On deductive systems associated with some equationally orderable quasivarieties

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Outline

- Preliminaries
- Part I: General results
  - Equationally orderable quasivarieties.
  - The deductive system of the order of an equationally orderable quasivariety.
  - When it is congruential (or fully selfextensional).
- Part II: Discussion of some examples.
  - BCK algebras.
  - BCK algebras with infimum and BCK algebras with supremum.
  - Hilbert algebras.
  - Hilbert algebras with infimum and Hilbert algebras with supremum.

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A set $F \subseteq A$ is an $\mathcal{S}$-filter if for every valuation v on **A**, and every $\Gamma \cup \{\varphi\} \subseteq Fm$

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We denote by $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ the set of $\mathcal{S}$-filters of **A** (which is a complete lattice).

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Proposition

A deductive system $\mathcal{S}$ is congruential if and only if for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ the relation $\Lambda_{\mathcal{S}}^{\mathbf{A}}$ is the identity.

Theorem (Font, J. (1996))

Let $\mathcal{S}$ be a deductive system.

1. *If $\mathcal{S}$ has the property of conjunction for a term $\wedge$ and the congruence property, then it is congruential and* **Alg**$\mathcal{S}$ *is a variety.*
2. *If $\mathcal{S}$ has the deduction-detachment theorem for a term $\rightarrow$ and the congruence property, then it is congruential and* **Alg**$\mathcal{S}$ *is a variety.*

In both cases the algebras in **Alg**$\mathcal{S}$ carry an equationally definable partial order, defined by

- $x \wedge y \approx x$, in the first case
- $x \rightarrow y \approx x \rightarrow x$, in the second case.

In the first case the deductive system is given by the order (in a sense we will make precise), but not necessarily in the second.

PART I

General results

Equationally orderable quasivarieties

Equationally orderable quasivarieties

Definition

Let K be a class of algebras of a fixed algebraic similarity type $\mathcal{L}$.
Let $\mu(x, y)$ be a finite set of $\mathcal{L}$-equations in two variables.
We say that K is *μ-equationally orderable*, or *admits a μ-order*, if for every $\mathbf{A} \in \mathsf{K}$

$$\leq_{\mu}^{\mathbf{A}} := \{\langle a, b\rangle \in A^2 : \mathbf{A} \models \mu(x, y)[a, b]\}$$

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We say that K is *properly equationally orderable* if it is μ-equationally orderable for some finite set $\mu(x, y)$ of $\mathcal{L}$-equations different from $\{x \approx y\}$.

Proposition

Let K *be a class of algebras and* $\mu(x, y)$ *a finite set of equations in two variables.* K *is* μ*-equationally orderable if and only if the following holds:*

1. $\models_{\mathsf{K}} \mu(x, x)$,
2. $\mu(x, y) \cup \mu(y, z) \models_{\mathsf{K}} \mu(y, z)$,
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If K *is* μ*-equationally orderable, the quasivariety generated by* K *is also* μ*-equationally orderable.*

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$$\Gamma \vdash_{\mathcal{S}_{\mathsf{Q}}^{\leq\mu}} \varphi \quad \text{iff} \quad \forall \mathbf{A} \in \mathsf{Q}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})\ \forall a \in A$$
$$((\forall \psi \in \Gamma)\ a \leq_{\mu}^{\mathbf{A}} v(\psi)) \Longrightarrow a \leq_{\mu}^{\mathbf{A}} v(\varphi)),$$

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It is easy to check that:

- The relation $\vdash_{\mathcal{S}_Q^{\leq\mu}}$ is a substitution-invariant consequence relation.
- Since Q is closed under ultraproducts and μ is finite, $\vdash_{\mathcal{S}_{Q^{\leq\mu}}}$ is finitary.

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Proposition

$\mathcal{S}_{\mathsf{Q}}^{\leq_\mu}$ has theorems if and only if every $\mathbf{A} \in \mathsf{Q}$ has an upper-bound w.r.t. $\leq_\mu$ and this largest element is term definable.

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Note that

$$\mathsf{Q} \models \varphi \approx \psi \quad \text{iff} \quad \varphi \dashv\vdash_{\mathcal{S}_{\mathsf{Q}}^{\leq_{\mu}}} \psi \quad \text{iff} \quad \psi \dashv\vdash_{\mathcal{S}_{\mathsf{Q}}^{\leq_{\mu^{\partial}}}} \varphi.$$

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Example

Let SL be the variety of semilattices.

SL is $\{x \cdot y \approx x\}$-equationally orderable.

The logic $\mathcal{S}^{\leq}_{\mathsf{SL}}$ is a logic of conjunction and the logic $\mathcal{S}^{\leq^{\partial}}_{\mathsf{SL}}$ is a logic of disjunction. They are different. For example

$$x \cdot y \vdash_{\mathcal{S}^{\leq}_{\mathsf{SL}}} x \quad \text{but} \quad x \cdot y \nvdash_{\mathcal{S}^{\leq^{\partial}}_{\mathsf{SL}}} x.$$

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Let Q be a μ-equationally orderable quasivariety and let $\mathbf{A} \in \mathsf{Q}$. Then every down-directed up-set $F \subseteq A$ of the poset $\langle A, \leq^{\mathbf{A}}_{\mu} \rangle$ is an $\mathcal{S}^{\leq}_{\mathsf{Q}}$-filter.

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Thus,

$$\{\langle \mathbf{A}, F \rangle : \mathbf{A} \in \mathsf{Q} \text{ and } F \text{ is a downdirected up-set}\}$$

is a matrix semantics for $\mathcal{S}^{\leq_{\mu}}_{\mathsf{Q}}$.

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$$a = b \quad \text{iff} \quad \forall F \in \mathrm{Fi}_{\mathcal{S}^{\leq}_{\mathsf{Q}}}\mathbf{A}(a \in F \Leftrightarrow b \in F) \quad \text{iff} \quad \langle a, b \rangle \in \Lambda^{\mathbf{A}}_{\mathcal{S}^{\leq}_{\mathsf{Q}}}.$$

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▶ Let Q be a μ-equationally orderable quasivariety. Then for every $\mathbf{A} \in \mathsf{Q}$ and every $a, b \in A$,

$$a = b \quad \text{iff} \quad \forall F \in \mathrm{Fi}_{\mathcal{S}_{\mathsf{Q}}^{\leq}}\mathbf{A}(a \in F \Leftrightarrow b \in F) \quad \text{iff} \quad \langle a, b \rangle \in \Lambda_{\mathcal{S}_{\mathsf{Q}}^{\leq}}^{\mathbf{A}}.$$

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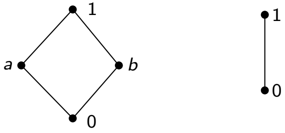
Then $\mathbf{A}^c_P \in \mathsf{Q}_{\rightarrow}$.

Let L and L' be the following posets (bounded lattices).



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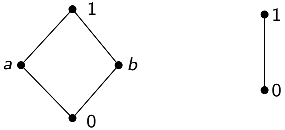

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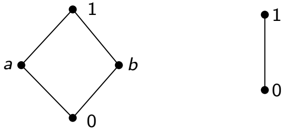

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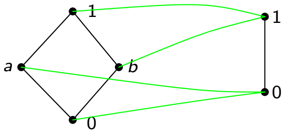

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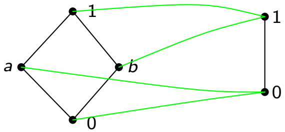

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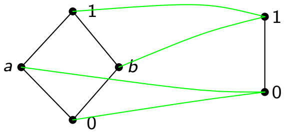

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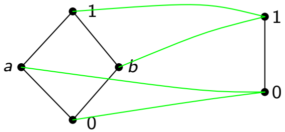

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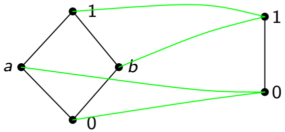

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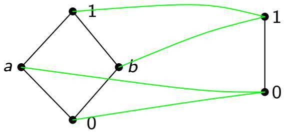

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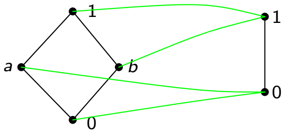

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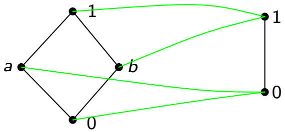

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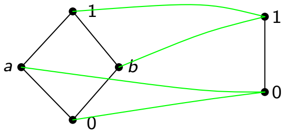

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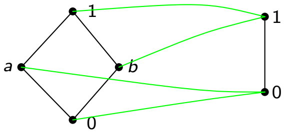

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On congruential deductive systems

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If Q *is a* μ*-equationally orderable quasivariety and* $\mathbf{Alg}\mathcal{S}^{\leq}_{\mathsf{Q}} = \mathsf{Q}$, *then* $\mathcal{S}^{\leq}_{\mathsf{Q}}$ *is congruential.*

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Proof.

The *intrinsic variety* of $\mathcal{S}_{\mathsf{Q}}^{\leq}$ is the variety $\mathsf{V}(\mathcal{S}_{\mathsf{Q}}^{\leq})$ axiomatized by the equations $\varphi \approx \psi$ such that

$$\varphi \dashv_{\mathcal{S}_{\mathsf{Q}}^{\leq}} \vdash \psi.$$

We recall: $\mathsf{Q} \models \varphi \approx \psi$ iff $\varphi \dashv_{\mathcal{S}_{\mathsf{Q}}^{\leq}} \vdash \psi$.

Therefore $\mathsf{V}(\mathcal{S}_{\mathsf{Q}}^{\leq})$ is the variety generated by Q. Also $\mathsf{V}(\mathcal{S}_{\mathsf{Q}}^{\leq})$ is the variety generated by $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq}$. Since $\mathsf{Q} \subseteq \mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq}$, Q and $\mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq}$ generate the same variety. $\square$

As a corollary:

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If Q *is a* μ*-equationally orderable variety, then* $\mathcal{S}^{\leq}_{\mathsf{Q}}$ *and* $\mathcal{S}^{\leq^{\partial}}_{\mathsf{Q}}$ *are congruential.*

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Theorem

If Q is a μ-equationally orderable variety, then $\mathcal{S}_{\mathsf{Q}}^{\leq}$ and ${\mathcal{S}_{\mathsf{Q}}^{\leq}}^{\partial}$ are congruential.

There exists a μ-equationally orderable quasivariety Q such that

- Q is not a variety,
- $\mathsf{Q} \subsetneq \mathbf{Alg}\mathcal{S}_{\mathsf{Q}}^{\leq}$,
- $\mathcal{S}_{\mathsf{Q}}^{\leq}$ congruential.

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The example discussed to show that $\mathsf{Q}_{\rightarrow}$ is not a variety shows also this.

Let $\mathsf{Q}_{\to}^{\wedge}$ be the quasivariety of algebras $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$, in the language $\{\to, \wedge, 1\}$, such that

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Open problem: In general, if $\mathcal{S}^{\leq}_{\mathsf{Q}}$ is congruential, is $\mathbf{Alg}\mathcal{S}^{\leq}_{\mathsf{Q}}$ a variety?

PART II

Discussion of some examples:
BCK algebras and Hilbert algebras, possibly with extra lattice operations.

BCK algebras

Definition

The quasivariety BCK of *BCK-algebras* is axiomatized by the following equations and quasiequation:

1. $(x \to y) \to ((y \to z) \to (x \to z)) \approx 1$,
2. $x \to x \approx 1$,
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- BCK is $\{x \to y \approx 1\}$-equationally orderable.

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- $\mathcal{S}^{\leq}_{\mathrm{BCK}}$ is not protoalgebraic and has theorems.

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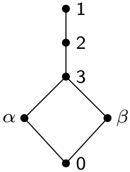
- $\mathcal{S}^{\leq}_{\mathsf{BCK}}$ is not protoalgebraic and has theorems.
- $\mathcal{S}^{\leq^{\partial}}_{\mathsf{BCK}}$ does not have theorems.
- The three logics $\mathcal{S}^{1}_{\mathsf{BCK}}$, $\mathcal{S}^{\leq}_{\mathsf{BCK}}$, $\mathcal{S}^{\leq^{\partial}}_{\mathsf{BCK}}$ are different.

Let us consider the BCK algebra we obtain by defining in the lattice below the operation $\to$ by the next table



$\to$	0	3	2	1	α	β
0	1	1	1	1	1	1
3	2	1	1	1	2	2
2	3	2	1	1	2	2
1	0	3	2	1	α	β
α	2	1	1	1	1	2
β	2	1	1	1	2	1

The principal up-sets are obviously $\mathcal{S}^{\leq}_{\mathsf{BCK}}$-filters.
Let

$$F = \{2, 1\} \subseteq G = \{\beta, 3, 2, 1\}.$$

It is not difficult to see that

$$\langle \alpha, \beta \rangle \in \mathbf{\Omega}(F), \text{ but } \langle \alpha, \beta \rangle \notin \mathbf{\Omega}(G).$$

Thus $\mathcal{S}^{\leq}_{\mathsf{BCK}}$ is not ptrotoalgebraic.

Problems.

- Are $\mathcal{S}^{\leq}_{\mathsf{BCK}}$ and $\mathcal{S}^{\leq^{\partial}}_{\mathsf{BCK}}$ congruential ?
- Is $\mathbf{Alg}\mathcal{S}^{\leq}_{\mathsf{BCK}} = \mathsf{BCK}$?
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An algebra $\mathbf{A} = \langle A, \to, \wedge, 1 \rangle$ in the language $\{\to, \wedge, 1\}$ is a BCK meet-semilattice if $\langle A, \to, 1 \rangle$ is a BCK algebra and the following equations and quasiequation are valid on $\mathbf{A}$

1. $(x \wedge y) \to x \approx 1$,
2. $(x \wedge y) \to y \approx 1$,
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In a dual way, we have BCK-join-semilattices. They also form a variety (P. Idziak). Thus the logic of the order is congruential as well as the logic of the dual order.

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Hilbert algebras

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These facts imply:

- $\mathcal{S}^1_{\mathsf{H}}$ is congruential.

- $\mathcal{S}^1_{\mathsf{H}}$ = the deductive system $\mathcal{S}^{\rightarrow}_{\mathsf{H}}$ associated with H by the following definition

$$\Gamma \vdash_{\mathcal{S}^{\rightarrow}_{\mathsf{K}}} \varphi \quad \text{iff} \quad (\forall \mathbf{A} \in \mathsf{K})(\forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}))\ v(\varphi) = 1^{\mathbf{A}} \text{ or}$$

$$(\exists \varphi_0, \ldots, \varphi_n \in \Gamma)(\forall \mathbf{A} \in \mathsf{K})(\forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}))$$
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Proposition

The deductive system $\mathcal{S}^{\leq}_{\mathsf{H}}$ enjoys Modus Ponens for $\to$.

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Proof.

Let **A** be any Hilbert algebra. We show that for every $a, b, c \in A$

$$a \leq b \ \& \ a \leq b \to c \Longrightarrow a \leq c.$$

This implies that $\mathcal{S}_{\mathsf{H}}^{\leq}$ enjoys Modus Ponens for $\to$.
Suppose that $a \leq b$ and $a \leq b \to c$. Then $a \to b = 1$ and $a \to (b \to c) = 1$. Therefore, $(a \to b) \to (a \to c) = 1$; hence, $1 \to (a \to c) = 1$. This implies that $a \to c = 1$ and so $a \leq c$. □

Proposition

The deductive system $\mathcal{S}_{\mathsf{H}}^{\leq}$ *is equal to* $\mathcal{S}_{\mathsf{H}}^{1}$.

Proof.

$\mathcal{S}_{\mathsf{H}}^{1}$ is an extension of $\mathcal{S}_{\mathsf{H}}^{\leq}$ with the same theorems, because 1 is a maximum element in every Hilbert algebra. Now note that all axioms of the axiomatization of $\mathcal{S}_{\mathsf{H}}^{1}$ are theorems of $\mathcal{S}_{\mathsf{H}}^{\leq}$. Since Modus Ponens is valid in $\mathcal{S}_{\mathsf{H}}^{\leq}$, $\mathcal{S}_{\mathsf{H}}^{\leq}$ is an extension of $\mathcal{S}_{\mathsf{H}}^{1}$. Thus $\mathcal{S}_{\mathsf{H}}^{\leq} = \mathcal{S}_{\mathsf{H}}^{1}$. □

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Therefore,

Proposition

The deductive system $\mathcal{S}_{\mathsf{H}}^{1}$ is congruential.

Hilbert algebras with supremum

Definition

A Hilbert algebra with supremum is an algebra $\mathbf{A} = \langle A, \to, \vee, 1 \rangle$ in the language $\{\to, \wedge, 1\}$ which is a BCK-join-semilattice such that $\langle A, \to, 1 \rangle$ is a Hilbert algebra.

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- As with Hilbert algebras, $\mathcal{S}^{\leq}_{\mathsf{H}^{\vee}}$, $\mathcal{S}^{1}_{\mathsf{H}^{\vee}}$ and $\mathcal{S}^{\to}_{\mathsf{H}^{\vee}}$ are equal and congruential.

Hilbert algebras with infimum

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A Hilbert algebra with infimum is an algebra $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ in the language $\{\to, \wedge, 1\}$ which is a BCK-meet-semilattice such that $\langle A, \to, 1\rangle$ is a Hilbert algebra.

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Hilbert algebras with infimum

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A Hilbert algebra with infimum is an algebra $\mathbf{A} = \langle A, \rightarrow, \wedge, 1 \rangle$ in the language $\{\rightarrow, \wedge, 1\}$ which is a BCK-meet-semilattice such that $\langle A, \rightarrow, 1 \rangle$ is a Hilbert algebra.

- The class of Hilbert algebras with infimum $\mathsf{H}^{\wedge}$ is a variety.
- We have four deductive systems.
  - The 1-assertional logic $\mathcal{S}^{1}_{\mathsf{H}^{\wedge}}$.
  - The logic $\mathcal{S}^{\leq}_{\mathsf{H}^{\wedge}}$ of the $\{x \rightarrow y \approx 1\}$-order of $\mathsf{H}^{\wedge}$.

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  - The logic $\mathcal{S}^{\to}_{\mathsf{H}^{\wedge}}$ defined by

$$\Gamma \vdash_{\mathcal{S}^{\to}_{\mathsf{H}^{\wedge}}} \varphi \quad \text{iff} \quad (\forall \mathbf{A} \in \mathsf{H}^{\wedge})(\forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}))\ v(\varphi) = 1^{\mathbf{A}} \text{ or}$$

$$(\exists \varphi_0, \ldots, \varphi_n \in \Gamma)(\forall \mathbf{A} \in \mathsf{K})(\forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}))$$

$$v(\varphi_0 \to (\ldots(\varphi_n \to \varphi)\ldots)) = 1^{\mathbf{A}}.$$

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1. 1,
2. $\varphi \to (\psi \to \varphi)$
3. $(\varphi \to (\psi \to \delta)) \to ((\varphi \to \psi) \to (\varphi \to \delta))$,
4. $(\varphi \wedge \psi) \to \psi$,
5. $(\varphi \wedge (\varphi \to \psi)) \to \psi$,
6. $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$,
7. $((\varphi \wedge \psi) \wedge \delta) \to ((\varphi \wedge \delta) \wedge \psi)$,

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5. $(\varphi \wedge (\varphi \to \psi)) \to \psi$,
6. $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$,
7. $((\varphi \wedge \psi) \wedge \delta) \to ((\varphi \wedge \delta) \wedge \psi)$,

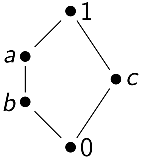
and the rules

$$\frac{\varphi, \varphi \to \psi}{\psi} \quad (MP) \qquad\qquad \frac{\varphi \to \psi}{\varphi \to (\varphi \wedge \psi)} \quad (AB).$$

Proposition

The four deductive systems $\mathcal{S}^1_{\mathsf{H}^\wedge}$, $\mathcal{S}^{\to}_{\mathsf{H}^\wedge}$, $\mathcal{S}^{\leq}_{\mathsf{H}^\wedge}$ and $\mathcal{S}^{\leq^\partial}_{\mathsf{H}^\wedge}$ are different.

- $\mathcal{S}^1_{\mathsf{H}^\wedge}$, $\mathcal{S}^{\to}_{\mathsf{H}^\wedge}$ and $\mathcal{S}^{\leq}_{\mathsf{H}^\wedge}$ have the same theorems.
- $\mathcal{S}^{\leq^\partial}_{\mathsf{H}^\wedge}$ does not have theorems.
- $\mathcal{S}^1_{\mathsf{H}^\wedge}$ does not have the deduction theorem for $\to$. If it would have it, since $p, q \vdash_{\mathcal{S}^1_{\mathsf{H}^\wedge}} p \wedge q$, it would follow that $\vdash_{\mathcal{S}^1_{\mathsf{H}^\wedge}} p \to (q \to (p \wedge q))$. But this is not a theorem of $\mathcal{S}^1_{\mathsf{H}^\wedge}$. In the Hilbert algebra with infimum given by the lattice



and $\to$ defined by setting

$$x \to y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases}$$

we have $b \to^{\mathbf{A}} (c \to^{\mathbf{A}} (b \wedge^{\mathbf{A}} c)) = 0$.

- $\mathcal{S}^{\rightarrow}_{\mathsf{H}^{\wedge}}$ has the deduction-detachment theorem for $\rightarrow$. All logics defined from a quasivariety of algebras with a Hilbert algebra reduct using the schema of definition we used to define $\mathcal{S}^{\rightarrow}_{\mathsf{H}^{\wedge}}$ have it.
- $\mathcal{S}^{\leq}_{\mathsf{H}^{\wedge}}$ does not have the deduction-detachment therorem for $\rightarrow$. If it had it, then every $\mathbf{A} \in \mathsf{H}^{\wedge}$ would be an implicative semilattice.
- The rule (AB) does not hold for $\mathcal{S}^{\leq}_{\mathsf{H}^{\wedge}}$. In the algebra Go holds:

  $c \rightarrow b = b$,
  $c \rightarrow (c \wedge b) = c \rightarrow 0 = 0$, but
  $b \nleq 0$.

CONJECTURE: $\mathcal{S}^{\leq}_{\mathsf{H}^{\wedge}}$ is not protoalgebraic.