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CO-ROTATION,  
CO-ROTATION-ANNIHILATION,  
AND INVOLUTIVE ORDINAL  
SUM CONSTRUCTIONS OF  
RESIDUATED SEMIGROUPS

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SROP-4.2.2.C-11/1/  
KONV-2012-0005 grant.

# DEFINITIONS

- *FL<sub>e</sub>-algebra* = comm. RL +  $f$ ,  
 $f$  is an arbitrary constant
- *involutive* =  $x'' = x$ , where  $x' = x \rightarrow f$   
(observe  $f = t$ )
- *integral* =  $t$  is its greatest element
- *Group-like* = involutive +  $f = t$

# CONIC REPRESENTATION

- Conic representation: For any conic, involutive  $\text{FL}_e$ -algebra

$$x * y = \begin{cases} x \otimes y & \text{if } x, y \in X_1 \\ x \oplus y & \text{if } x, y \in X_2 \\ (x \rightarrow_{\oplus} y')' & \text{if } x \in X_2, y \in X_1, \text{ and } x \leq y' \\ (y \rightarrow_{\oplus} x')' & \text{if } x \in X_1, y \in X_2, \text{ and } x \leq y' \\ (y \rightarrow_{\otimes} (x' \wedge t))' & \text{if } x \in X_2, y \in X_1, \text{ and } x \not\leq y' \\ (x \rightarrow_{\otimes} (y' \wedge t))' & \text{if } x \in X_1, y \in X_2, \text{ and } x \not\leq y' \end{cases}$$

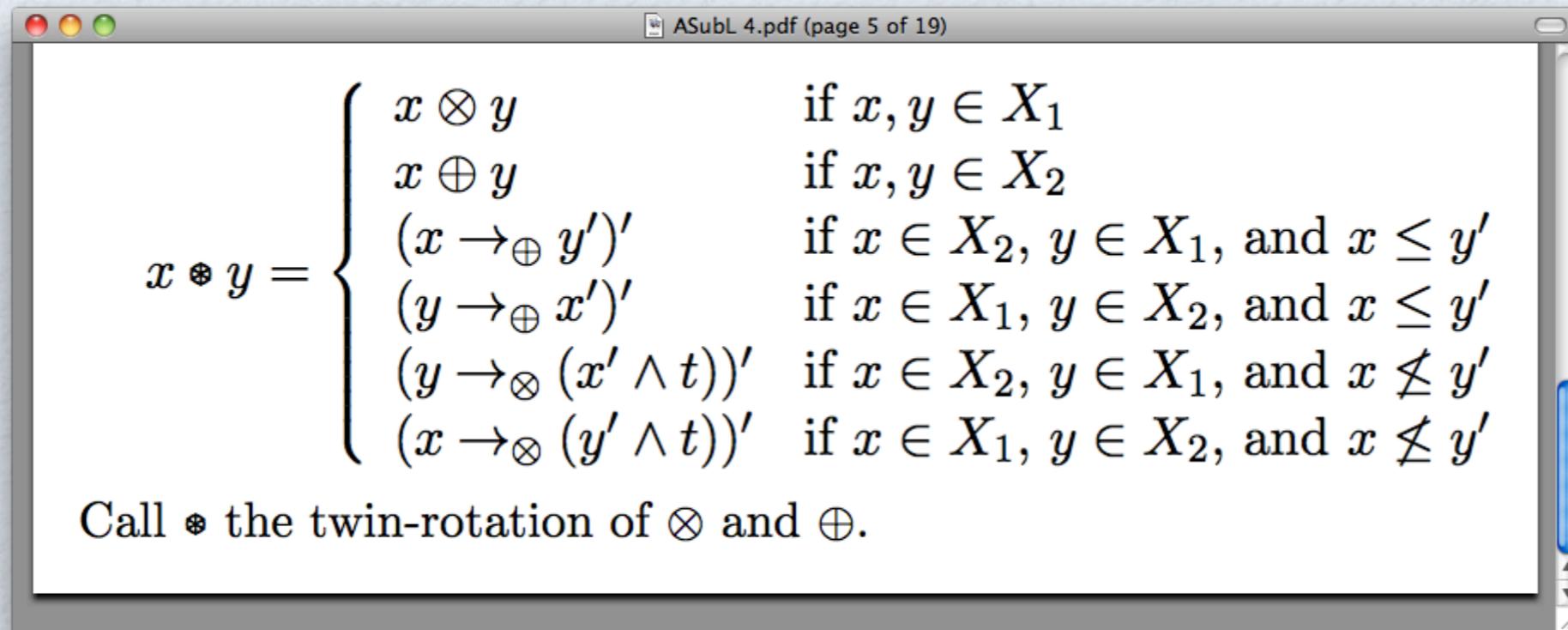
- [S. Jenei, H. Ono, On Involutive  $\text{FL}_e$ -monoids, Archive for Mathematical Logic, 51 (7-8), 719-738 (2012)]

# TWIN ROTATION

$$x \otimes y = \begin{cases} x \otimes y & \text{if } x, y \in X_1 \\ x \oplus y & \text{if } x, y \in X_2 \\ (x \rightarrow_{\oplus} y')' & \text{if } x \in X_2, y \in X_1, \text{ and } x \leq y' \\ (y \rightarrow_{\oplus} x')' & \text{if } x \in X_1, y \in X_2, \text{ and } x \leq y' \\ (y \rightarrow_{\otimes} (x' \wedge t))' & \text{if } x \in X_2, y \in X_1, \text{ and } x \not\leq y' \\ (x \rightarrow_{\otimes} (y' \wedge t))' & \text{if } x \in X_1, y \in X_2, \text{ and } x \not\leq y' \end{cases}$$

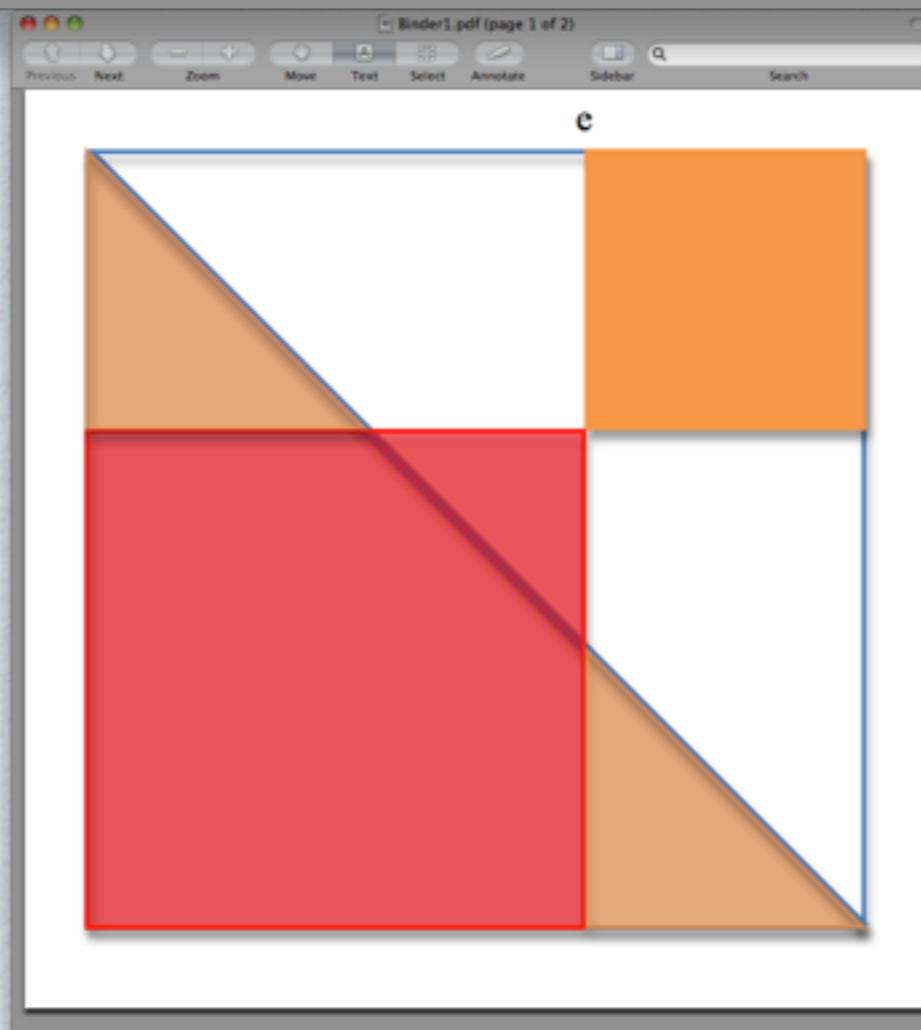
Call  $\otimes$  the twin-rotation of  $\otimes$  and  $\oplus$ .

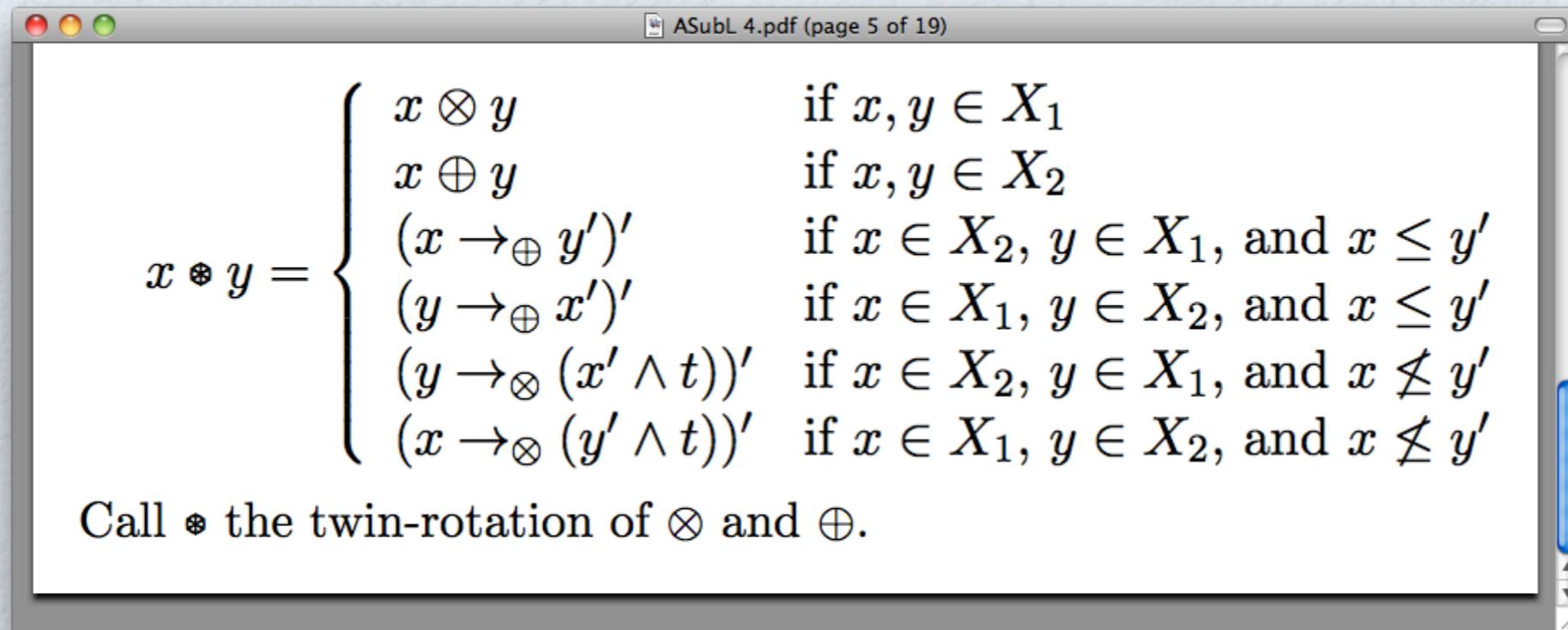
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ASubL 4.pdf (page 5 of 19)

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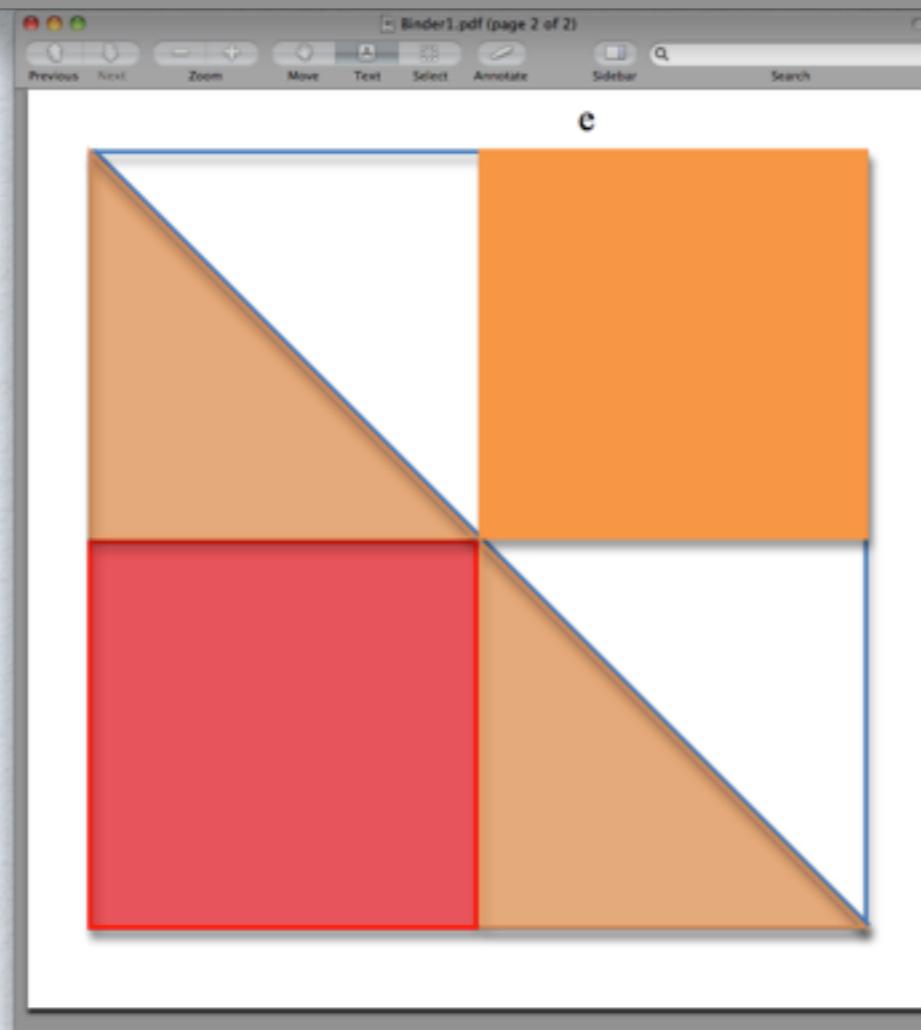
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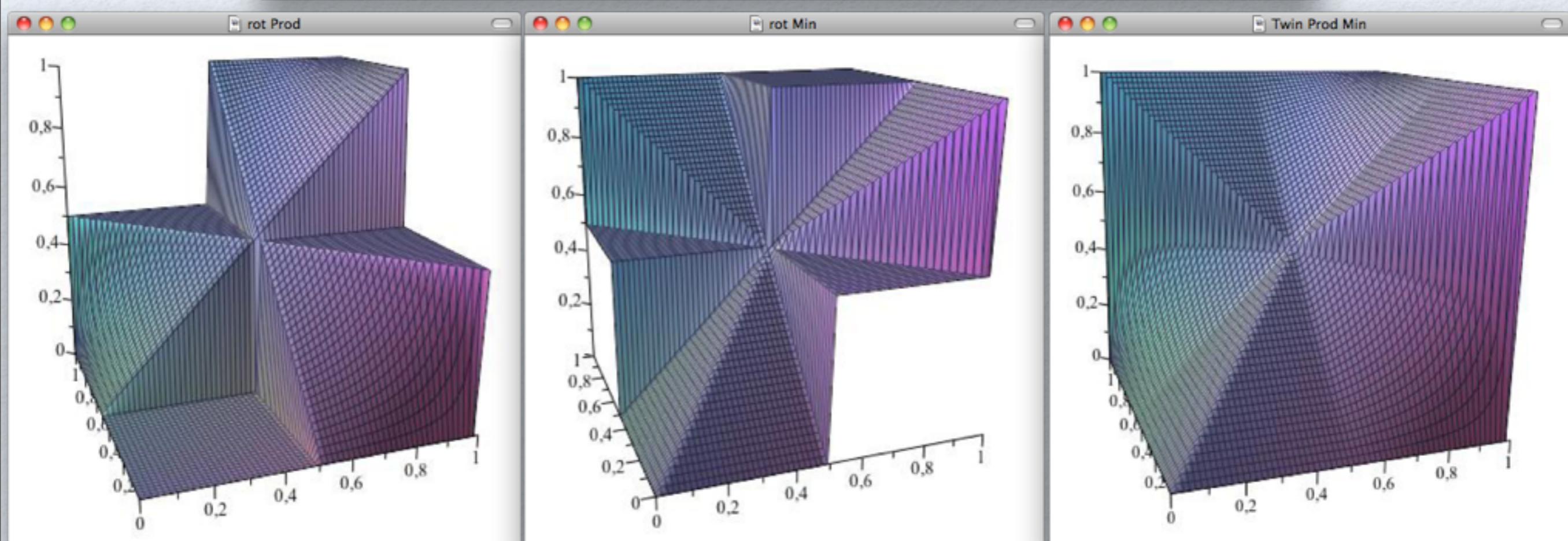
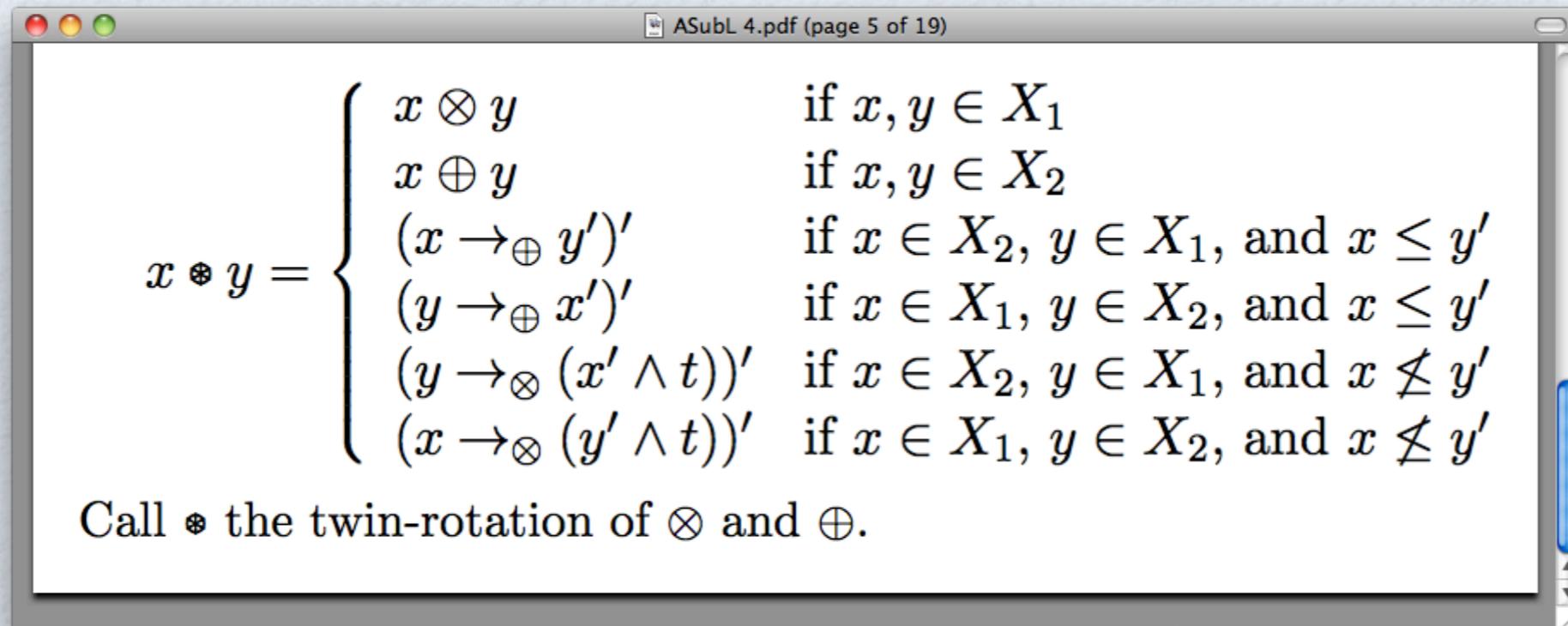


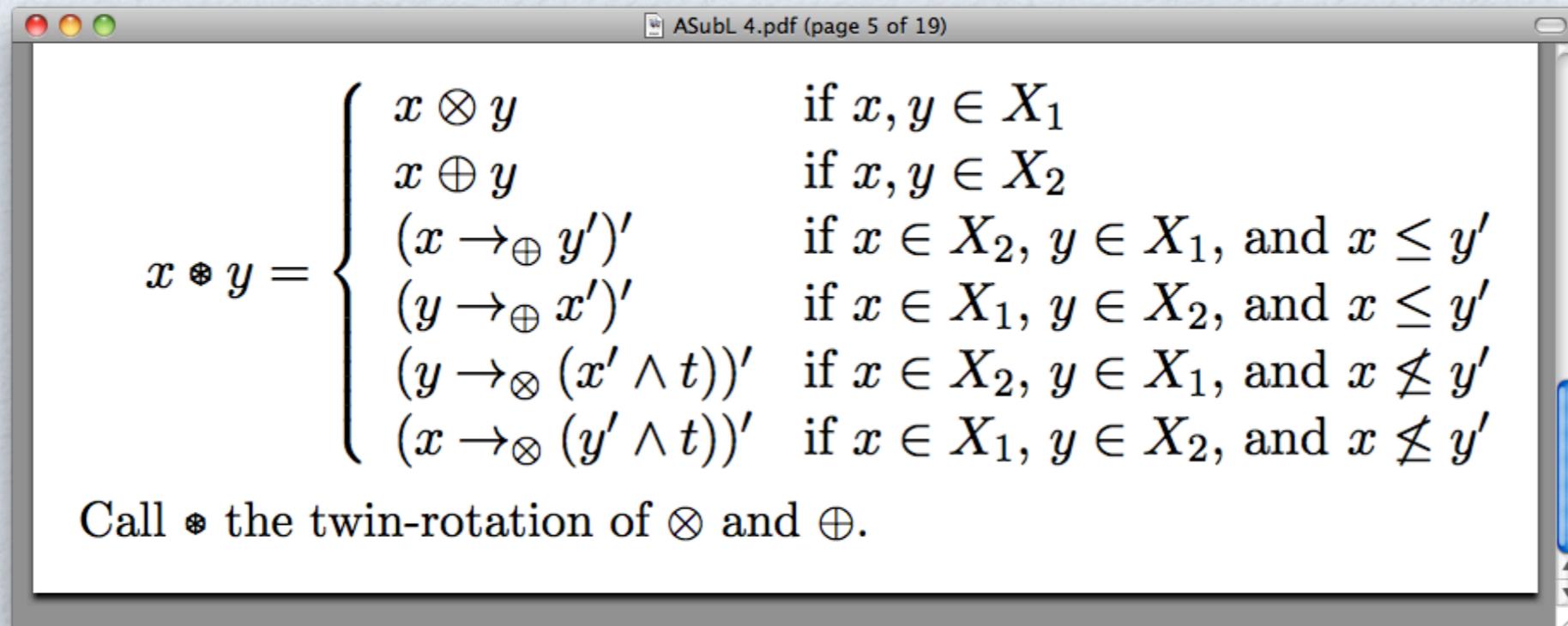
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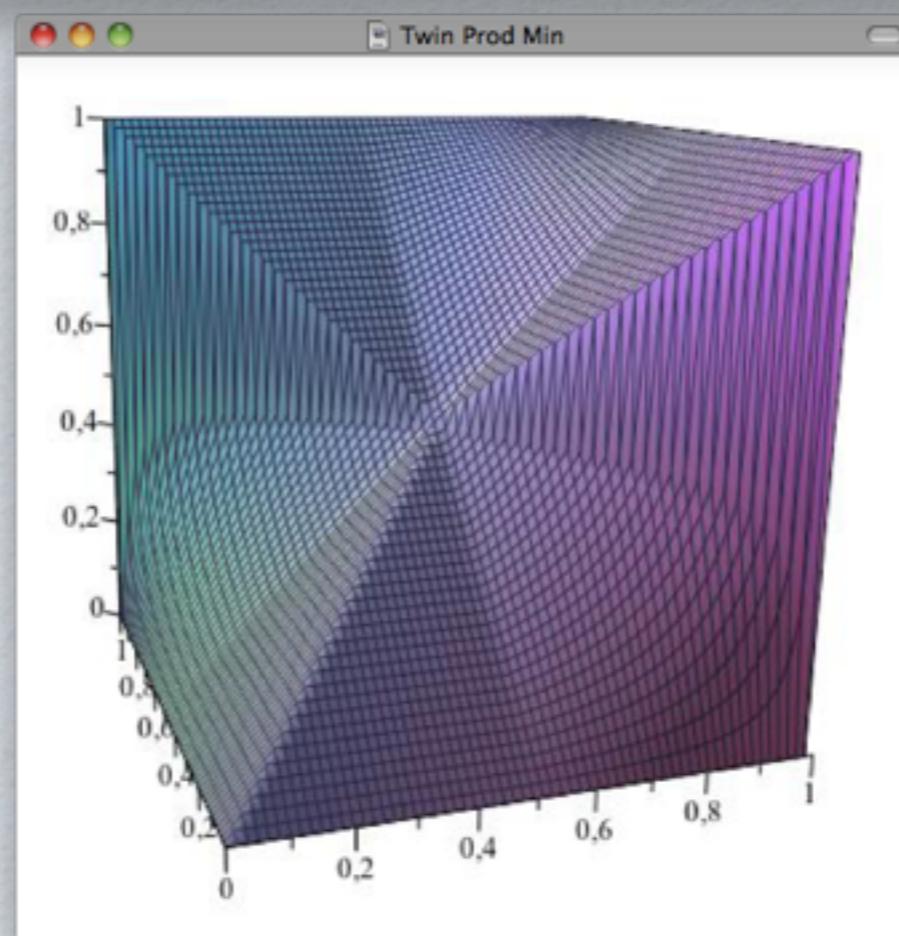




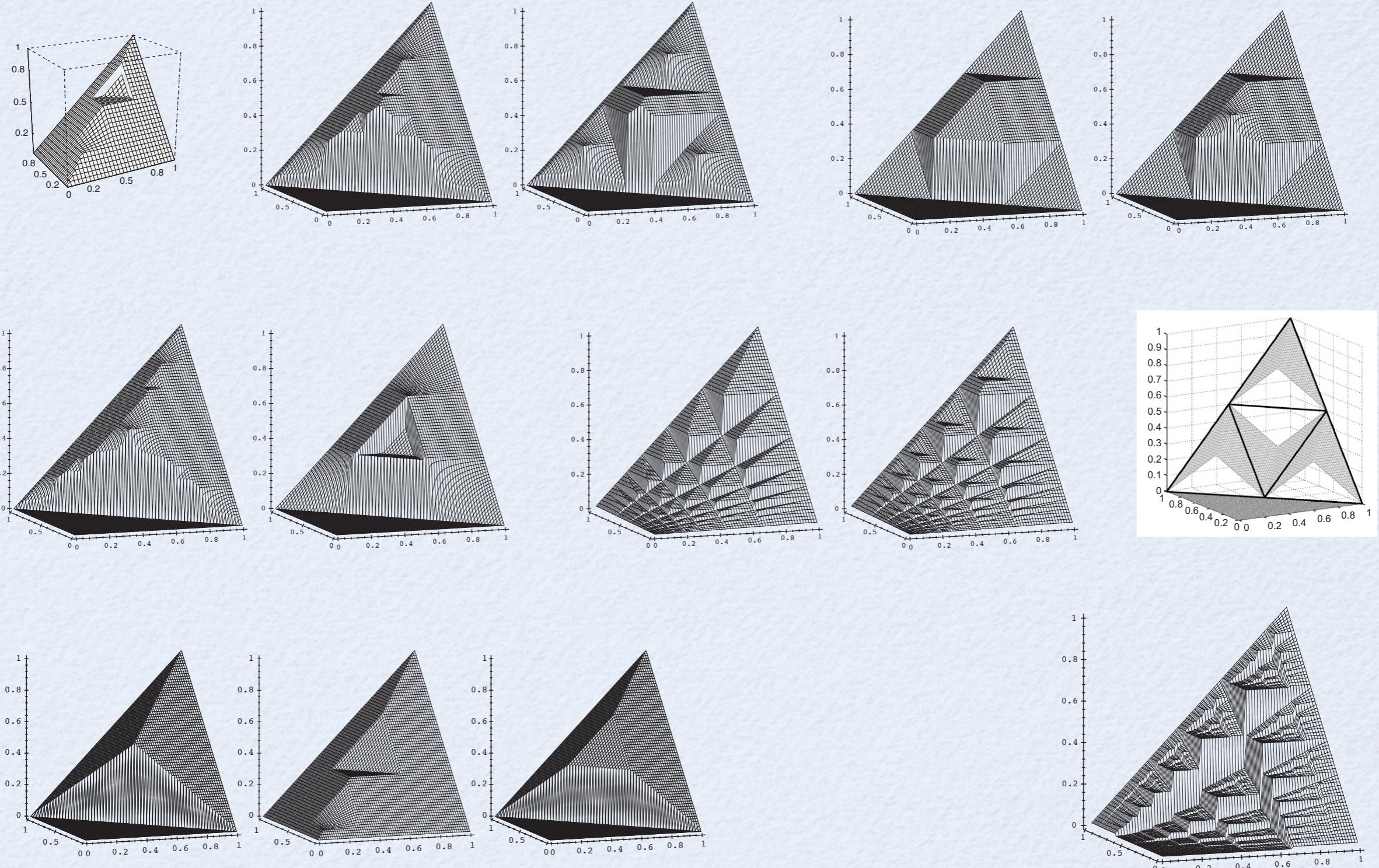
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$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \in X_1 \\ x \oplus y & \text{if } x, y \in X_2 \\ (x \rightarrow_{\oplus} y')' & \text{if } x \in X_2, y \in X_1, \text{ and } x \leq y' \\ (y \rightarrow_{\oplus} x')' & \text{if } x \in X_1, y \in X_2, \text{ and } x \leq y' \\ (y \rightarrow_{\otimes} (x' \wedge t))' & \text{if } x \in X_2, y \in X_1, \text{ and } x \not\leq y' \\ (x \rightarrow_{\otimes} (y' \wedge t))' & \text{if } x \in X_1, y \in X_2, \text{ and } x \not\leq y' \end{cases}$$

Call  $\circledast$  the twin-rotation of  $\otimes$  and  $\oplus$ .

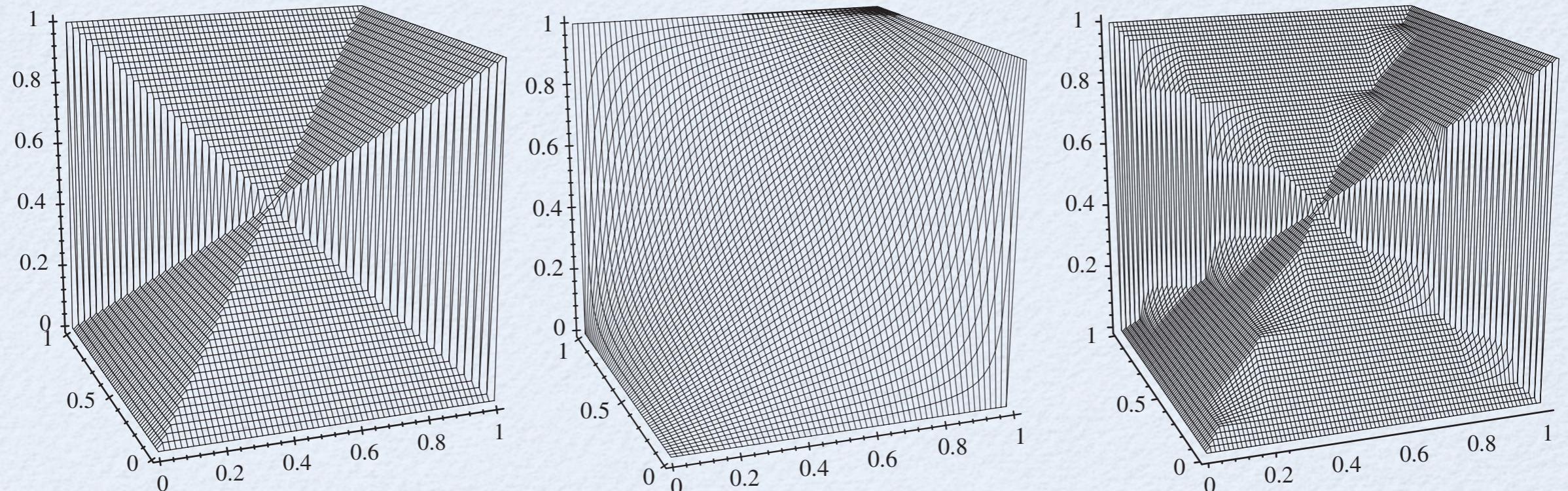


# AN UNCHARTABLE WILDERNESS



# GROUP-LIKE CASE

# ABSORBENT-CONTINUOUS GROUP-LIKE FL<sub>e</sub>-ALGEBRAS ON SUBREAL CHAINS



- [S. Jenei, F. Montagna, A classification of certain group-like FL<sub>e</sub>-chains, submitted]

# ABSORBENT-CONTINUOUS GROUP-LIKE FL<sub>e</sub>-ALGEBRAS ON SUBREAL CHAINS

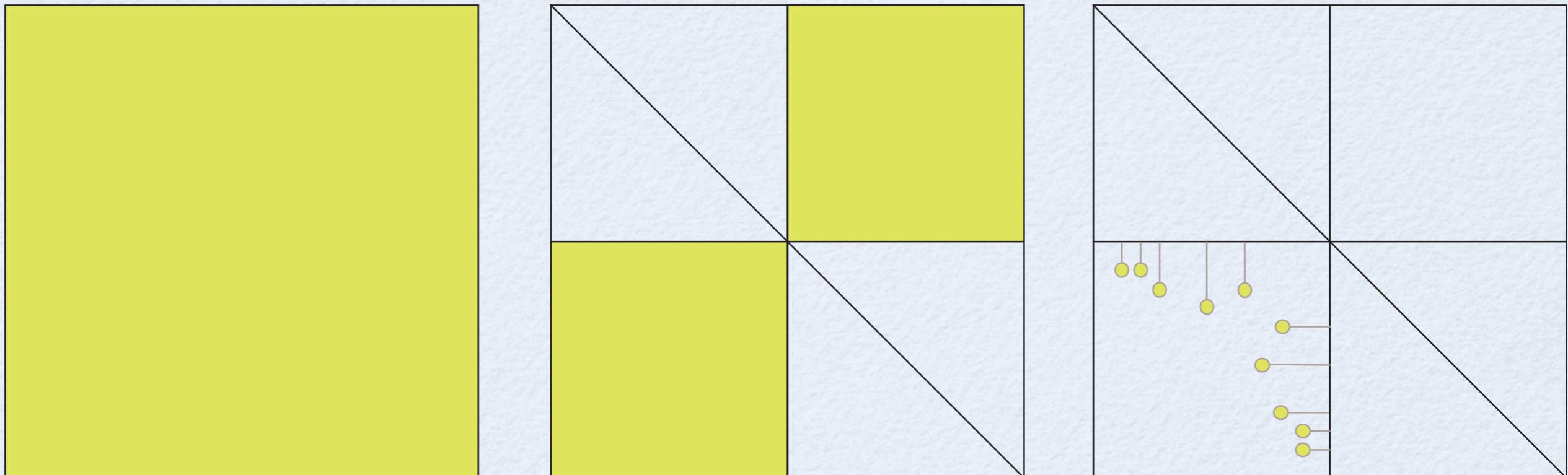
- Call a chain  $\langle X, \leq \rangle$  *weakly real* if  $X$  is order-dense and complete, there exists a dense  $Y \subset X$  with  $|Y| < |X|$ , and for any  $x, y \in Y$  there exist  $u, v \in Y$  such that  $u > x, v > y$ , and there exists a strictly increasing function from  $[x, u]$  into  $[y, v]$ .
- An order dense chain is said to be *subreal* if its Dedekind-MacNeille completion is weakly real.
- Absorbent continuity = for  $x \in X^-$ ,  
 $a(x) \otimes x = x$ , where  $a(x) = \inf\{u \in X^- : u \otimes x = x\}$

# ABSORBENT-CONTINUOUS GROUP-LIKE FL<sub>e</sub>-ALGEBRAS ON SUBREAL CHAINS

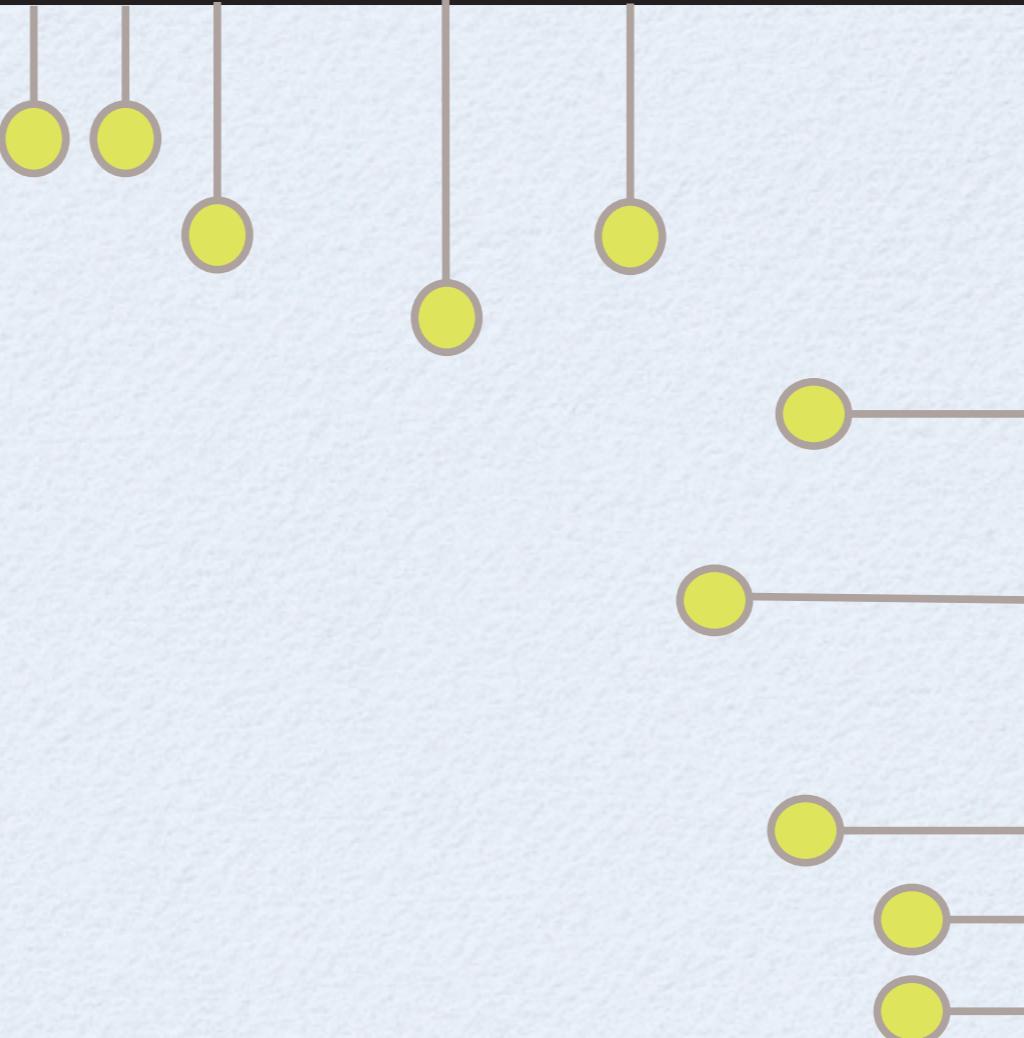
- BL-algebras = divisibility (continuity) **everywhere**
- Absorbent continuity = continuity only **at a few point** of the domain of  $\otimes$  (viewed as a two-place function)

# ABSORBENT-CONTINUOUS GROUP-LIKE FL<sub>e</sub>-ALGEBRAS ON SUBREAL CHAINS

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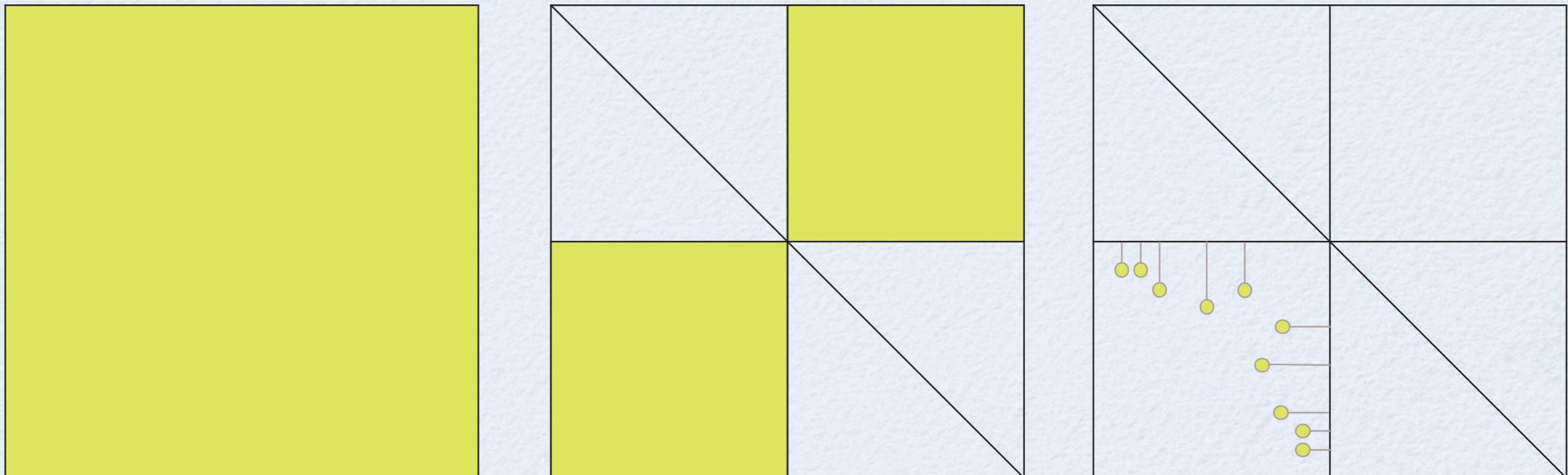


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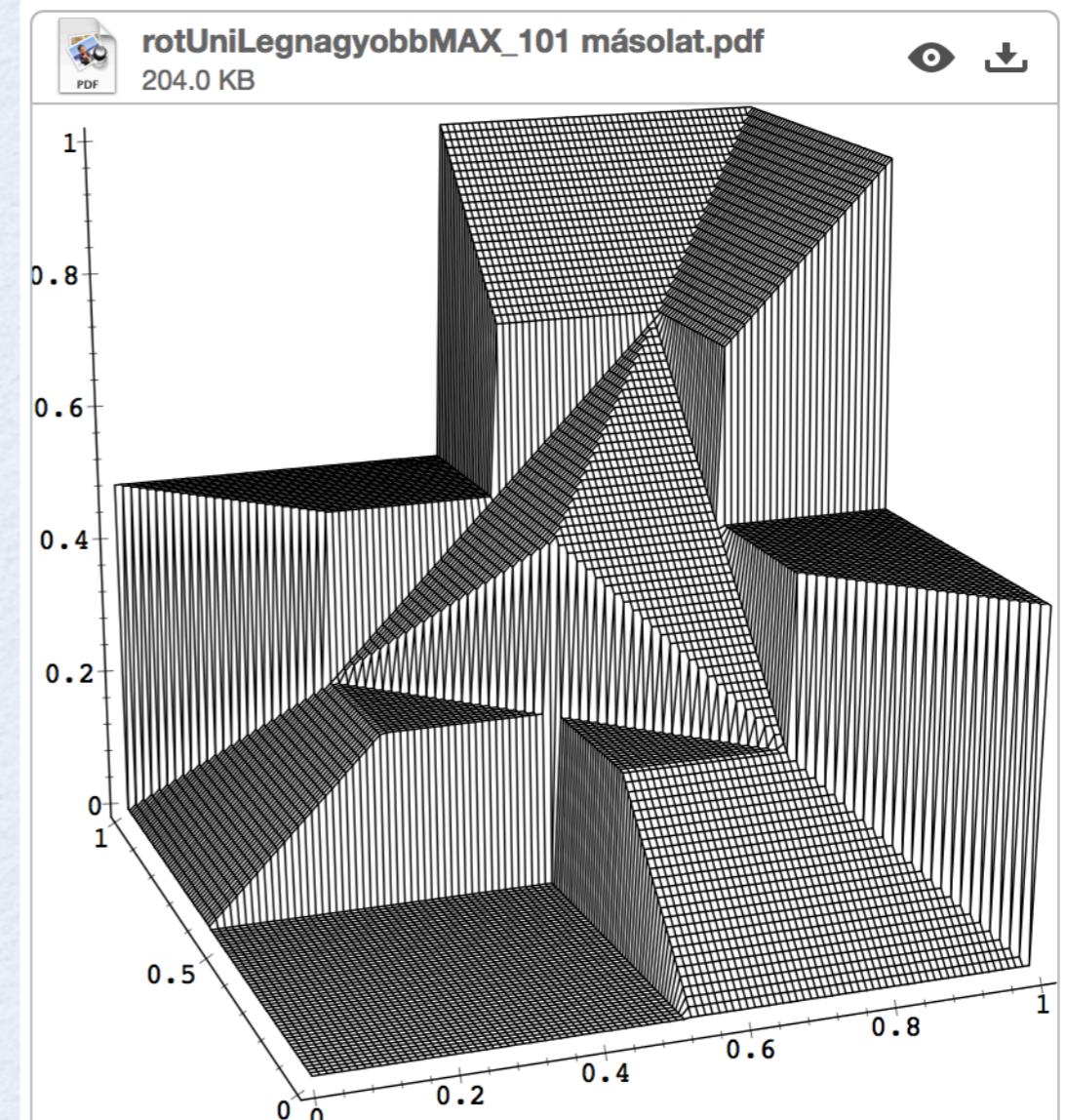
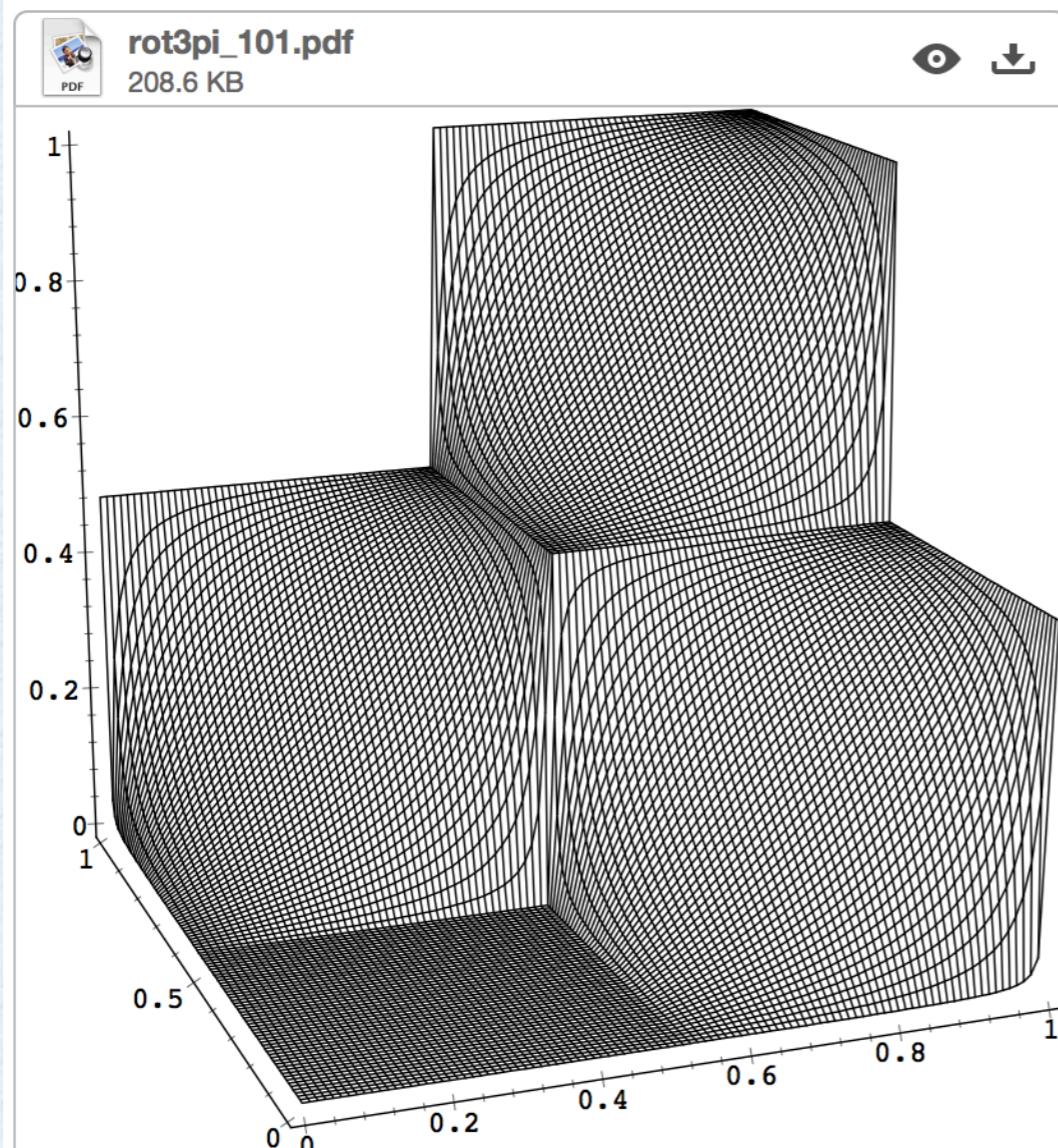
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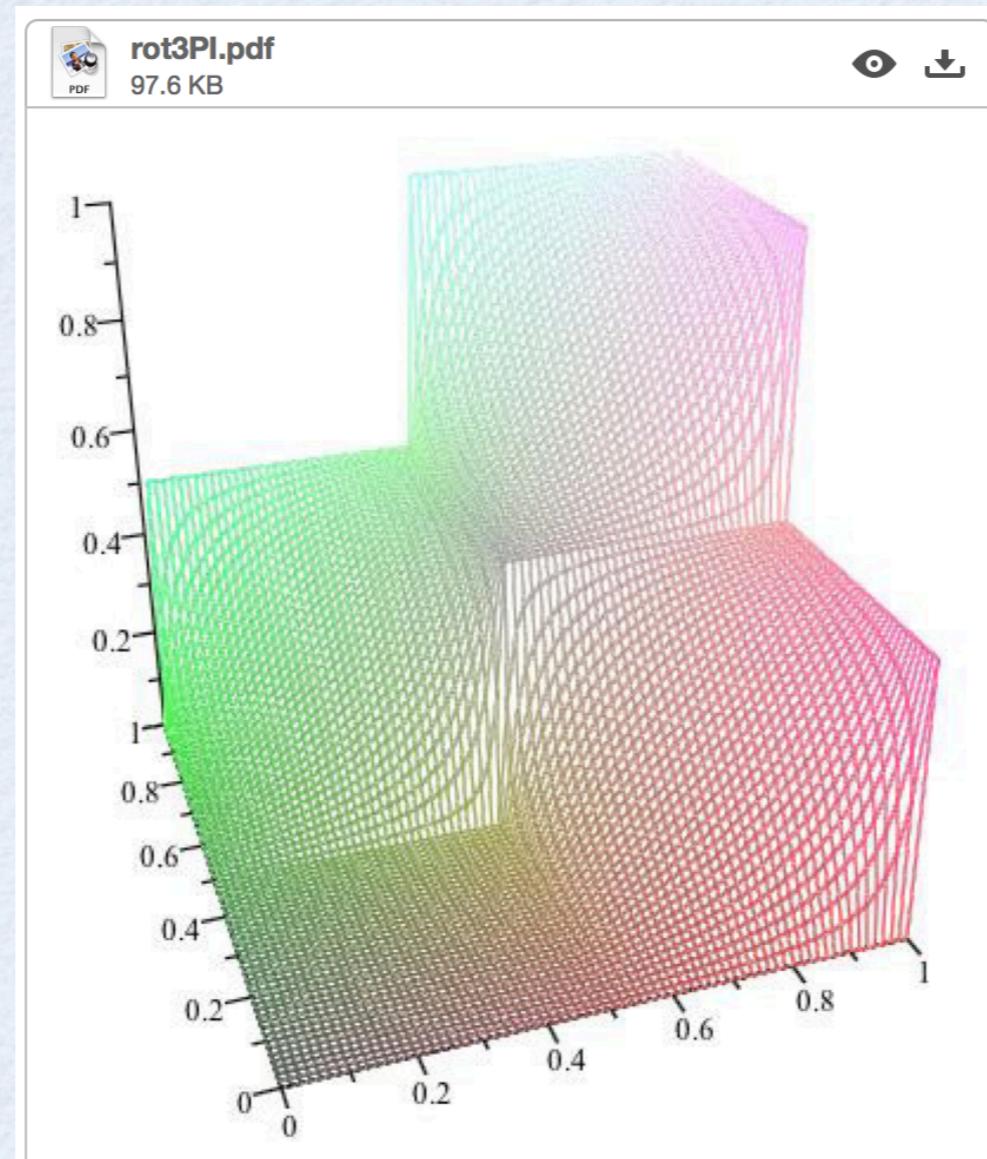
# I: INVOLUTIVE ORDINAL SUMS

- Theorem: The twin-rotation of the Clifford-style ordinal sum of any family of negative cones of group-like  $FL_e$ -chains and their skew-duals is a group-like  $FL_e$ -chain.

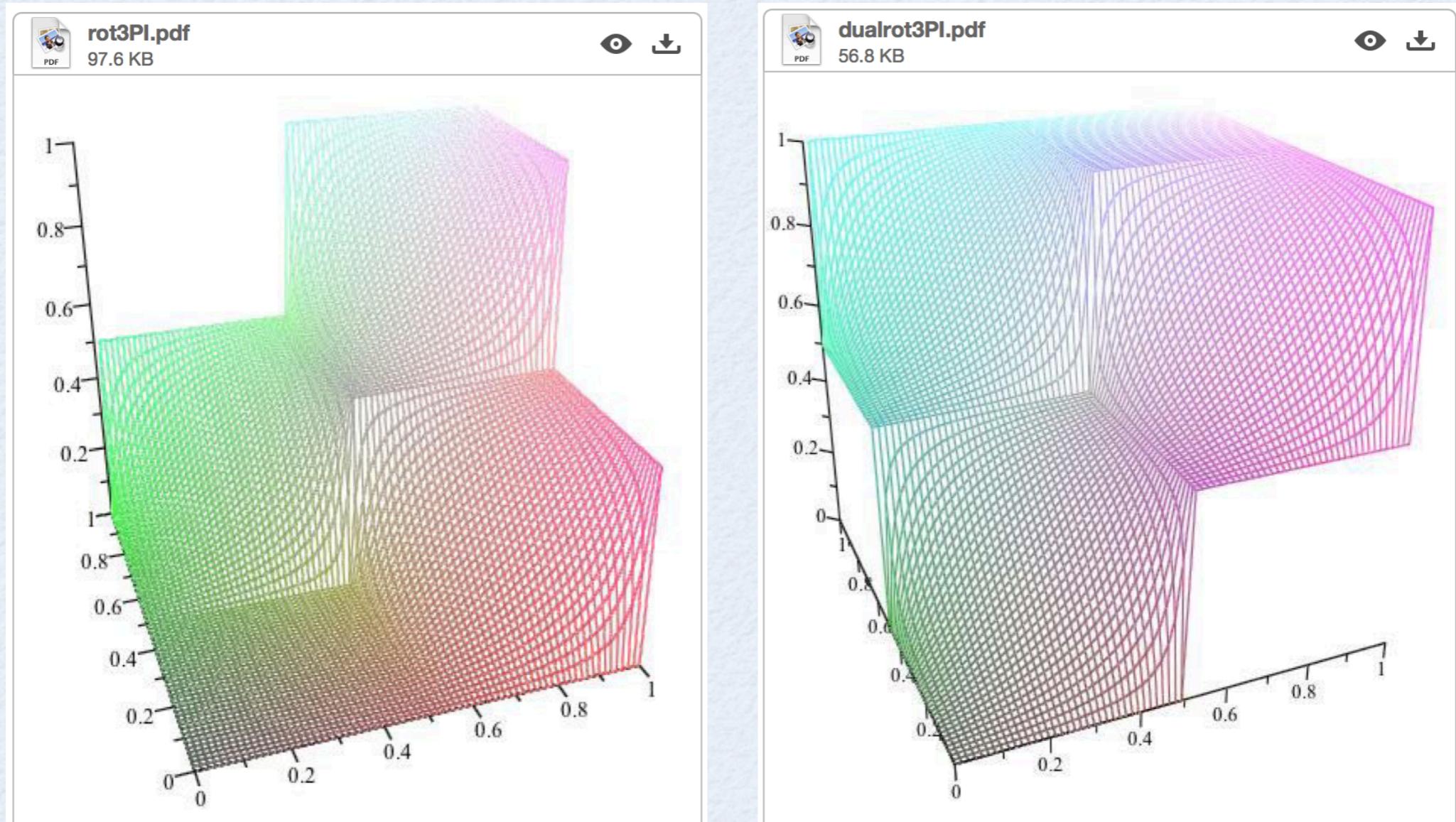
# MOTIVATION



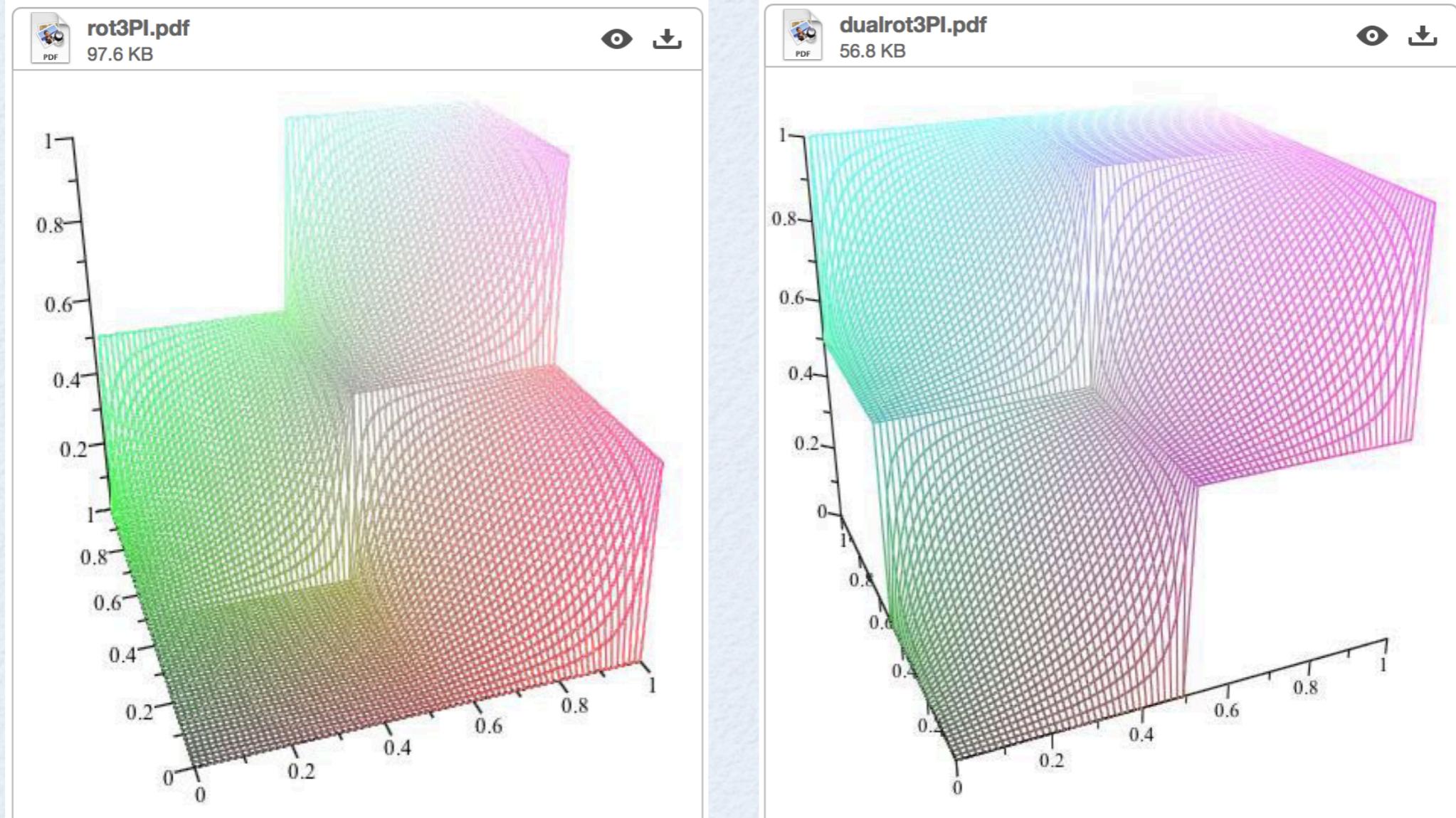
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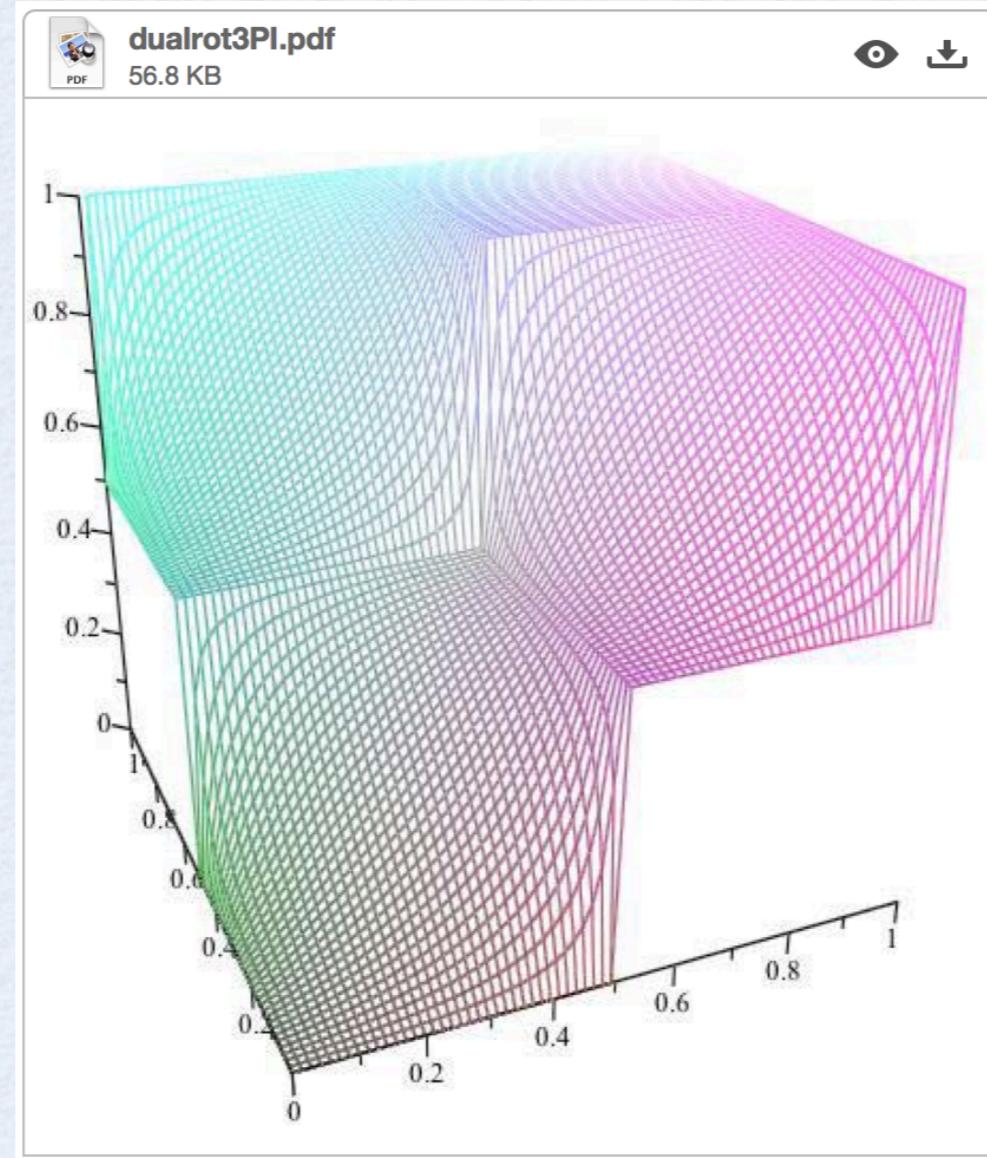
# MOTIVATION



$$x \odot y = \sup\{u \odot v \mid u < x, v < y\}$$

*skewed modification of  $\odot$ .*

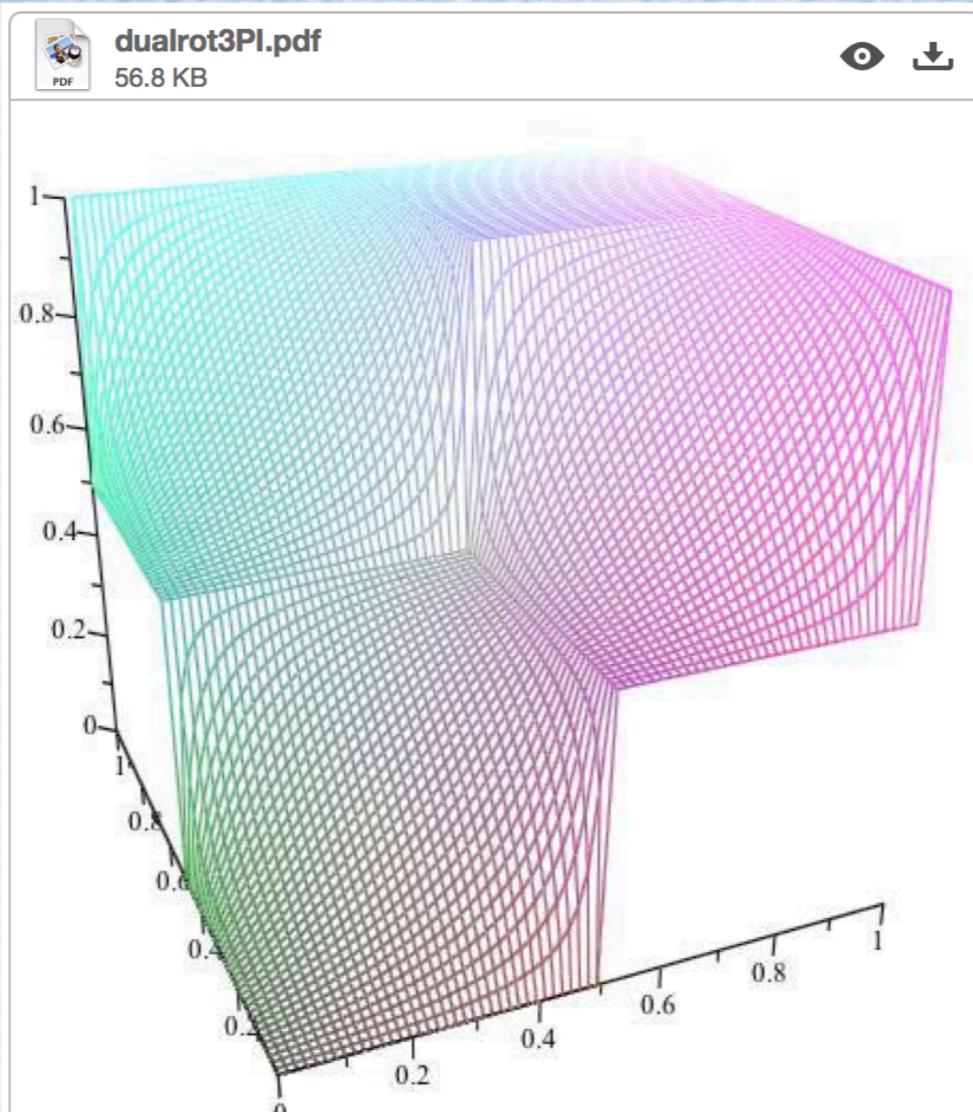
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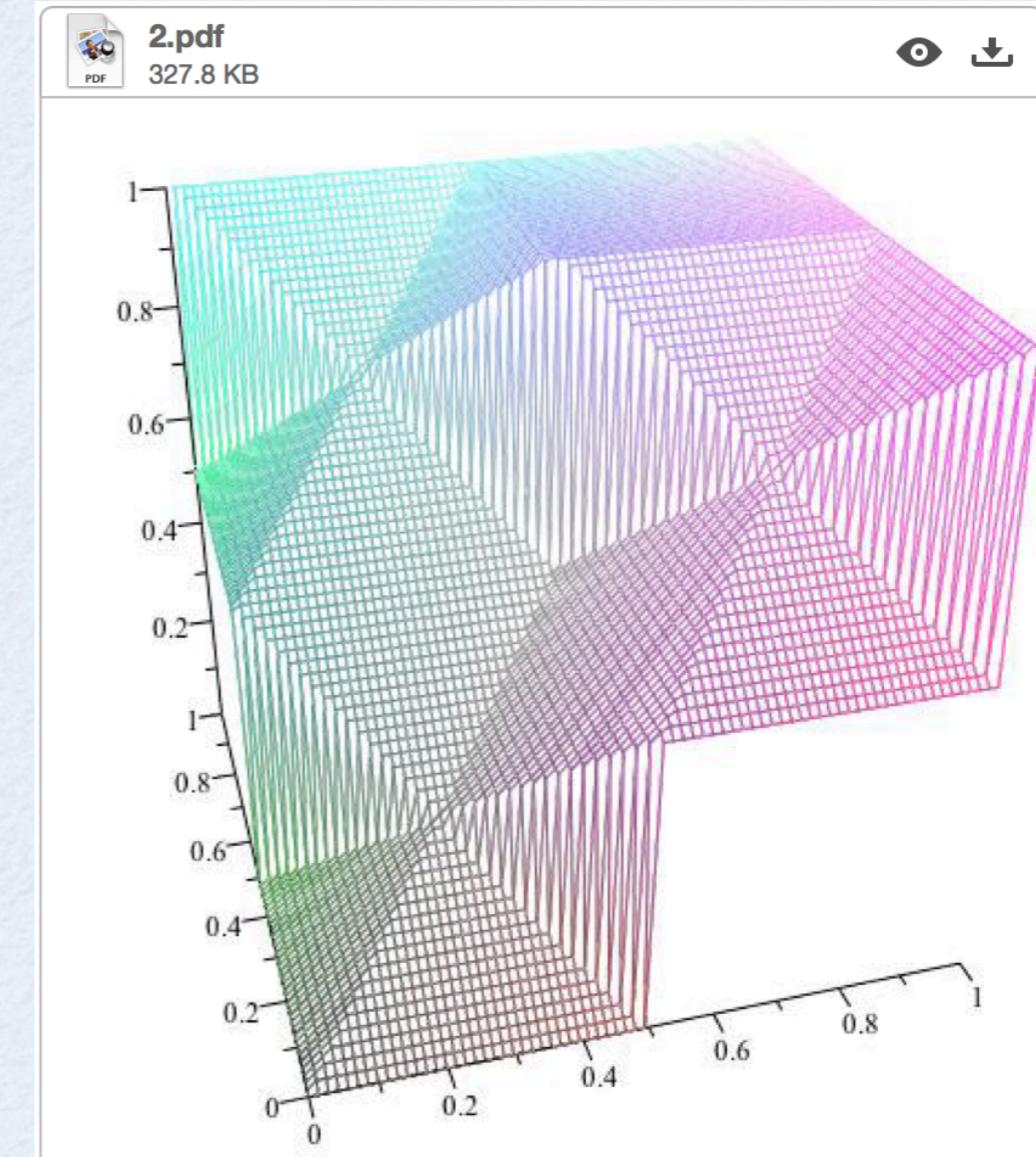
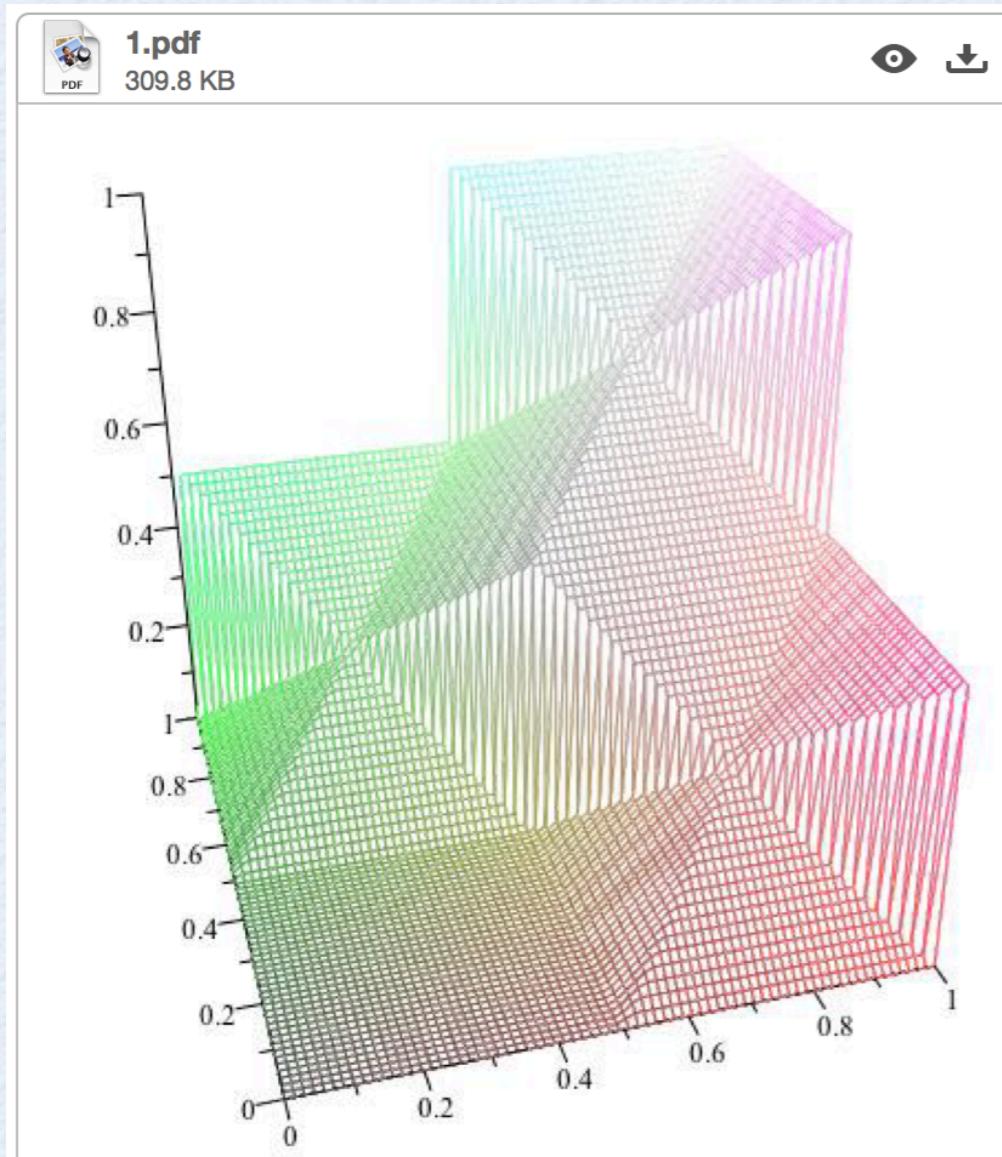
*skewed modification of  $\odot$ .*

# MOTIVATION

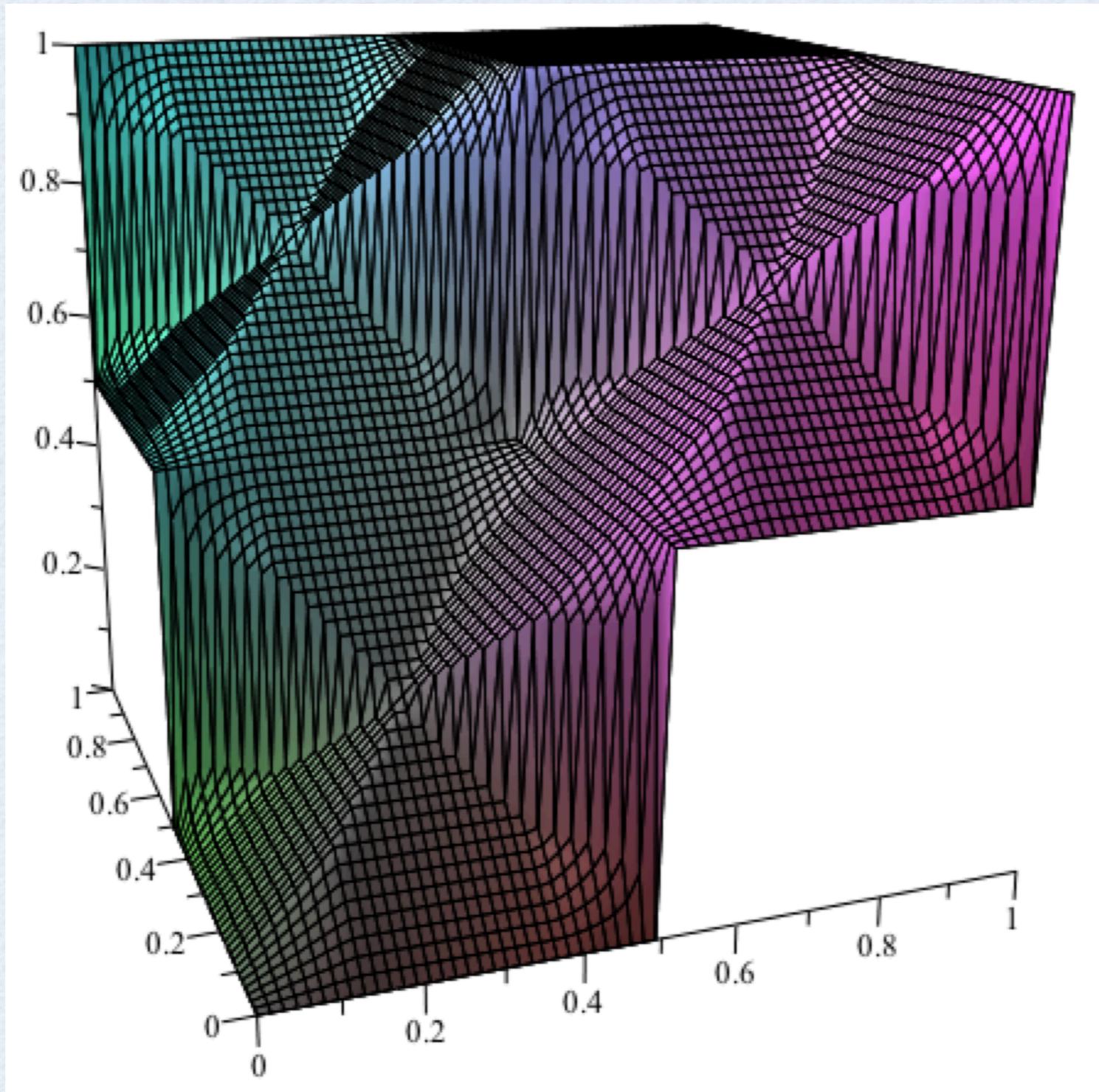


Csak ott similt az asszociativitás, ahol  $x(yz) = 0$  is  
 $(xy)z = \frac{1}{2}$ , az is csak akkor lehetséges, ha  $x(yz)$  minden  $y, z$  esetén  $\frac{1}{2}$  lesz. Íme, ha  $x = \frac{1}{2}$  akkor  $xy$  szigetleípítés  $= \frac{1}{2}$  (0 minden esetben, mert minden  $(xy)z$  is 0 lenne) minden  $y, z \geq \frac{1}{2}$ , mivel  $\frac{1}{2} \cdot z = \frac{1}{2}$  - bár  $z \geq \frac{1}{2}$  is áll. Ekkor  $\frac{1}{2}(\frac{1}{2} \cdot \frac{1}{2}) = \frac{1}{2}$  is minden esetben 0, az elvethetetlenség.

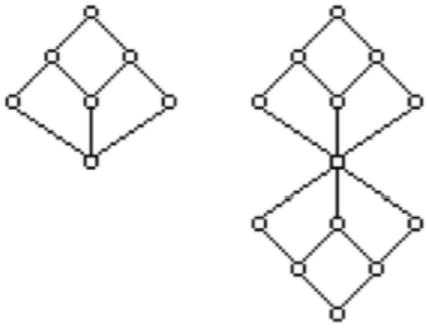
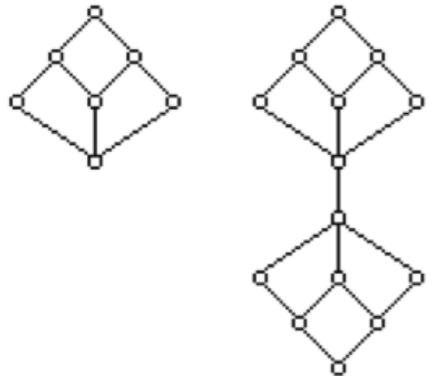
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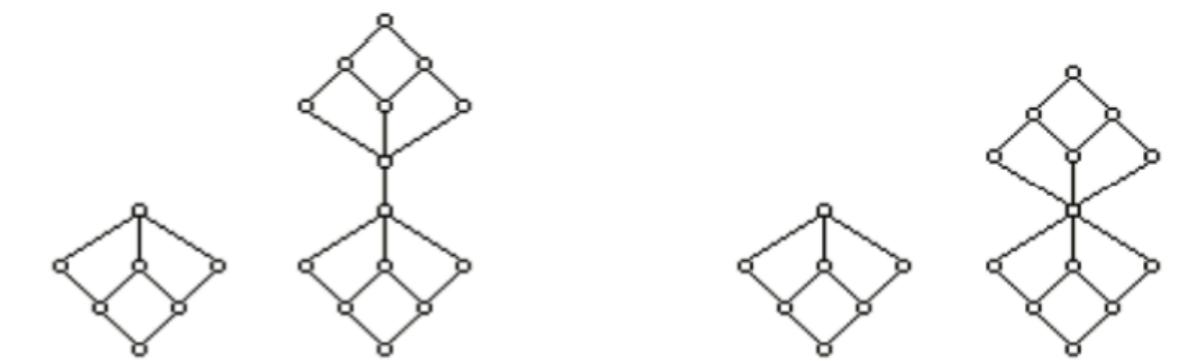


# 2: CO-ROTATIONS



*let*  $x \otimes_{\vartheta} y =$

$$\begin{cases} x \otimes y & \text{if } x, y \in M^+ \\ (x \rightarrow_{\circ} y')' & \text{if } x \in M^+ \text{ and } y \in M^- \\ (y \rightarrow_{\circ} x')' & \text{if } x \in M^- \text{ and } y \in M^+ \\ \perp & \text{if } x, y \in M^- \end{cases}$$



*let*  $x \otimes_{\eta} y =$

$$\begin{cases} \perp & \text{if } x \text{ or } y \text{ is } \perp \\ x \otimes y & \text{if } x, y \in M^- \setminus \{\perp\} \\ (x \rightarrow_{\circ} y')' & \text{if } x \in M^- \setminus \{\perp\} \text{ and } y \in M^+ \\ (y \rightarrow_{\circ} x')' & \text{if } x \in M^+ \text{ and } y \in M^- \setminus \{\perp\} \\ \top & \text{if } x, y \in M^+ \end{cases}$$

- disconnected

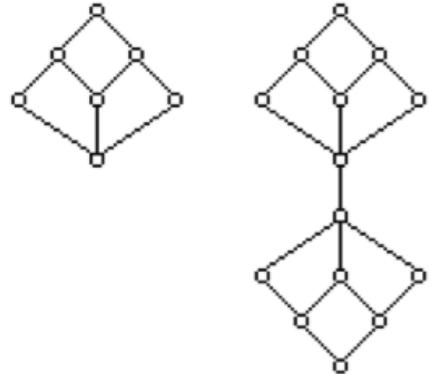
*commutative, residuated po-semigroup*

- connected

*commutative, residuated po-semigroup either*

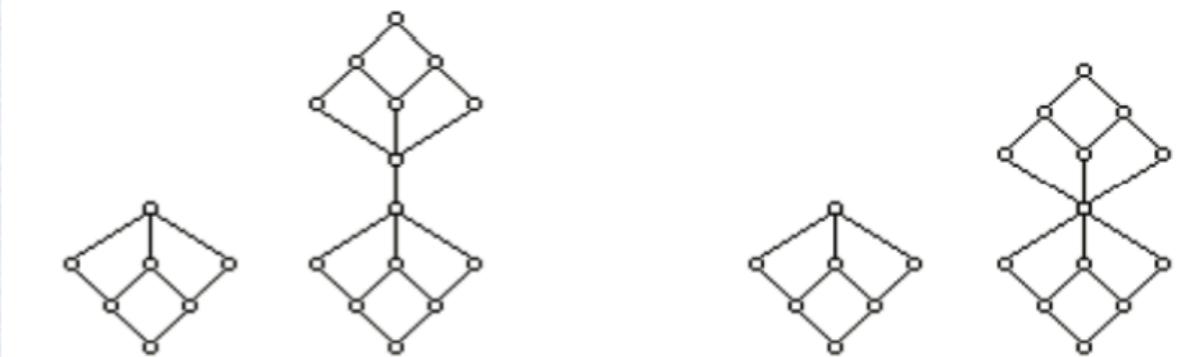
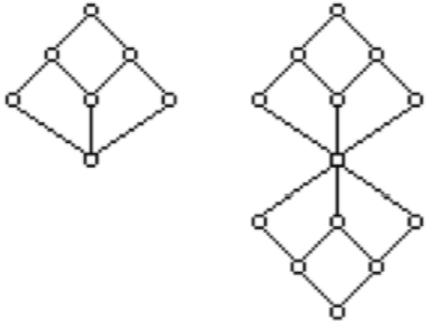
1. *without zero divisors or*
2. *with zero divisors. In this case suppose that there exist  $c \in M$  such that for any zero divisor  $x$ ,  $x \rightarrow_{\circ} \perp = c$  holds.*

# 2: CO-ROTATIONS



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- disconnected

*commutative, residuated po-semigroup*

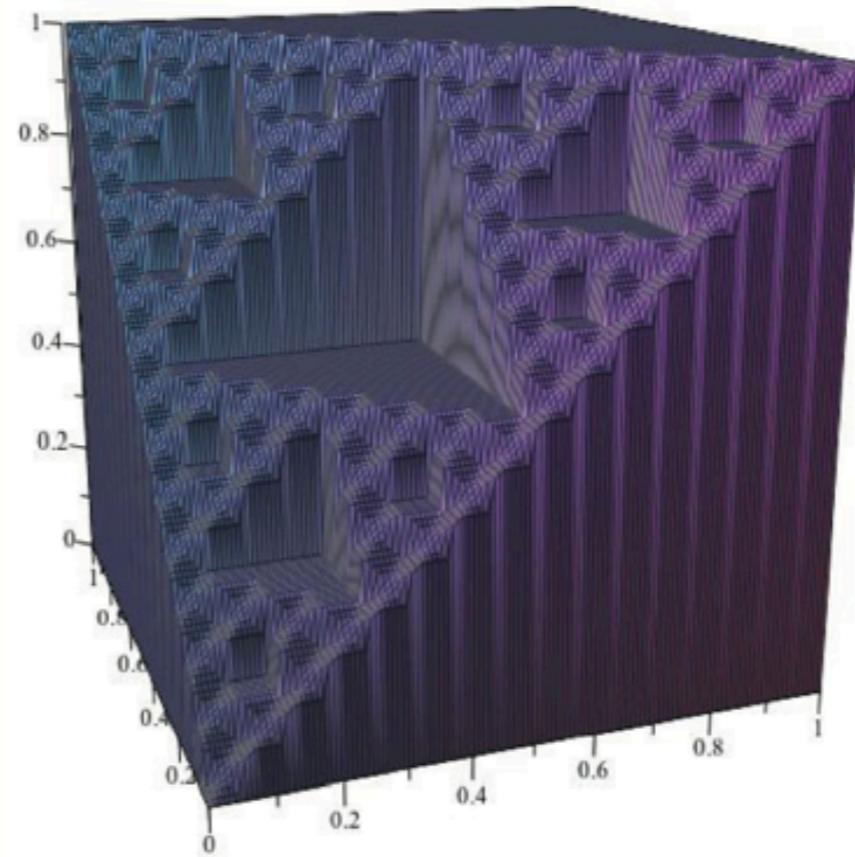
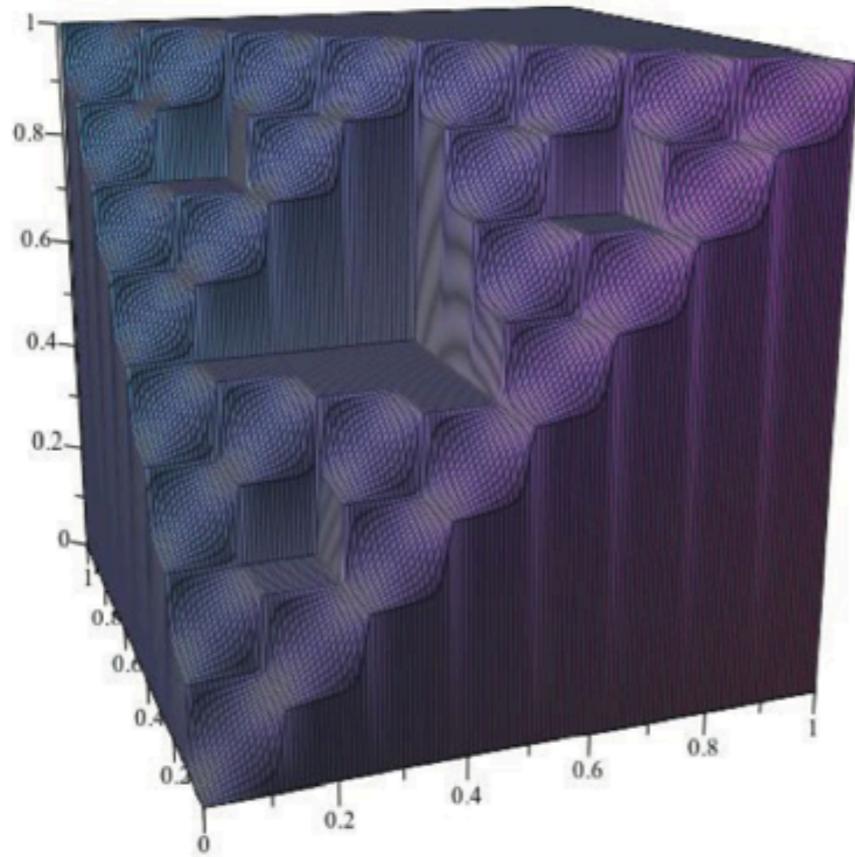
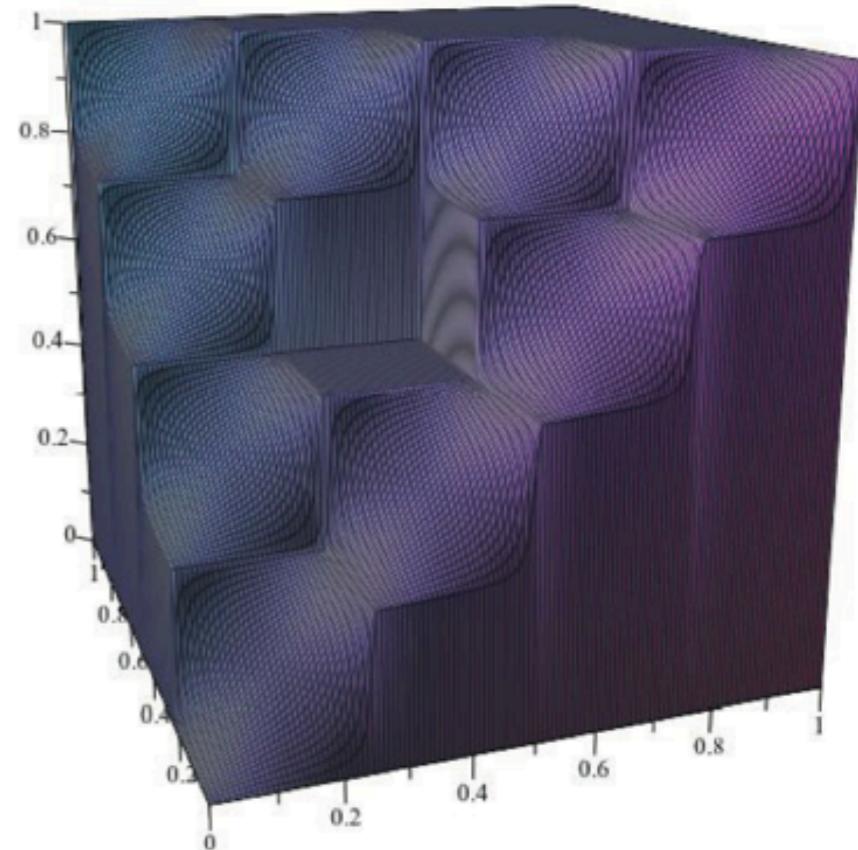
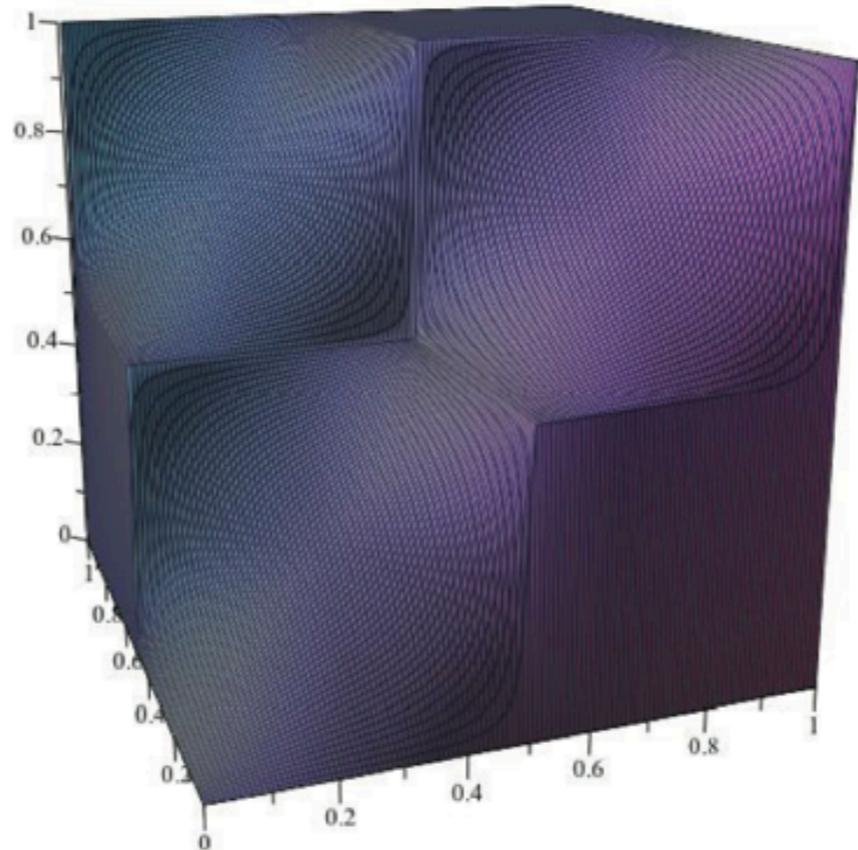
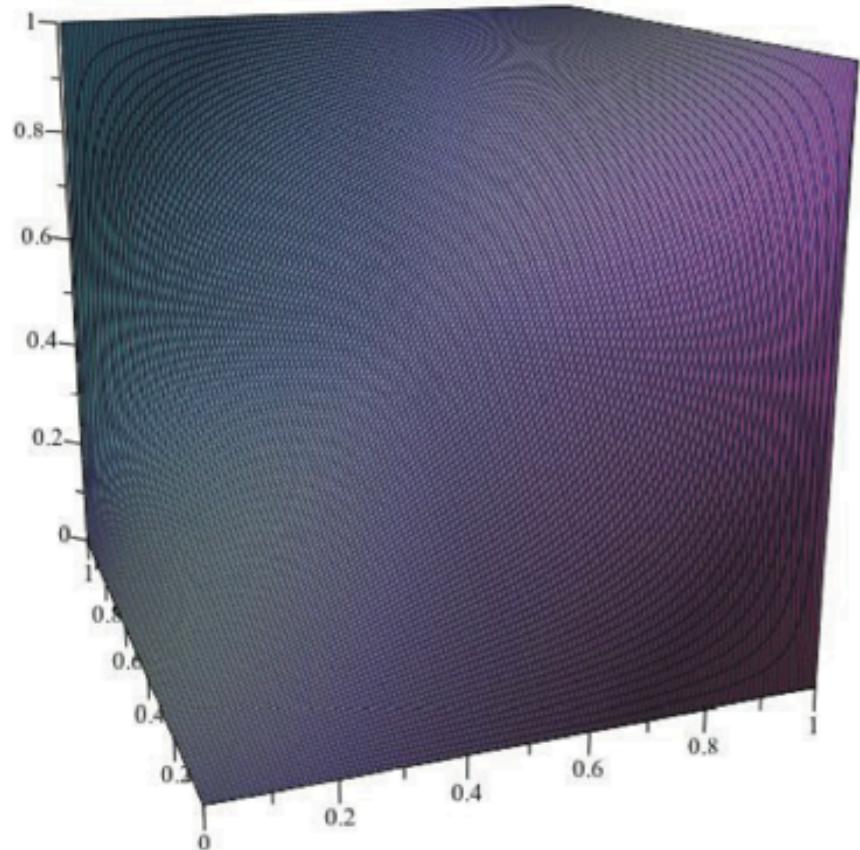
*without zero divisors.*

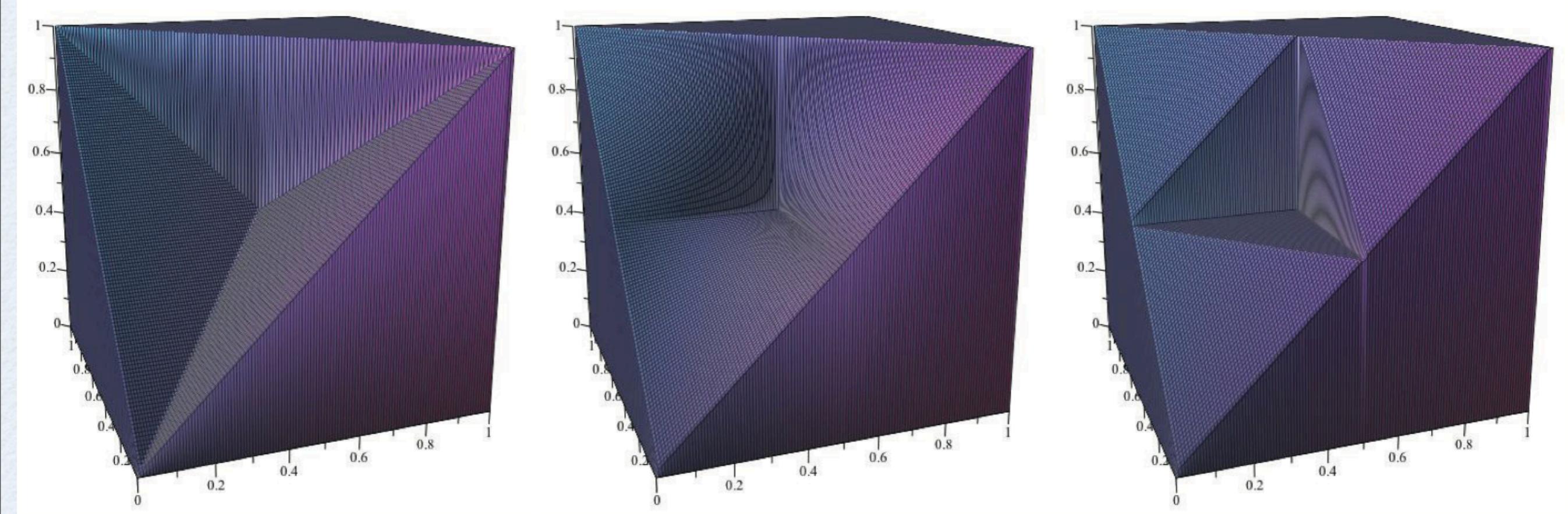
- connected

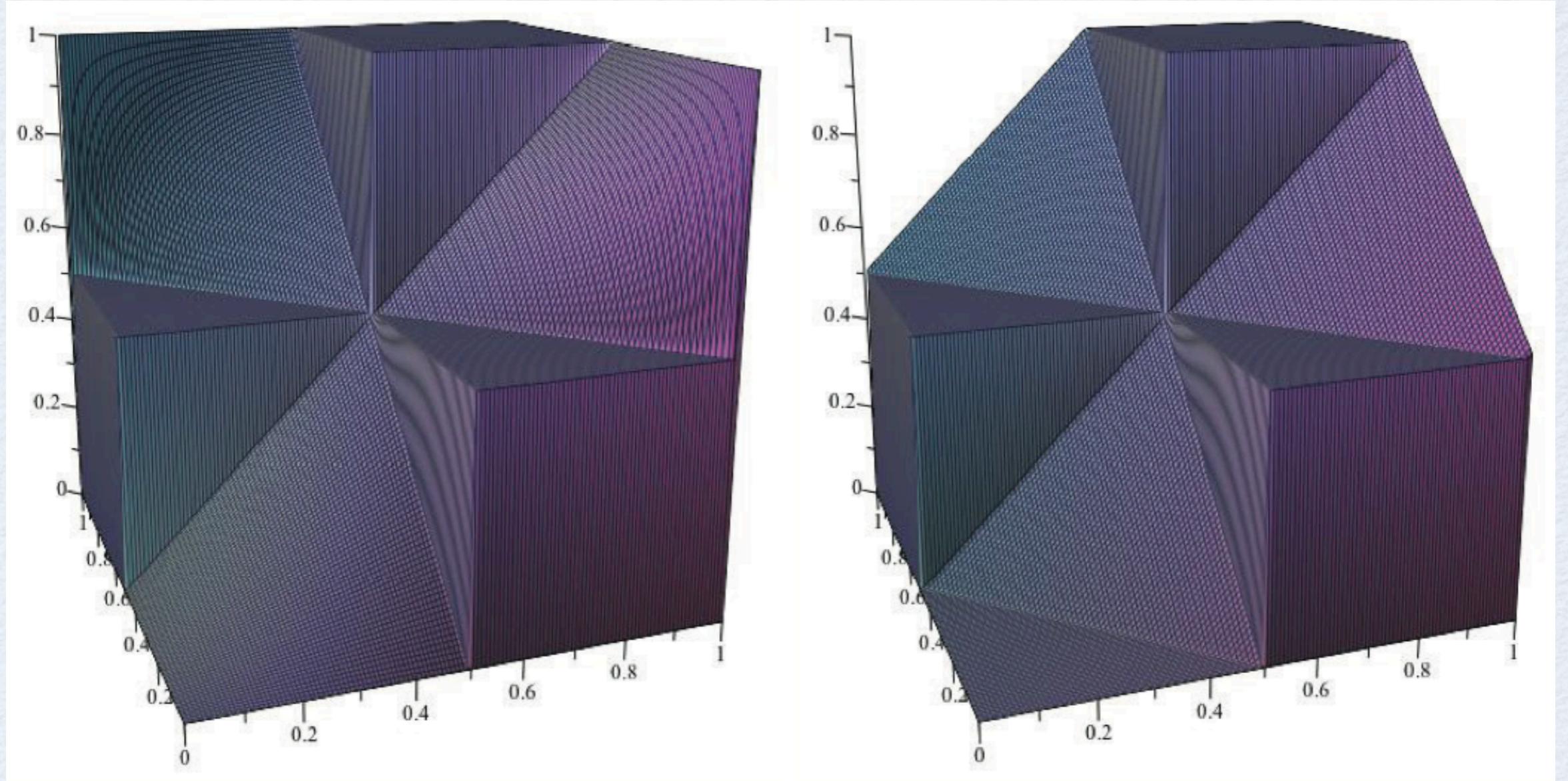
*commutative, residuated po-semigroup without zero divisors and satisfying*

$$\iota \otimes x = \iota \text{ for } x > \perp.$$

(8)







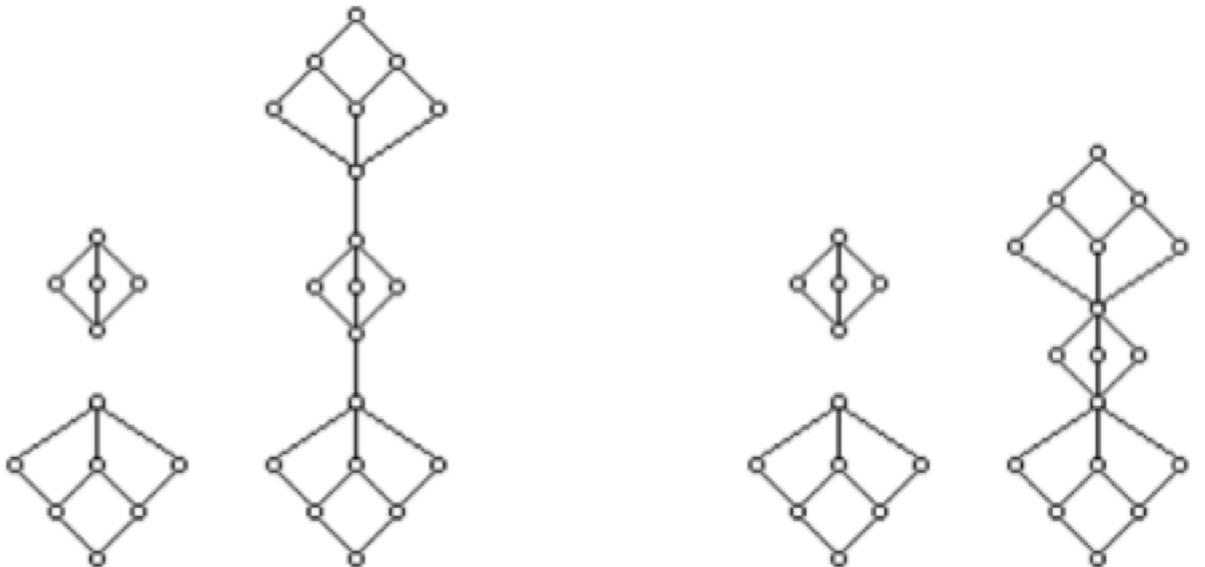
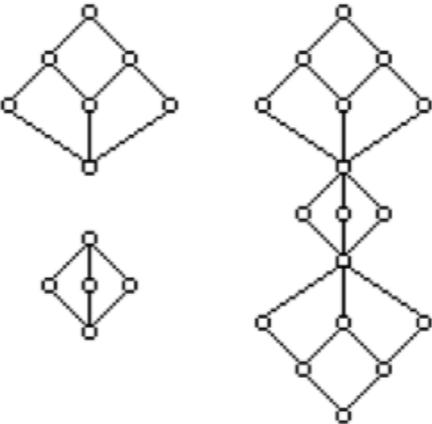
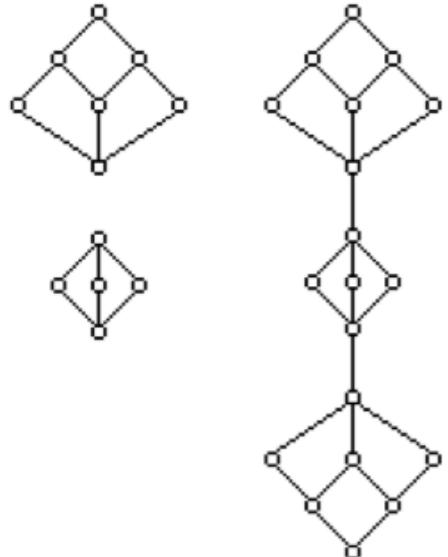
# APPLICATIONS OF THE ROTATION CONSTRUCTION

- in the structural description of
  - Perfect and bipartite IMTL-algebras  
[C. Noguera, F. Esteva, J. Gispert, Perfect and bipartite IMTL-algebras and disconnected rotations of basic semihoops, Archive for Mathematical Logic, 44 (2005), 869–886. ]
  - Free nilpotent minimum algebras  
[M. Busaniche, Free nilpotent minimum algebras, Mathematical Logic Quarterly 52 (3) (2006) 219–236. ]
  - Free Glivenko MTL-algebras  
[R. Cignoli, A. Torrens, Free algebras in varieties of Glivenko MTL-algebras satisfying the equation  $2(x2) = (2x)2$ , Studia Logica 83 (1-3) (2006) 157-181]

# APPLICATIONS OF THE ROTATION CONSTRUCTION

- Nelson algebras  
[M. Busaniche, R. Cignoli, Constructive Logic with Strong Negation as a Substructural Logic, *Journal of Logic and Computation* 20 (4) (2010) 761–793.]
- in establishing a spectral duality for finitely generated nilpotent minimum algebras  
[S. Aguzzoli, M. Busaniche, Spectral duality for finitely generated nilpotent minimum algebras, with applications, *Journal of Logic and Computation* 17 (4) (2007) 749–765.]

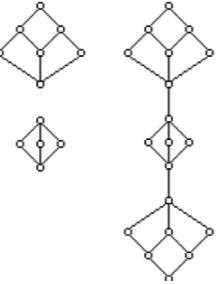
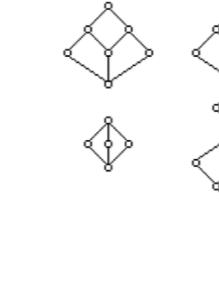
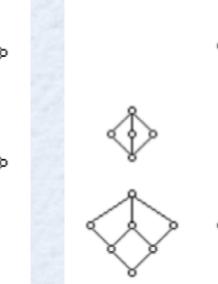
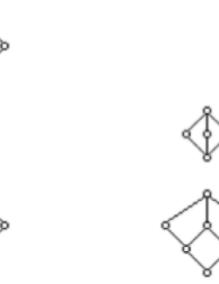
# 3: CO-ROTATION-ANNIHILATIONS



$$\left\{ \begin{array}{ll} x \oplus y & \text{if } x, y \in M^+ \\ (x \rightarrow_* y')' & \text{if } x \in M^+, y \in M^- \\ (y \rightarrow_* x')' & \text{if } x \in M^-, y \in M^+ \\ 0 & \text{if } x, y \in M^- \\ x \odot y & \text{if } x, y \in M^0 \text{ and } x > y' \\ T & \text{if } x, y \in M^0 \text{ and } x \leq y' \\ y & \text{if } x \in M^+, y \in M^0 \\ x & \text{if } x \in M^0, y \in M^+ \\ 0 & \text{if } x \in M^-, y \in M^0 \\ 0 & \text{if } x \in M^0, y \in M^- \end{array} \right.$$

$$\left\{ \begin{array}{ll} \perp & \text{if } x \text{ or } y \text{ is } \perp \\ x \oplus y & \text{if } x, y \in M^- \setminus \{\perp\} \\ (x \rightarrow_* y')' & \text{if } x \in M^- \setminus \{\perp\} \text{ and } y \in M^+ \\ (y \rightarrow_* x')' & \text{if } x \in M^+ \text{ and } y \in M^- \setminus \{\perp\} \\ T & \text{if } x, y \in M^+ \\ x \odot y & \text{if } x, y \in M^0 \text{ and } x \leq y' \\ T & \text{if } x, y \in M^0 \text{ and } x > y' \\ T & \text{if } x \in M^+, y \in M^0 \\ T & \text{if } x \in M^0, y \in M^+ \\ y & \text{if } x \in M^- \setminus \{\perp\}, y \in M^0 \\ x & \text{if } x \in M^0, y \in M^- \setminus \{\perp\} \end{array} \right.$$

# 3: CO-ROTATION-ANNIHILATIONS

 	 
$\left\{ \begin{array}{ll} x * y & \text{if } x, y \in M^+ \\ (x \rightarrow_* y')' & \text{if } x \in M^+, y \in M^- \\ (y \rightarrow_* x')' & \text{if } x \in M^-, y \in M^+ \\ 0 & \text{if } x, y \in M^- \\ x \circ y & \text{if } x, y \in M^0 \text{ and } x > y' \\ \top & \text{if } x, y \in M^0 \text{ and } x \leq y' \\ y & \text{if } x \in M^+, y \in M^0 \\ x & \text{if } x \in M^0, y \in M^+ \\ 0 & \text{if } x \in M^-, y \in M^0 \\ 0 & \text{if } x \in M^0, y \in M^- \end{array} \right.$	$\left\{ \begin{array}{ll} \perp & \text{if } x \text{ or } y \text{ is } \perp \\ x * y & \text{if } x, y \in M^- \setminus \{\perp\} \\ (x \rightarrow_* y')' & \text{if } x \in M^- \setminus \{\perp\} \text{ and } y \in M^+ \\ (y \rightarrow_* x')' & \text{if } x \in M^+ \text{ and } y \in M^- \setminus \{\perp\} \\ \top & \text{if } x, y \in M^+ \\ x \circ y & \text{if } x, y \in M^0 \text{ and } x \leq y' \\ \top & \text{if } x, y \in M^0 \text{ and } x > y' \\ \top & \text{if } x \in M^+, y \in M^0 \\ y & \text{if } x \in M^- \setminus \{\perp\}, y \in M^0 \\ x & \text{if } x \in M^0, y \in M^- \setminus \{\perp\} \end{array} \right.$

- disconnected

*commutative, residuated, po-semigroup,*

*commutative, conjunctive, rotation-invariant po-semigroup,*

- connected

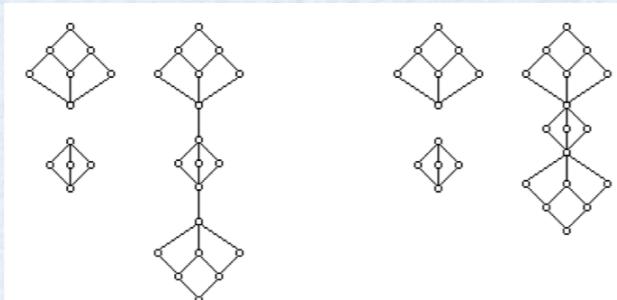
*commutative, residuated, po-semigroup,*

*commutative, rotation-invariant, integral po-monoid,*

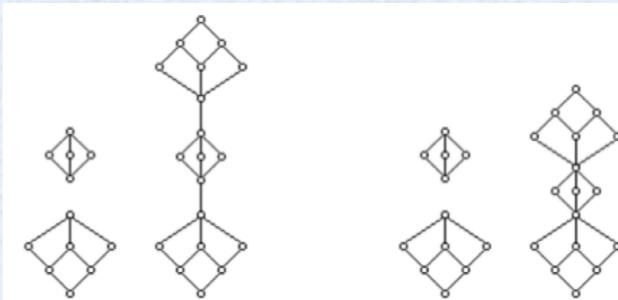
*commutative, residuated, po-semigroup without zero divisors,*

*commutative, conjunctive, rotation-invariant po-semigroup,*

# 3: CO-ROTATION-ANNIHILATIONS



$$\begin{cases} x \otimes y & \text{if } x, y \in M^+ \\ (x \rightarrow_* y')' & \text{if } x \in M^+, y \in M^- \\ (y \rightarrow_* x')' & \text{if } x \in M^-, y \in M^+ \\ 0 & \text{if } x, y \in M^- \\ x \circ y & \text{if } x, y \in M^0 \text{ and } x > y' \\ \top & \text{if } x, y \in M^0 \text{ and } x \leq y' \\ y & \text{if } x \in M^+, y \in M^0 \\ x & \text{if } x \in M^0, y \in M^+ \\ 0 & \text{if } x \in M^-, y \in M^0 \\ 0 & \text{if } x \in M^0, y \in M^- \end{cases}$$



$$\begin{cases} \perp & \text{if } x \text{ or } y \text{ is } \perp \\ x \otimes y & \text{if } x, y \in M^- \setminus \{\perp\} \\ (x \rightarrow_* y')' & \text{if } x \in M^- \setminus \{\perp\} \text{ and } y \in M^+ \\ (y \rightarrow_* x')' & \text{if } x \in M^+ \text{ and } y \in M^- \setminus \{\perp\} \\ \top & \text{if } x, y \in M^+ \\ x \circ y & \text{if } x, y \in M^0 \text{ and } x \leq y' \\ \top & \text{if } x, y \in M^0 \text{ and } x > y' \\ \top & \text{if } x \in M^+, y \in M^0 \\ y & \text{if } x \in M^- \setminus \{\perp\}, y \in M^0 \\ x & \text{if } x \in M^0, y \in M^- \setminus \{\perp\} \end{cases}$$

- disconnected

*commutative, residuated po-semigroup without zero divisors.*

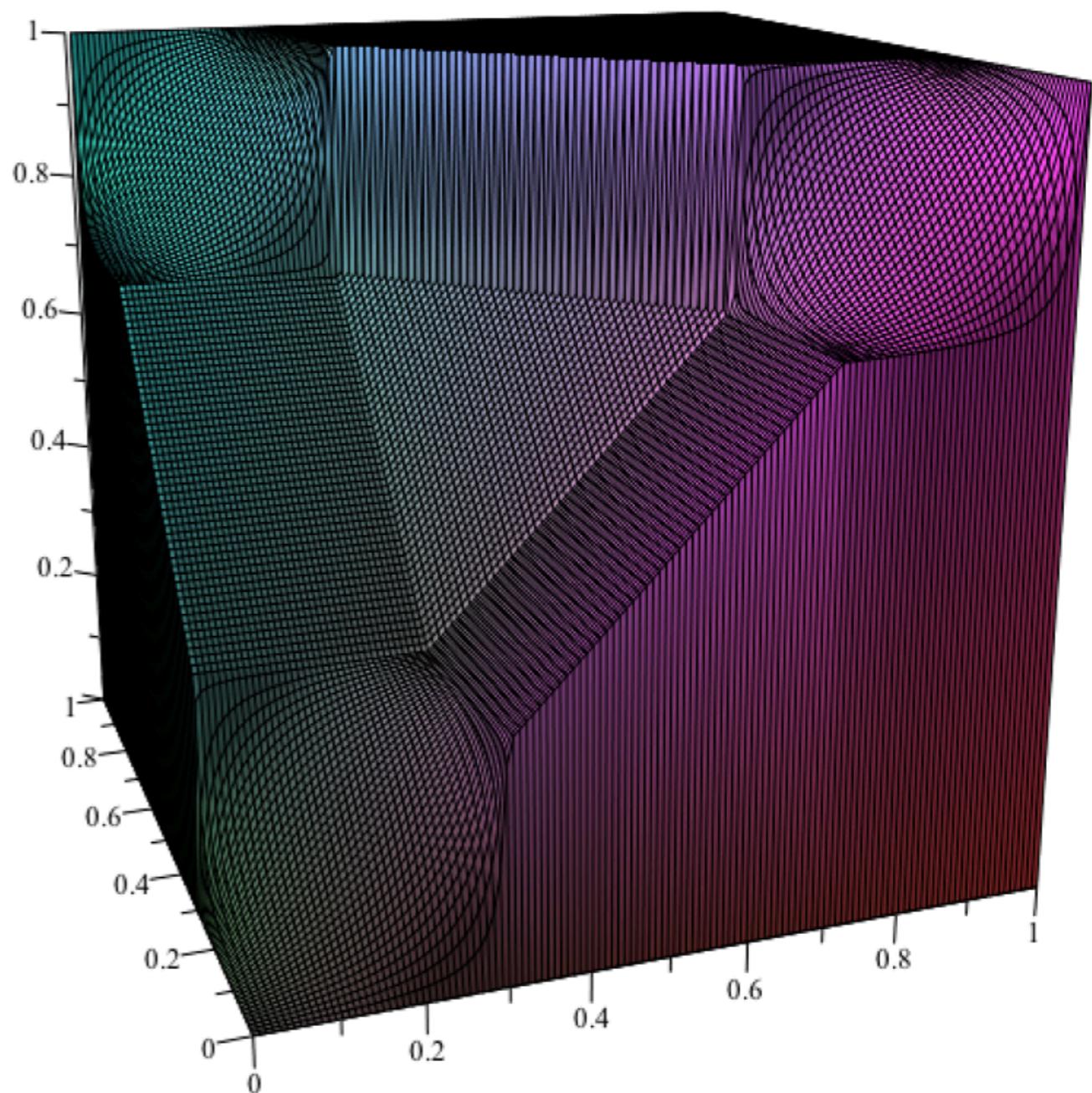
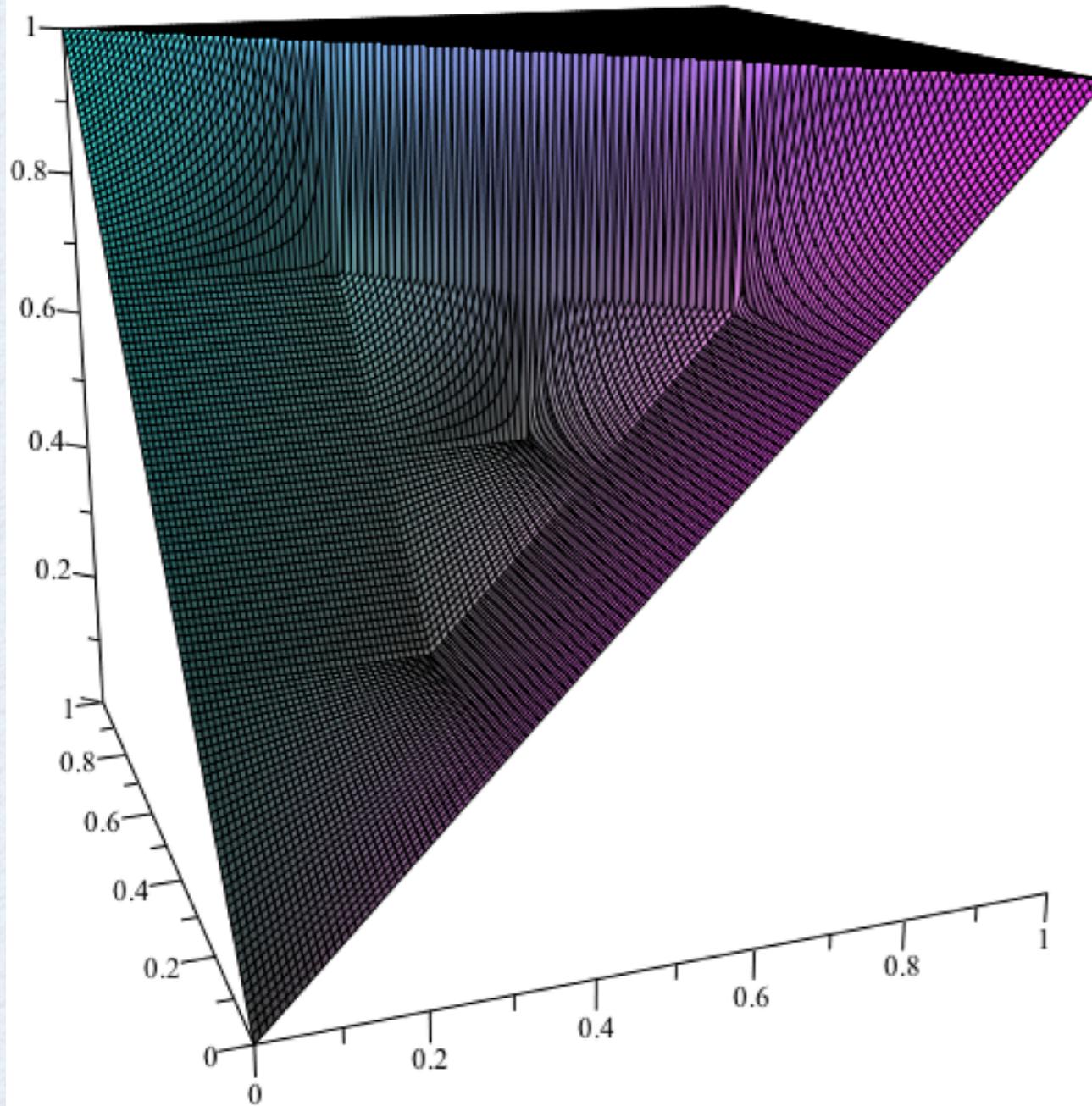
*commutative, weakly disjunctive, rotation-invariant po-semigroup,*

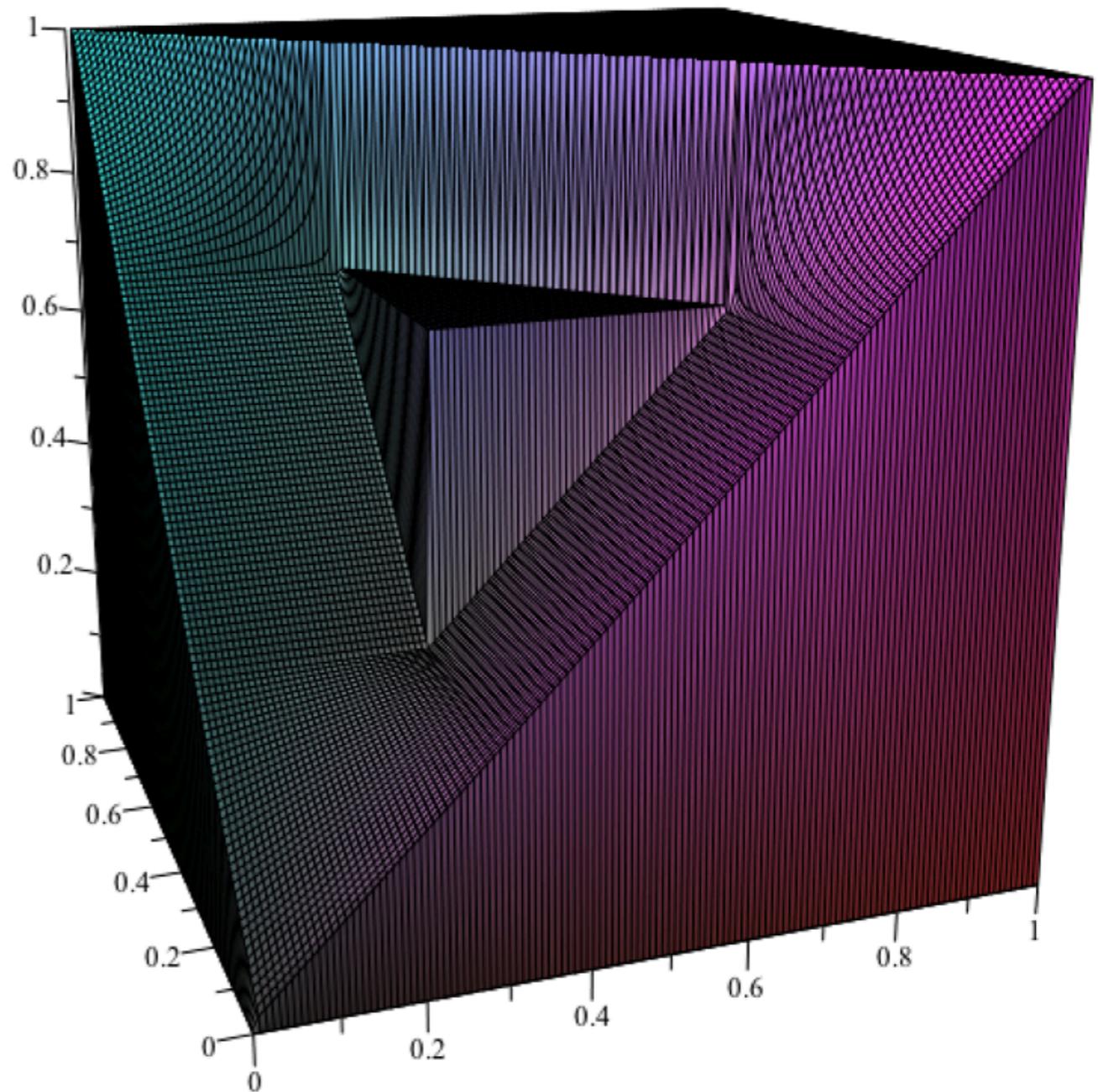
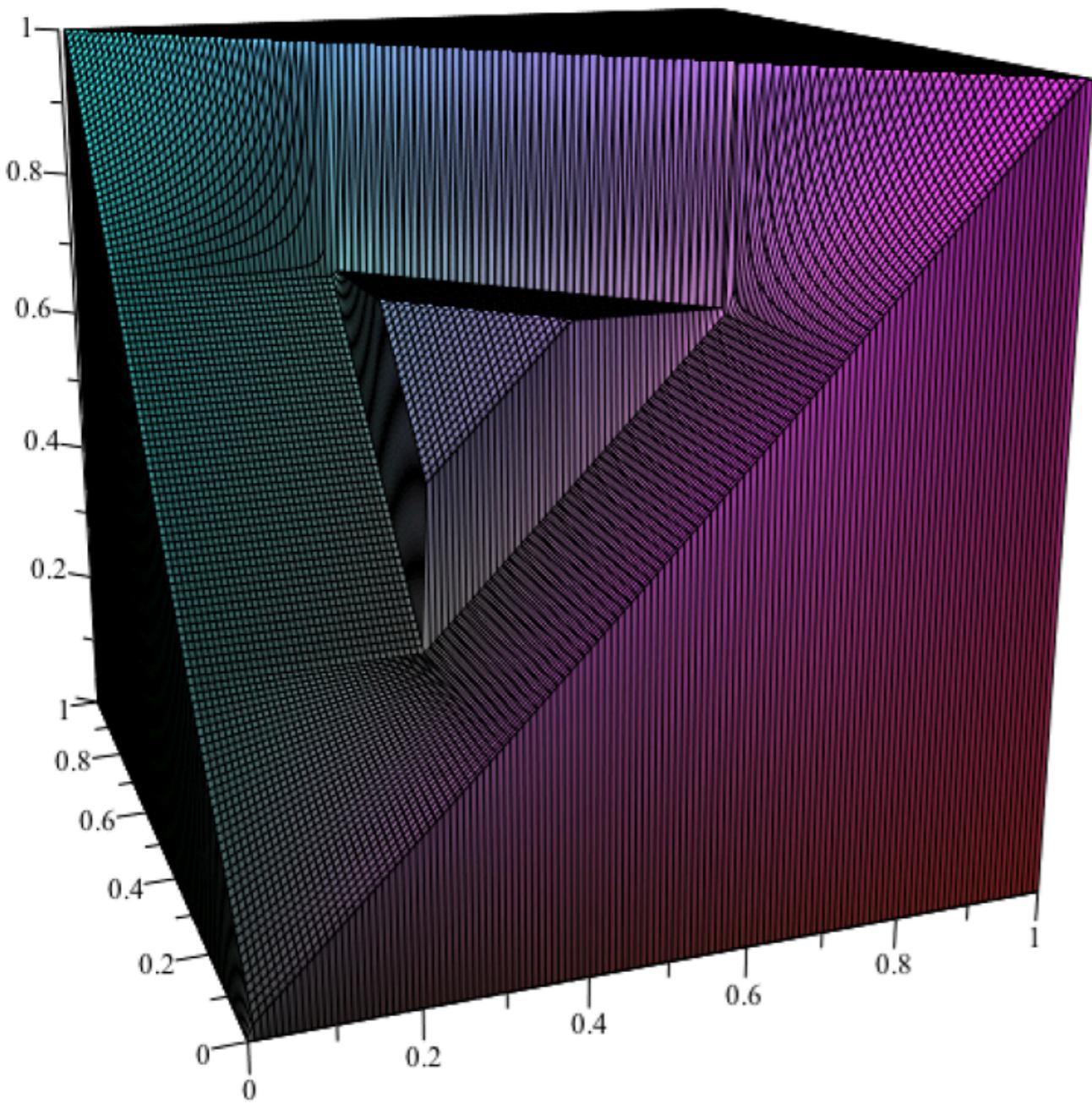
- connected

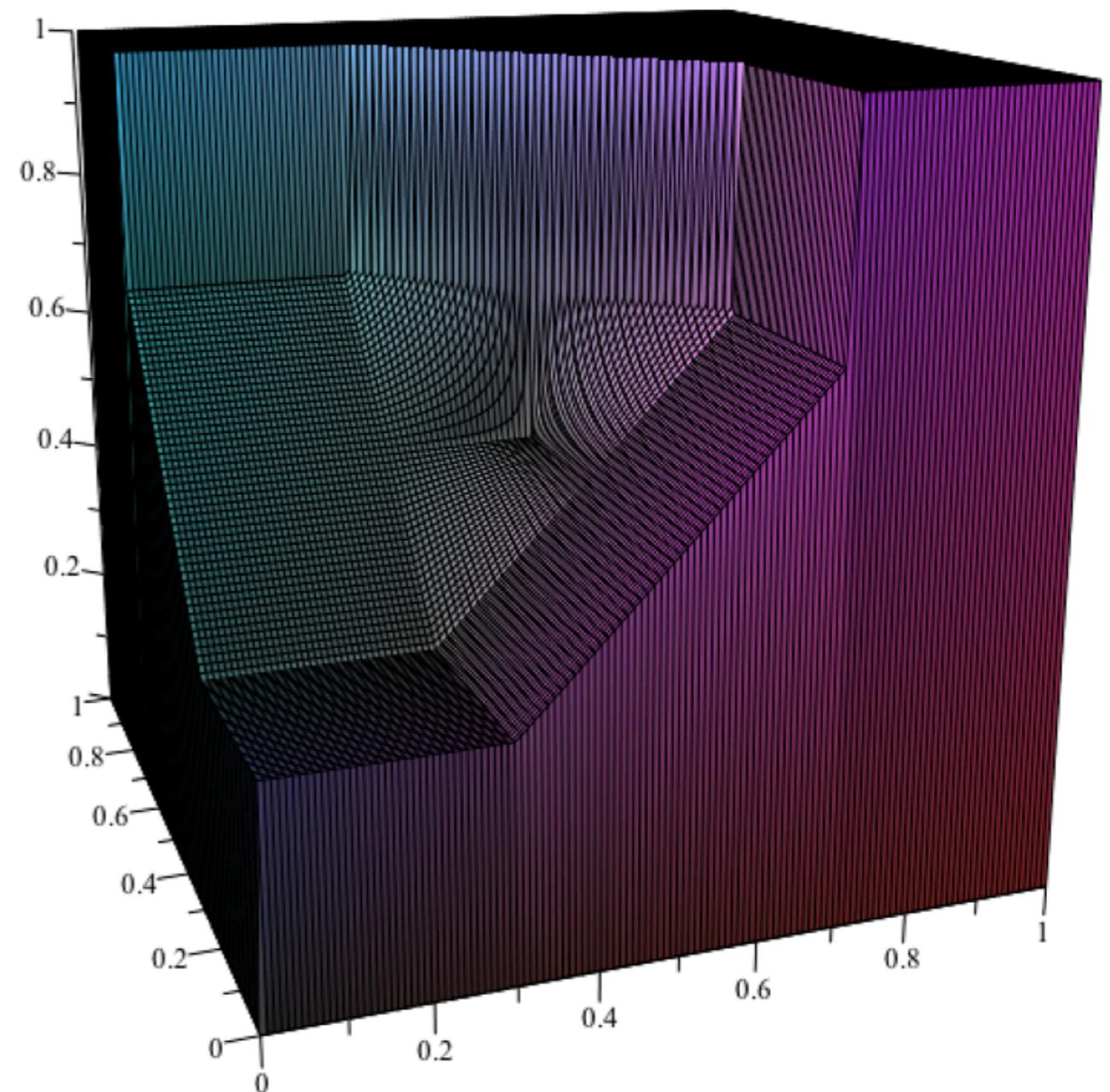
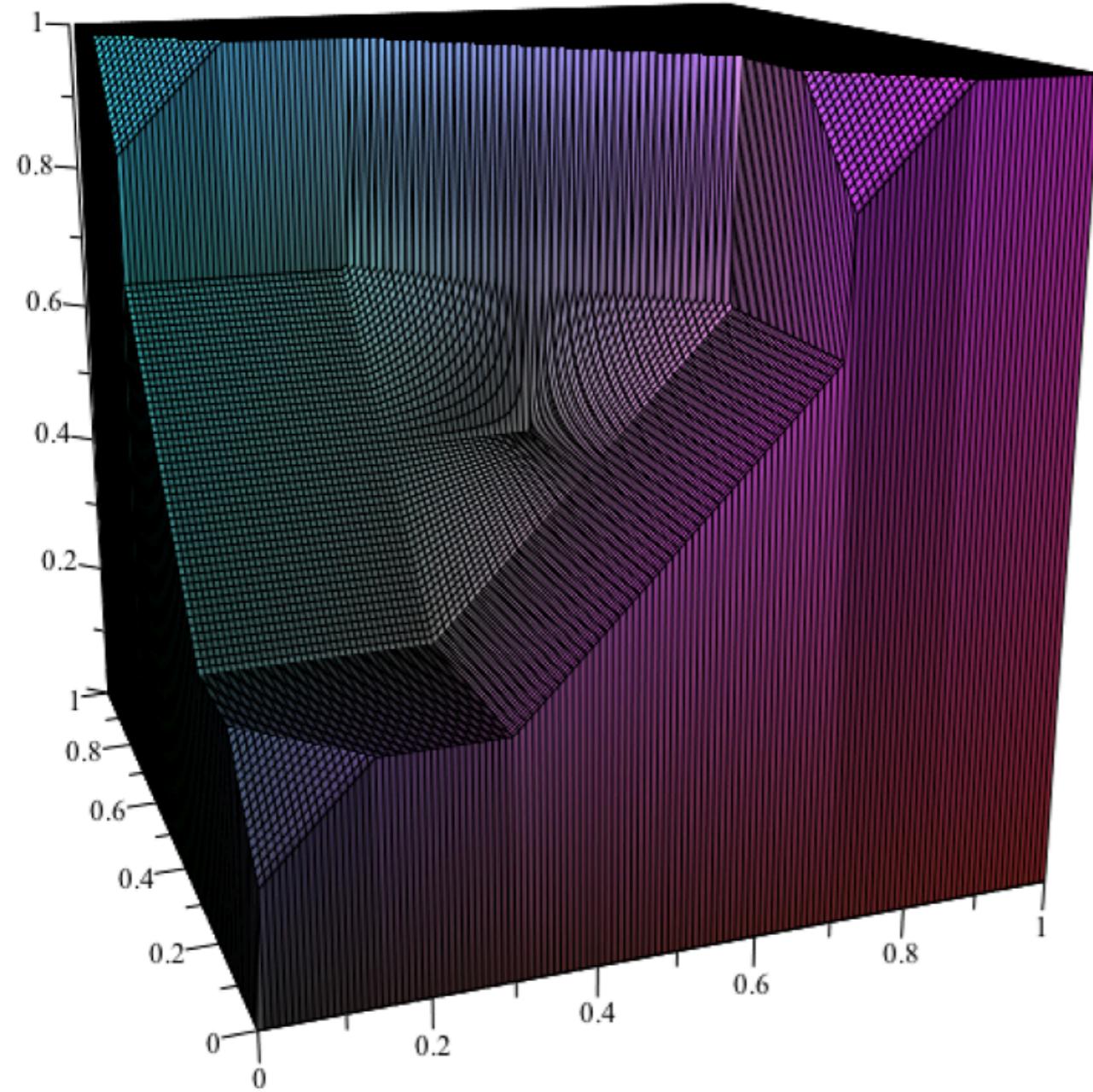
*commutative, residuated, po-semigroup satisfying*

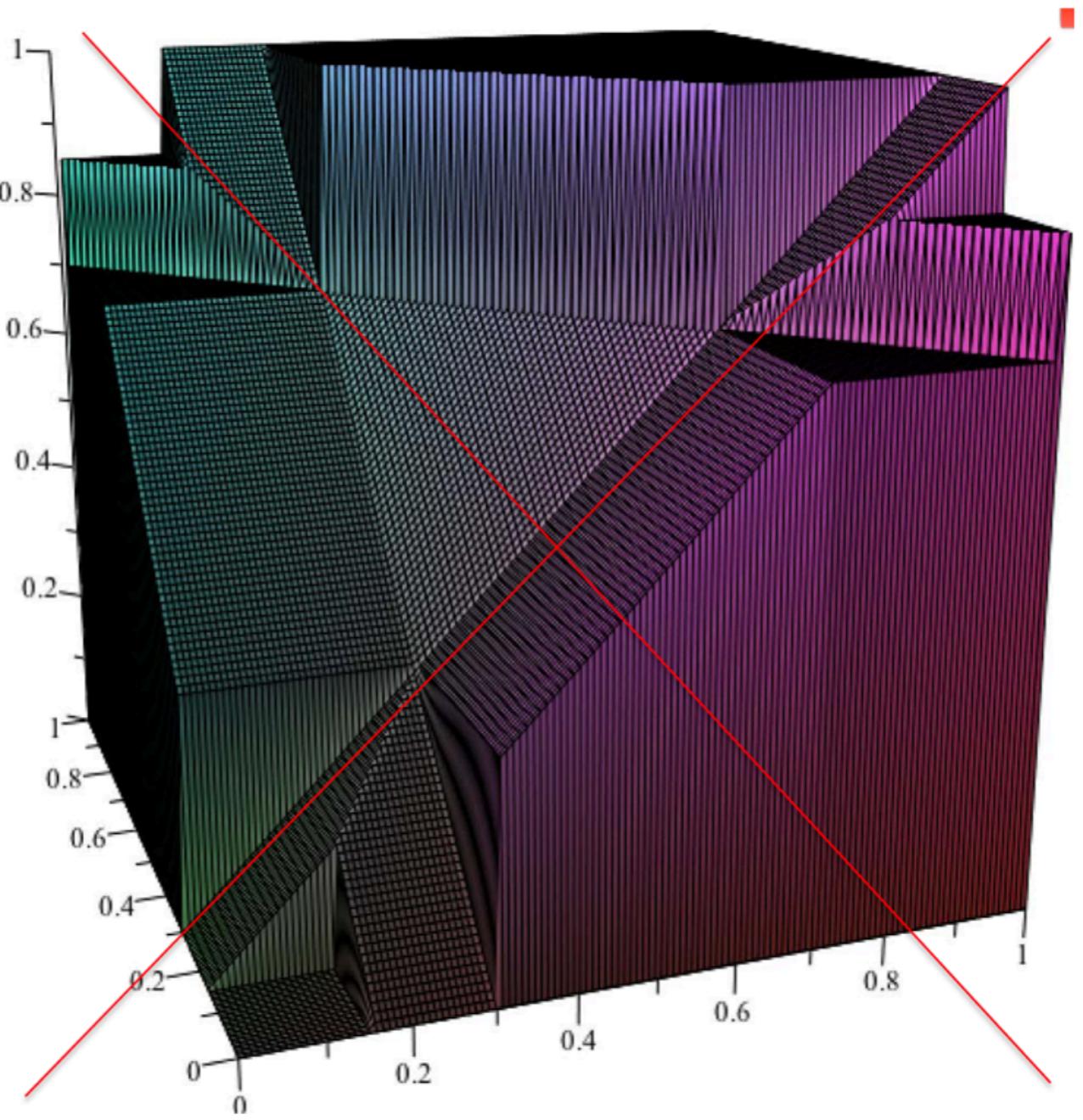
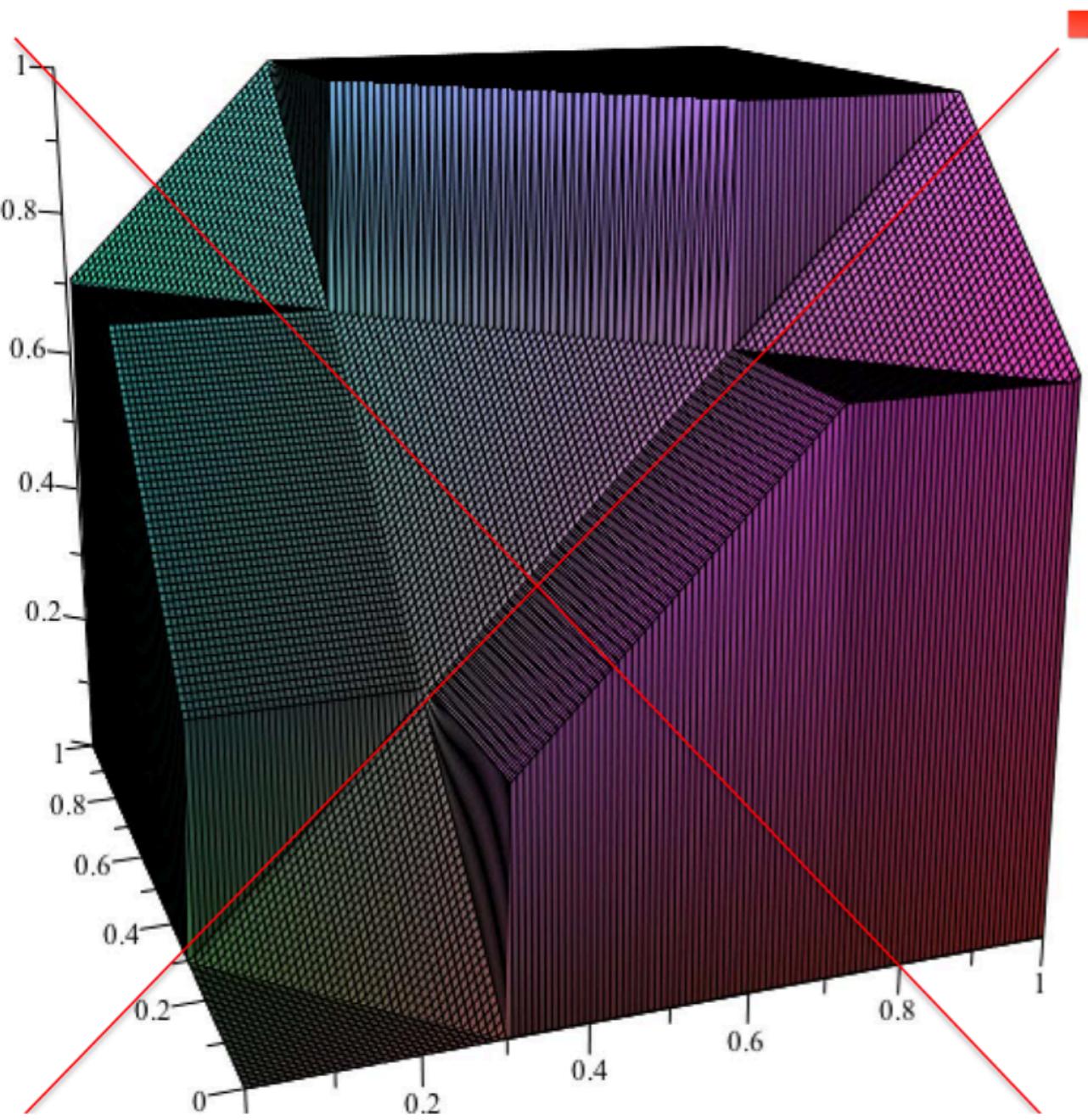
$$\iota \otimes x = \iota \text{ for } x > \perp.$$

*commutative, rotation-invariant, weakly disjunctive monoid*









THANK YOU FOR  
YOUR ATTENTION!