$ and is axiomatized by $\varphi \leftrightarrow \varphi$, Modus Ponens for $\leftrightarrow$, and all the rules:

$$\varphi, \psi \vdash \Box^n \varphi \leftrightarrow \Box^n \psi$$

$$\varphi \leftrightarrow \psi, \varphi' \leftrightarrow \psi' \vdash \Box^n (\varphi \leftrightarrow \varphi') \leftrightarrow \Box^n (\psi \leftrightarrow \psi')$$

for each $n \geq 0$

F. The logic of ortholattices is weakly regularly algebraizable but not equivalential.

Leibniz operator characterization of Leibniz hierarchy

Characterization using properties of $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}$ (or equivalently via $\Omega_{\boldsymbol{A}}$ for each $\boldsymbol{A}$)

A logic is ...	iff	$\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}$ is ...
protoalgebraic		monotone
equivalential		monotone and commutes with preimages of substitutions
finitely equivalential		continuous
weakly algebraizable		an isomorphism
algebraizable		an isomorphism and commutes with preimages of substitutions
finitely algebraizable		a continuous isomorphism

Syntactic characterization of Leibniz hierarchy

Recall:

(R) $\vdash_{\mathrm{L}} \varphi \Leftrightarrow \varphi$ (MP) $\varphi, \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \psi$ (Reg) $p, q \vdash_{\mathrm{L}} p \Leftrightarrow q$

(Cng) $\varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots) \Leftrightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots)$

(Alg) $p \dashv\vdash_{\mathrm{L}} \{\mu(p) \Leftrightarrow \nu(p) \mid \mu \approx \nu \in \mathcal{T}\}$.

A logic is ...	iff	there is $\Leftrightarrow$ which is ...	and satisfies ...
protoalgebraic		parameterized set	(R),(MP),(Cng)
equivalential		set	(R),(MP),(Cng)
finitely equivalential		finite set	(R),(MP),(Cng)
weakly algebraizable		parameterized set	(R),(MP),(Cng), (Alg)
regularly weakly algebraizable		parameterized set	(R),(MP),(Cng), (Reg)

Model-theoretic characterization of Leibniz hierarchy

A logic is ...	iff	$\mathbf{MOD}^*(\mathrm{L})$ is closed under ...
protoalgebraic		$\mathbf{P}_{\mathrm{SD}}$
equivalential		$\mathbf{S}$ and $\mathbf{P}$
weakly algebraizable		$\mathbf{P}_{\mathrm{SD}}$ and the filters in $\mathbf{MOD}^*(\mathrm{L})$ are equationally definable
regularly weakly algebraizable		$\mathbf{P}_{\mathrm{SD}}$ and the filters in $\mathbf{MOD}^*(\mathrm{L})$ are singletons
algebraizable		$\mathbf{S}$ and $\mathbf{P}$ and the filters in $\mathbf{MOD}^*(\mathrm{L})$ are equationally definable
regularly algebraizable		$\mathbf{S}$ and $\mathbf{P}$ and the filters in $\mathbf{MOD}^*(\mathrm{L})$ are singletons

Reduced products of matrices

Recall: a filter $\mathcal{F}$ on an index set I is a lattice filter on its powerset (i.e., a non-empty subset of $\mathcal{P}(I)$, closed under intersections and supersets.)

Reduced product: take matrices $\{\langle \boldsymbol{A}_i, F_i\rangle \mid i \in I\}$ and a filter $\mathcal{F}$ over I . We define a congruence $\theta_{\mathcal{F}}$ on $\prod_{i\in I} \boldsymbol{A}_i$ as:

$$\langle a, b\rangle \in \theta_{\mathcal{F}} \quad \text{iff} \quad \{i \in I \mid a(i) = b(i)\} \in \mathcal{F}.$$

The reduced product modulo $\mathcal{F}$ is $\langle (\prod_{i\in I} \boldsymbol{A}_i)/\theta_{\mathcal{F}}, (\prod_{i\in I} F_i)/\theta_{\mathcal{F}}\rangle$.

We say that reduced product is

- σ-filtered if $\mathcal{F}$ is closed under arbitrary intersections
- ultraproduct if $\mathcal{F}$ is an ultrafilter ($X \in \mathcal{F}$ iff $I \setminus X \notin \mathcal{F}$).

We use operators $\mathbf{P}_{\mathrm{R}}$, $\mathbf{P}_{\sigma\text{-}f}$, and $\mathbf{P}_{\mathrm{U}}$.

Finitarity and ultraproducts

Recall Lemma 1.4: If $\mathbb{K}$ is a finite class of finite matrices, then the logic $\models_{\mathbb{K}}$ is finitary.

Theorem 3.20

If $\mathbf{P}_{\mathrm{U}}(\mathbb{K}) \subseteq \mathbb{K}$*, then* $\models_{\mathbb{K}}$ *is finitary.*

Theorem 3.21

Let L *be an arbitrary logic. Then:*

$$\mathbf{P}_{\mathrm{U}}(\mathbf{MOD}(\mathrm{L})) = \mathbf{MOD}(\mathrm{L}) \quad \textit{iff} \quad \mathrm{L} \textit{ is finitary.}$$

Theorem 3.23

Let L *be an arbitrary logic. TFAE:*

1. $\mathbf{MOD}^*(\mathrm{L})$ *is closed under* $\mathbf{S}$, $\mathbf{P}$, $\mathbf{P}_{\mathrm{U}}$ *(i.e. it is quasivariety)*
2. L *is finitary and finitely equivalential*

Finitarity and ultraproducts

Recall Lemma 1.4: If $\mathbb{K}$ is a finite class of finite matrices, then the logic $\models_{\mathbb{K}}$ is finitary.

Theorem 3.20

If $\mathbf{P}_{\mathrm{U}}(\mathbb{K}) \subseteq \mathbb{K}$, then $\models_{\mathbb{K}}$ is finitary.

Theorem 3.22

Let L be a finitely equivalential logic. Then:

$$\mathbf{P}_{\mathrm{U}}(\mathbf{MOD}^*(\mathrm{L})) = \mathbf{MOD}^*(\mathrm{L}) \quad \textit{iff} \quad \mathrm{L} \textit{ is finitary.}$$

Theorem 3.23

Let L be an arbitrary logic. TFAE:

1. $\mathbf{MOD}^*(\mathrm{L})$ *is closed under* $\mathbf{S}$, $\mathbf{P}$, $\mathbf{P}_{\mathrm{U}}$ *(i.e. it is quasivariety)*
2. L *is finitary and finitely equivalential*

Equational consequence

An equation in the language $\mathcal{L}$ is a formal expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in Fm_{\mathcal{L}}$.

We say that an equation $\varphi \approx \psi$ is a consequence of a set of equations Π w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-algebras if for each $\boldsymbol{A} \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e we have $e(\varphi) = e(\psi)$ whenever $e(\alpha) = e(\beta)$ for each $\alpha \approx \beta \in \Pi$; we denote it by $\Pi \models_{\mathbb{K}} \varphi \approx \psi$.

Characterizations of algebraically implicative logics

Recall the translations:

$$\rho[\Pi] = \bigcup_{\alpha \approx \beta \in \Pi} (\alpha \leftrightarrow \beta)$$

$$\tau[\Gamma] = \{\alpha(\varphi) \approx \beta(\varphi) \mid \varphi \in \Gamma \text{ and } \alpha \approx \beta \in \mathcal{T}\}$$

Recall

Given any weakly implicative logic L*, TFAE:*

1. L *is algebraically implicative with the truth definition* $\mathcal{T}$*.*
2. *There is a set of equations* $\mathcal{T}$ *in one variable such that:*
   1. $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ *iff* $\rho[\Pi] \vdash_{\mathrm{L}} \rho(\varphi \approx \psi)$
   2. $p \dashv\vdash_{\mathrm{L}} \rho[\tau(p)]$
3. *There is a set of equations* $\mathcal{T}$ *in one variable such that:*
   1. $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\tau[\Gamma] \models_{\mathbf{ALG}^*(\mathrm{L})} \tau(\varphi)$
   2. $p \approx q \mathrel{=\mid\models}_{\mathbf{ALG}^*(\mathrm{L})} \tau[\rho(p \approx q)]$

Redefine the translations:

$$\rho[\Pi] = \bigcup_{\alpha\approx\beta\in\Pi} (\alpha \Leftrightarrow \beta)$$

$$\tau[\Gamma] = \{\alpha(\varphi) \approx \beta(\varphi) \mid \varphi \in \Gamma \text{ and } \alpha \approx \beta \in \mathcal{T}\}$$

Theorem 3.24

Given any protoalgebraic logic L*, TFAE:*

1. L *is weakly algebraizable with the truth definition* $\mathcal{T}$*.*
2. *There is a set of equations* $\mathcal{T}$ *in one variable such that:*
   1. $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ *iff* $\rho[\Pi] \vdash_{\mathrm{L}} \rho(\varphi \approx \psi)$
   2. $p \dashv\vdash_{\mathrm{L}} \rho[\tau(p)]$
3. *There is a set of equations* $\mathcal{T}$ *in one variable such that:*
   1. $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\tau[\Gamma] \models_{\mathbf{ALG}^*(\mathrm{L})} \tau(\varphi)$
   2. $p \approx q \mathrel{=\joinrel\mid\models}_{\mathbf{ALG}^*(\mathrm{L})} \tau[\rho(p \approx q)]$

Characterizations of (finitely) algebraizable logics

Redefine the translations:

$$\rho[\Pi] = \bigcup_{\alpha\approx\beta\in\Pi} (\alpha \Leftrightarrow \beta)$$

$$\tau[\Gamma] = \{\alpha(\varphi) \approx \beta(\varphi) \mid \varphi \in \Gamma \text{ and } \alpha \approx \beta \in \mathcal{T}\}$$

Theorem 3.25

Given any (finitely) equivalential logic L*, TFAE:*

1. L *is (finitely) algebraizable with the truth definition* $\mathcal{T}$*.*
2. *There is a set of equations* $\mathcal{T}$ *in one variable such that:*
    1. $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ *iff* $\rho[\Pi] \vdash_{\mathrm{L}} \rho(\varphi \approx \psi)$
    2. $p \dashv\vdash_{\mathrm{L}} \rho[\tau(p)]$
3. *There is a set of equations* $\mathcal{T}$ *in one variable such that:*
    1. $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\tau[\Gamma] \models_{\mathbf{ALG}^*(\mathrm{L})} \tau(\varphi)$
    2. $p \approx q \;=\!\mid\models_{\mathbf{ALG}^*(\mathrm{L})} \tau[\rho(p \approx q)]$

Traditional account of algebraizability

Theorem 3.26

Given an arbitrary logic L*, TFAE:*

- L *is algebraizable with the truth definition* $\mathcal{T}$ *and equivalence* $\Leftrightarrow$*.*
- *There exists a class of algebras* $\mathbb{K}$*, a set* $\mathcal{T}$ *of equations in one variable, and a set* $\Leftrightarrow$ *of formulae in two variables such that for the translations* τ *and* ρ *defined as before we have:*
  1. $\Pi \models_{\mathbb{K}} \varphi \approx \psi$ *iff* $\rho[\Pi] \vdash_{\mathrm{L}} \rho(\varphi \approx \psi)$
  2. $p \dashv\vdash_{\mathrm{L}} \rho[\tau(p)]$
  3. $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\tau[\Gamma] \models_{\mathbb{K}} \tau(\varphi)$
  4. $p \approx q \mathrel{=\!\mid\!\models}_{\mathbb{K}} \tau[\rho(p \approx q)]$

In this case $\mathbb{K}$ *is called an equivalent algebraic semantics of* L*.*

Note that it suffices to assume conditions 1 and 2 (or 3 and 4).

Theorem 3.27

Let L be an algebraizable logic. Then the following hold:

- *If L is finitary, then τ can be chosen finite*
- *If $\models_{\mathbf{ALG}^*(\mathrm{L})}$ is finitary, then ρ can be chosen finite*
- *If L is finitary and ρ is finite, then $\models_{\mathbf{ALG}^*(\mathrm{L})}$ is finitary.*
- *If $\models_{\mathbf{ALG}^*(\mathrm{L})}$ is finitary and τ is finite, then L is finitary.*

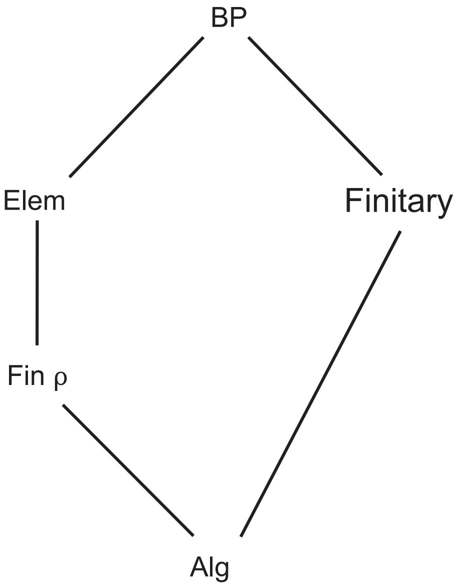
More kinds of algebraizable logics

Definition 3.28

An algebraizable logic L is

- finitely algebraizable if ρ can be taken finite
- elementarily algebraizable if $\mathbf{ALG}^*(\mathrm{L})$ is a quasivariety, i.e., $\models_{\mathbf{ALG}^*(\mathrm{L})}$ is finitary
- algebraizable in the sense of Blok-Pigozzi if it is finitary and finitely algebraizable

More kinds of algebraizable logics



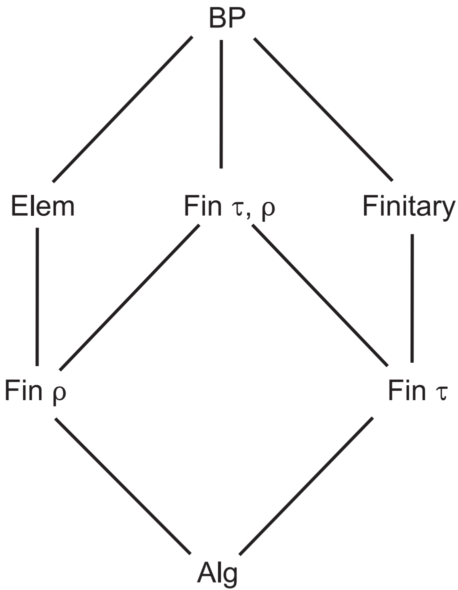
More on finiteness issues

Theorem 3.29

Let L *be an algebraizable logic. Then the following hold:*

- *If* L *is finitary, then it has a finite truth definition.*
- *If* L *is elementarily algebraizable, then* L *is finitely algebraizable.*
- *If* L *is finitary and finitely algebraizable, then* L *is elementarily algebraizable.*
- *If* L *is elementarily algebraizable with a finite truth definition, then* L *is finitary.*

Extending the hierarchy



Separating example – 1

Raftery's logic is elementarily finitely algebraizable but not finitary. It has the language $\{\Box, \leftrightarrow, \pi_1, \pi_2\}$, axioms:

$$\varphi \leftrightarrow \varphi \quad \varphi \leftrightarrow \pi_1(\varphi \leftrightarrow \psi) \quad \psi \leftrightarrow \pi_2(\varphi \leftrightarrow \psi) \quad (\varphi \leftrightarrow \psi) \leftrightarrow \Box(\varphi \leftrightarrow \psi)$$

and rules

$$\varphi, \varphi \leftrightarrow \psi \vdash \psi$$

$$\chi \leftrightarrow \delta, \varphi \leftrightarrow \psi \vdash (\chi \leftrightarrow \varphi) \leftrightarrow (\delta \leftrightarrow \psi)$$

$$\varphi \leftrightarrow \psi \vdash *\varphi \leftrightarrow *\psi \qquad * \in \{\pi_1, \pi_2, \Box\}$$

$$\varphi \vdash \pi_1(\Box^i\varphi) \leftrightarrow \pi_2(\Box^i\varphi) \qquad i \in \omega$$

$$\{\pi_1(\Box^i\varphi) \leftrightarrow \pi_2(\Box^i\varphi) \mid i \in \omega\} \vdash \varphi$$

Separating example – 2

Dellunde's logic is finitary regularly algebraizable but not finitely algebraizable. It has the language $\{\Box, \leftrightarrow\}$ and is axiomatized by:

$$\vdash \varphi \leftrightarrow \varphi$$

$$\varphi, \varphi \leftrightarrow \psi \vdash \psi$$

$$\varphi, \psi \vdash \Box^n\varphi \leftrightarrow \Box^n\psi$$

$$\varphi \leftrightarrow \psi, \varphi' \leftrightarrow \psi' \vdash \Box^n(\varphi \leftrightarrow \varphi') \leftrightarrow \Box^n(\psi \leftrightarrow \psi')$$

for each $n \in \omega$

Separating example – 3

Łukasiewicz logic $Ł_\infty$ is regularly finitely algebraizable but not finitary and not elementarily algebraizable. It has the language $\{\to, \neg\}$, axioms:

$$\varphi \to (\psi \to \varphi) \quad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$

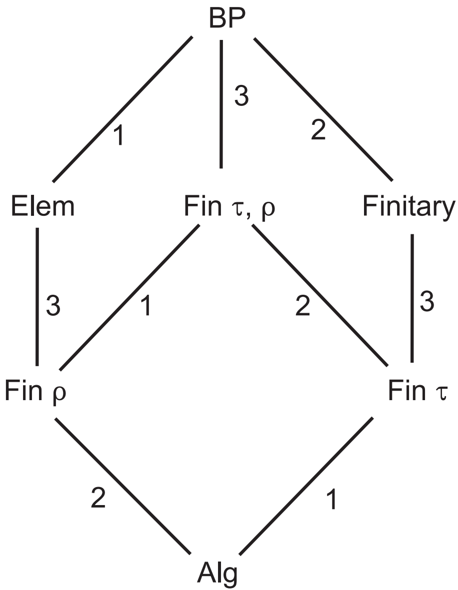
$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi) \quad (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

and rules:

$$\varphi, \varphi \to \psi \vdash \psi$$

$$\{i\varphi \to \psi \mid i \in \omega\} \cup \{\neg\varphi \to \psi\} \vdash \psi$$

Extending the hierarchy



Order algebraizability [Raftery]

Partially ordered algebra: $\langle \boldsymbol{A}, \leq \rangle$.

A logic L is order algebraizable if it is equivalent to an inequational consequence $\models^{\leq}_{\mathbb{K}}$ for some class of partially ordered algebras $\mathbb{K}$, i.e. there exist translations $\tau : \boldsymbol{Fm}_{\mathcal{L}} \to \mathrm{Ineq}_{\mathcal{L}}$ and $\rho : \mathrm{Ineq}_{\mathcal{L}} \to \boldsymbol{Fm}_{\mathcal{L}}$ such that:

- $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\tau[\Gamma] \models^{\leq}_{\mathbb{K}} \tau(\varphi)$
- $\alpha \preccurlyeq \beta \models^{\leq}_{\mathbb{K}} \tau[\rho(\alpha \preccurlyeq \beta)]$ and $\tau[\rho(\alpha \preccurlyeq \beta)] \models^{\leq}_{\mathbb{K}} \alpha \preccurlyeq \beta$.

{algebraizable logics} $\subseteq$ {order algebraizable logics} $\subseteq$ {equivalential logics}

Order algebraizability [Raftery]

A logic L is order algebraizable iff there is a set of binary formulae $\rho(x, y)$ and a set $\mathcal{T}(p)$ of pairs of formulae in the variable p such that:

$$\begin{array}{ll} \text{(R)} & \vdash_{\mathrm{L}} \rho(\varphi, \varphi) \\ \text{(T)} & \rho(\varphi, \psi), \rho(\psi, \chi) \vdash_{\mathrm{L}} \rho(\varphi, \chi) \\ \text{(Subst)} & \rho(\varphi, \psi), \rho(\psi, \varphi), \gamma(\varphi, \overrightarrow{\alpha}) \vdash_{\mathrm{L}} \gamma(\psi, \overrightarrow{\alpha}) \\ \text{(Alg)'} & p \dashv\vdash_{\mathrm{L}} \bigcup_{\langle \varphi, \psi \rangle \in \mathcal{T}(p)} \rho(\varphi, \psi) \end{array}$$

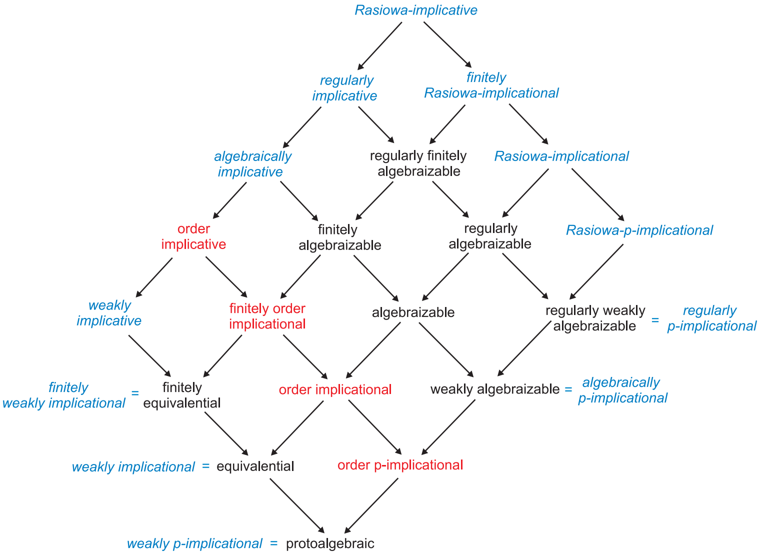
$\rho(x, y)$ defines a partial order in every $\langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$:

$$a \leq_F b \text{ iff } \rho^{\boldsymbol{A}}(a, b) \subseteq F.$$

The ρ-ordered model class is: $\mathbb{K} = \{\langle \boldsymbol{A}, \leq_F \rangle \mid \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})\}$.

Differences: absence of parameters and absence of *modus ponens*.

The enriched implicational hierarchy



Separating examples

Exercise 8

BCI is order implicative ($\rho(x, y) = \{x \to y\}$, $\tau(p) = \{\langle p \to p, p\rangle\}$) but it is not weakly algebraizable.

Separating examples

Theorem 3.30

If L *is weakly order algebraizable with* ρ*, then for every algebra* $\boldsymbol{A}$*:*
$F \mapsto \rho^{-1}[F] = \{\langle a, b\rangle \in A^2 \mid \rho^{\boldsymbol{A}}(a, b, \overrightarrow{c}) \subseteq F$ *for every* $\overrightarrow{c} \in A^{\leq\omega}\}$
is injective on $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$*.*

Let E be the Entailment logic of Anderson and Belnap.

- $\mathrm{E}_{\to}$, $\mathrm{E}_{\neg,\to}$ and $\mathrm{E}_{\wedge,\to}$ are weakly implicative but not weakly order algebraizable.