Two-layer modal logics: from fuzzy logics to a general framework

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Probability and fuzzy logic . . .

Once upon a time, there was a logician . . .

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and he wrote a book . . .

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Hájek: *Metamathematics of fuzzy logic*, Kluwer 1998

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Many have read it

and he wrote a book . . .

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But just few have got to Section 8.4 . . .

Fuzzy logic for reasoning about probability

Let us take:

- the classical logic CL in language $\to, \neg, \vee, \wedge, \overline{0}$
- Łukasiewicz logic Ł in language $\to_{Ł}, \neg_{Ł}, \oplus, \ominus$
- an extra symbol $\Box$

We define three kinds of formulae of a two-level language over a fixed set of variables Var:

- non-modal: built from Var using $\to, \neg, \vee, \wedge, \overline{0}$
- atomic modal: of the form $\Box\varphi$, for each non-modal φ
- modal: built from atomic ones using $\to_{Ł}, \neg_{Ł}, \oplus, \ominus$

We use the following notational conventions:

	non-modal	modal
formulae	$\varphi, \psi, \dots$	$\Phi, \Psi, \dots$
sets of formulae	$T, S, \dots$	$\Gamma, \Delta, \dots$

Probability Kripke frames and Kripke models

Definition 1

A *probability Kripke frame* is a system $\mathbf{F} = \langle W, \mu \rangle$ where

- W is a set (of possible worlds)
- μ is a finitely additive probability measure defined on a sublattice of 2^W

Definition 2

A *Kripke model* $\mathbf{M}$ over a probability Kripke frame $\mathbf{F} = \langle W, \mu \rangle$ is a tuple $\mathbf{M} = \langle \mathbf{F}, (e_w)_{w \in W} \rangle$ where:

- e_w is a classical evaluation of non-modal formulae
- the domain of μ contains the set $\{w \mid e_w(\varphi) = 1\}$ for each non-modal formula φ

Truth definition

The truth values of modal formulae are defined uniformly:

$$
\begin{aligned}
||\Box\varphi||_{\mathbf{M}} &= \mu(\{w \mid e_w(\varphi) = 1\}) \\
||\neg_{Ł}\Phi||_{\mathbf{M}} &= 1 - ||\Phi||_{\mathbf{M}} \\
||\Phi \to_{Ł} \Psi||_{\mathbf{M}} &= \min\{1, 1 - ||\Phi||_{\mathbf{M}} + ||\Psi||_{\mathbf{M}}\} \\
||\Phi \oplus \Psi||_{\mathbf{M}} &= \min\{1, ||\Phi||_{\mathbf{M}} + ||\Psi||_{\mathbf{M}}\} \\
||\Phi \ominus \Psi||_{\mathbf{M}} &= \max\{0, ||\Phi||_{\mathbf{M}} - ||\Psi||_{\mathbf{M}}\}
\end{aligned}
$$

Axiomatization

Definition 3

The logic $\mathfrak{FP}$ of probability inside Łukasiewicz logic is given by the axiomatic system consisting of:

- the axioms and rules of CL for non-modal formulae,
- axioms and rules of Ł for modal formulae,
- modal axioms

$$\begin{array}{ll}
\text{(FP0)} & \neg_{Ł}\Box(\bar{0}) \\
\text{(FP1)} & \Box(\varphi \to \psi) \to_{Ł} (\Box\varphi \to_{Ł} \Box\psi) \\
\text{(FP2)} & \neg_{Ł}\Box(\varphi) \to_{Ł} \Box(\neg\varphi) \\
\text{(FP3)} & \Box(\varphi \vee \psi) \to_{Ł} (\Box\psi \oplus (\Box\varphi \ominus \Box(\varphi \wedge \psi)))
\end{array}$$

- a unary modal rule:

$$\varphi \vdash \Box\varphi$$

The notion of provability $\vdash_{\mathfrak{FP}}$ (from both modal and non-modal premises) is defined as usual.

Completeness theorem

Theorem 4

Let $\Gamma \cup \{\Psi\}$ *be a set of modal formulas. TFAE:*

- $\Gamma \vdash_{\mathfrak{FP}} \Psi$
- $||\Psi||_{\mathbf{M}} = 1$ *for each Kripke model* $\mathbf{M}$ *where* $||\Phi||_{\mathbf{M}} = 1$ *for each* $\Phi \in \Gamma$

Variations and our goal

Variations considered in the literature:

- changing the measure
- changing the 'upper' logic: replacing the Łukasiewicz logic by any other t-norm-based logic
- changing the 'lower' logic: e.g. replacing CL by the Łukasiewicz logic to speak about probability of 'fuzzy' events
- adding more modalities
- any combination of the above four options

The goal of this contribution: identify the common aspects of all existing approaches and recover particular completeness results as instances of a general theory.

Setting up the stage . . .

Propositional logics – an abstract way

$\mathcal{L}$: propositional language (a type)

$\varphi, \psi, \dots$: formulae from $Fm_{\mathcal{L}}$ (terms) defined as usual

L: protoalgebraic logic

E: a (parameterized) equivalence of L

We write $\varphi \leftrightarrow \psi$ for $\{\chi(\varphi, \psi, \vec{\delta}) \mid \chi \in E \text{ and } \vec{\delta} \in {Fm_{\mathcal{L}}}^{<\omega}\}$

$T \vdash S$ for $T \vdash \varphi$ for each $\varphi \in S$

Propositional logics – a 'fuzzy' way

$\mathcal{L}$: the language of MTL

$\varphi, \psi, \ldots$: formulae from $Fm_{\mathcal{L}}$ (terms) defined as usual

L: finitary extension of MTL

$\leftrightarrow$: the equivalence connective of MTL

We write $\varphi \leftrightarrow \psi$ for $\{\varphi \leftrightarrow \psi\}$

$T \vdash S$ for $T \vdash \varphi$ for each $\varphi \in S$

Semantics – a ‘fuzzy’ way

$\mathcal{L}$-algebra $\boldsymbol{A}$: just an algebra of type $\mathcal{L}$

$\mathbf{A}$-evaluation e: a homomorphism from the absolutely free $\mathcal{L}$-algebra into an $\mathcal{L}$-algebra $\boldsymbol{A}$.

$\boldsymbol{A}$ is an L-algebra if

- $\boldsymbol{A}$ an MTL-algebra
- $T \vdash_{\mathrm{L}} \varphi$ implies that for each $\boldsymbol{A}$-evaluation: if $e[T] \subseteq \{1^{\boldsymbol{A}}\}$, then $e(\varphi) = 1^{\boldsymbol{A}}$.

$\mathbb{L}$: the class of L-algebras (a quasivariety)

$\models_{\mathbb{K}}$: semantical consequence w.r.t. a class $\mathbb{K}$ of L-algebras

Theorem 5 (Completeness)

$$\vdash_{\mathrm{L}} = \models_{\mathbb{L}}$$

Semantics – an abstract way

$\mathcal{L}$-matrix $\mathbf{A}$: a pair $\langle A, F\rangle$, where A is an $\mathcal{L}$-algebra and $F \subseteq A$

$\mathbf{A}$-evaluation e: a homomorphism from the absolutely free $\mathcal{L}$-algebra into an $\mathcal{L}$-algebra A.

$\mathbf{A}$ is a reduced L-matrix if

- $x \leftrightarrow^{A} y \subseteq F_A$ implies $x = y$
- $T \vdash_{\mathrm{L}} \varphi$ implies that for each $\mathbf{A}$-evaluation: if $e[T] \subseteq F_{\mathbf{A}}$, then $e(\varphi) \in F_{\mathbf{A}}$.

$\mathbf{MOD}^*(\mathrm{L})$: the class of all reduced L-matrices

$\models_{\mathbb{K}}$: semantical consequence w.r.t. a class $\mathbb{K}$ of red. L-matrices

Theorem 6 (Completeness)

$$\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^*(\mathrm{L})}$$

Two-layer modal logics . . .

General notion of two level language

Let us fix two logics L_1 and L_2 in disjoint languages and an extra symbol $\Box$.

We define three kinds of formulae of a two-level language over a fixed set of variables Var:

- non-modal: built from Var using connectives of L_1
- atomic modal: of the form $\Box\varphi$, for each non-modal φ
- modal: built from atomic ones using connectives of L_2.

We use the following notational conventions:

	non-modal	modal
formulae	$\varphi, \psi, \ldots$	$\Psi, \Phi, \ldots$
sets of formulae	$T, S, \ldots$	$\Gamma, \Delta, \ldots$

The minimal logic and its extensions

An *n-ary modal rule* is a pair $T \vdash \Psi$, where T is a set of n non-modal formulae and Ψ is a modal formula.

Definition 7

The *minimal L_2-modal logic over L_1* is given by the axiomatic system consisting of

- the axioms and rules of L_1 for non-modal formulae,
- axioms and rules of L_2 for modal formulae,
- a modal rule:

$$\varphi \leftrightarrow \psi \vdash \Box\varphi \leftrightarrow \Box\psi \qquad \text{(CONGR)}$$

An *L_2-modal logic over* L_1 is an extension of the minimal one by some modal rules.

The notion of proof (from both modal and non-modal premises) is defined as usual.

Measured Kripke frames and Kripke models

We fix two classes of reduced matrices $\mathbb{K}_i \subseteq \mathbf{MOD}^*(\mathrm{L}_i)$

Definition 8

A *$\mathbb{K}_1$-based $\mathbb{K}_2$-measured Kripke frame* is a system $\mathbf{F} = \langle W, \langle \boldsymbol{A}_w \rangle_{w \in W}, \boldsymbol{B}, \mu \rangle$ where

- W is a set (of possible worlds)
- $\boldsymbol{A}_w \in \mathbb{K}_1$ for each $w \in W$
- $\boldsymbol{B} \in \mathbb{K}_2$
- μ is a *partial* mapping $\mu \colon \prod_{w \in W} A_w \to B$

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A *$\mathbb{K}_1$-based $\mathbb{K}_2$-measured Kripke frame* is a system

$$\mathbf{F} = \langle W, \langle \boldsymbol{A}_w \rangle_{w \in W}, \boldsymbol{B}, \mu \rangle \text{ where}$$

- W is a set (of possible worlds)
- $\boldsymbol{A}_w \in \mathbb{K}_1$ for each $w \in W$
- $\boldsymbol{B} \in \mathbb{K}_2$
- μ is a *partial* mapping $\mu \colon \prod_{w \in W} A_w \to B$

A *Kripke model* $\mathbf{M}$ over a $\mathbf{F}$ is a tuple $\mathbf{M} = \langle \mathbf{F}, \langle e_w \rangle_{w \in W} \rangle$ where:

- e_w is an $\boldsymbol{A}_w$-evaluation of formulae of L_1
- The domain of μ contains the element $\langle e_w(\varphi) \rangle_{w \in W}$ for each non-modal formula φ

Truth definition

Let us fix a Kripke model $\mathbf{M} = \langle\langle W, \langle \boldsymbol{A}_w\rangle_{w\in W}, \boldsymbol{B}, \mu\rangle, \langle e_w\rangle_{w\in W}\rangle$ and we define the truth value of

- non-modal formulae in each possible world using the evaluation e_w
- atomic modal formulae uniformly in $\mathbf{M}$ as:

$$||\Box\varphi||_{\mathbf{M}} = \mu(\langle e_w(\varphi)\rangle_{w\in W})$$

- non-atomic modal formulae using operations from $\boldsymbol{B}$

We say that $\mathbf{M}$ is a *model of*

- a non-modal formula ψ if $e_w(\psi) \in F_{\boldsymbol{A}_w}$ for each $w \in W$.
- a modal formula Ψ whenever $||\Psi||_{\mathbf{M}} \in F_{\boldsymbol{B}}$.

Semantical consequence and frames for a logic

Definition 9

A formula Φ is a semantical consequence of $T \cup \Gamma$ w.r.t. a class of measured Kripke frames K, $T, \Gamma \models_{\mathsf{K}} \Phi$, if for each frame $\mathbf{F} \in$ K and each Kripke model $\mathbf{M}$ over $\mathbf{F}$ holds that $\mathbf{M}$ is a model of Φ whenever it is a model of Γ and T.

Definition 10

A $\mathbb{K}_1$-based $\mathbb{K}_2$-measured Kripke frame $\mathbf{F}$ is a frame for an L_2-modal logic $\mathfrak{L}$ over L_1, $\mathbf{F} \in \mathsf{KF}^{\mathbb{K}_2}_{\mathbb{K}_1}(\mathfrak{L})$, if for each additional modal rule $T \vdash \Psi$ we have $T \models_{\mathbf{F}} \Psi$.

Example

Recall the logic $\mathfrak{FP}$ built over the classical logic CL; with 'upper' logic being the Łukasiewicz logic; and the modal rules:

(FP0) $\neg_{Ł}\Box(\bar{0})$

(FP1) $\Box(\varphi \to \psi) \to_{Ł} (\Box\varphi \to_{Ł} \Box\psi)$

(FP2) $\neg_{Ł}\Box(\varphi) \to_{Ł} \Box(\neg\varphi)$

(FP3) $\Box(\varphi \vee \psi) \to_{Ł} (\Box\psi \oplus (\Box\varphi \ominus \Box(\varphi \wedge \psi)))$

$\varphi \vdash \Box\varphi$

The rule (CONGR) is clearly derivable $\Rightarrow$

$\mathfrak{FP}$ is an Ł-modal logic over CL

Let us take $\mathbf{F} \in \mathrm{KF}_{\mathbf{2}}^{[0,1]_{Ł}}$; note that $\mathbf{F} = \langle W, \langle \mathbf{2} \rangle_{w \in W}, [0,1]_{Ł}, \mu \rangle$ and μ is a finitely additive probability measure

Definition 11

A logic L enjoys the

- strong $\mathbb{K}$-completeness, $\mathrm{S}\mathbb{K}\mathrm{C}$, if for each $T \cup \{\varphi\}$ holds: $T \vdash_{\mathrm{L}} \varphi$ iff $T \models_{\mathbb{K}} \varphi$.
- finite strong $\mathbb{K}$-completeness, $\mathrm{FS}\mathbb{K}\mathrm{C}$, if for each finite $T \cup \{\varphi\}$ holds: $T \vdash_{\mathrm{L}} \varphi$ iff $T \models_{\mathbb{K}} \varphi$.

Strong completeness theorem

Theorem 12

Let $\mathfrak{L}$ be an L_2-modal logic over a logic L_1 such that

- L_1 *has* $\mathrm{S}\mathbb{K}_1\mathrm{C}$.
- L_2 *has* $\mathrm{S}\mathbb{K}_2\mathrm{C}$.
-
-

Then for each non-modal theory T, modal theory Γ, and a modal formula Φ:

$$\Gamma, T \vdash_{\mathfrak{L}} \Phi \qquad \textit{iff} \qquad \Gamma, T \models_{\mathsf{KF}^{\mathbb{K}_2}_{\mathbb{K}_1}(\mathfrak{L})} \Phi$$

Finite strong completeness theorem

Theorem 13

Let $\mathfrak{L}$ be an L_2-modal logic over a logic L_1 such that

- L_1 *has* $\mathrm{FS}\mathbb{K}_1\mathrm{C}$.
- L_2 *has* $\mathrm{FS}\mathbb{K}_2\mathrm{C}$.
- $\mathfrak{L}$ *has only finitely many modal rules.*
- $\mathbf{MOD}^*(\mathrm{L}_1)$ *is locally finite.*

Then for each finite non-modal theory T, finite modal theory Γ, and a modal formula Φ:

$$\Gamma, T \vdash_{\mathfrak{L}} \Phi \qquad \textit{iff} \qquad \Gamma, T \models_{\mathsf{KF}_{\mathbb{K}_1}^{\mathbb{K}_2}(\mathfrak{L})} \Phi$$

A hint of the proof . . .

Let us fix

- protoalgebraic logics L_i in languages $\mathcal{L}_i$
- an L_2-modal logic $\mathfrak{L}$ be over L_1
- classes $\mathbb{K}_i$ of reduced L_i-matrices, s.t. L_i enjoys $\mathrm{S}\mathbb{K}_i\mathrm{C}$
- a modal theory Γ
- a non-modal theory T
- a modal formula Ψ such that $\Gamma, T \nvdash_{\mathfrak{L}} \Psi$

Translating $\mathfrak{L}$ into L_2

We set $Var_\Box = \{p_\varphi \mid \varphi$ a non-modal formula$\}$ and define:

- $(\Box\varphi)^* = p_\varphi$
- $(c(\Phi_1, \dots, \Phi_n))^* = c(\Phi_1^*, \dots, \Phi_n^*)$, for any n-ary $c \in \mathcal{L}_2$.
- $\Gamma^* = \{\Phi^* \mid \Phi \in \Gamma\}$
- $T^* = \{\Phi^* \mid$ there is a model rule $\langle S, \Phi\rangle$ of $\mathfrak{L}$ s.t. $T \vdash_{\mathrm{L}_1} S\}$

(i.e. T^* consists of $*$-translations conclusions of additional modal rules of $\mathfrak{L}$ with premises provable from T in L_1)

Lemma 14

$$\Gamma, T \vdash_{\mathfrak{L}} \Phi \quad \textit{iff} \quad \Gamma^*, T^* \vdash_{\mathrm{L}_2} \Phi^*$$

Constructing counterexample for $\Gamma, T \nvdash_{\mathfrak{L}} \Psi$

We know that $\Gamma^*, T^* \nvdash_{\mathrm{L}_2} \Psi^*$, then:

- let $\boldsymbol{B}$ be a $\mathbb{K}_2$-algebra and e an $\boldsymbol{B}$-evaluation s.t. $e[\Gamma^*, T^*] \subseteq F_{\boldsymbol{B}}$ and $e[\Psi^*] \notin F_{\boldsymbol{B}}$.
- $W = \{\varphi \mid T \nvdash \varphi\}$
- for each $\varphi \in W$ we take $\mathbb{K}_1$-algebra $\boldsymbol{A}_\varphi$ and an $\boldsymbol{A}_\varphi$-evaluation e_φ s.t. $e_\varphi[T] \subseteq F_{\boldsymbol{A}_\varphi}$ and $e_\varphi(\varphi) \notin F_{\boldsymbol{A}_\varphi}$
- $\mu(\langle a_\varphi \rangle_{\varphi \in W}) = \begin{cases} e(v_\chi) & \text{if } (\exists \chi)(\forall \varphi \in W)(a_\varphi = e_\varphi(\chi)) \\ \text{undefined} & \text{otherwise} \end{cases}$

Proposition 15

$\mathbf{F} = \langle W, \langle \boldsymbol{A}_\varphi \rangle_{\varphi \in W}, \boldsymbol{B}, \mu \rangle$ *is a Kripke frame*

Constructing counterexample for $\Gamma, T \nvdash_{\mathfrak{L}} \Psi$, cont.

Proposition 16

For each Kripke model $\mathbf{M} = \langle \mathbf{F}, \langle \hat{e}_\varphi \rangle_{\varphi \in W} \rangle$ there is a substitution σ such that for each non-modal ψ and modal Ψ:

$$\hat{e}_\varphi(\psi) = e_\varphi(\sigma\psi) \qquad \text{and} \qquad ||\Psi||_{\mathbf{M}} = e((\sigma\Psi)^*)$$

Furthermore, $\mathbf{M}$ is a model of ψ iff $T \vdash_{\mathrm{L}_1} \sigma\psi$

Proposition 17

$\mathbf{F}$ is a Kripke frame for $\mathfrak{L}$

Proof of the completeness theorem.

We know that $\mathbf{F}$ is a Kripke frame for $\mathfrak{L}$ and if we consider Kripke model $\mathbf{M} = \langle \mathbf{F}, \langle e_\varphi \rangle_{\varphi \in W} \rangle$, here the σ of Proposition 16 is the identity and thus $\mathbf{M}$ is a model of Γ, T and not of Ψ.