Colourings
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Intuitions
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Colourability/Representabilty
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Experimental results
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Ramsey-type colourings and Relation Algebras

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A COLOURING PROBLEM

For a given number n of colours: $c_1, \ldots, c_n$, is there a complete graph K_m that admits an edge colourings with n colours, such that:

1. There are no monochromatic triangles.
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The second requirement, formally stated is this:

▶ $\forall a, b \in K_m$, if $c_i(a, b)$, and the triangle c_i, c_j, c_k is not monochromatic, then there is a $c \in K_m$ such that $c_j(a, c)$ and $c_k(c, b)$.

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The first requirement gives the upper bound, via Ramsey theorem. The second requirement gives a lower bound.

Question

For which n is the upper bound greater than the upper bound?

In terms of Relation Algebras

Let **M**$_n$ be a finite relation algebra on $n + 1$ atoms $e, a_1, \ldots, a_n$, whose composition table is given by:

$$e \circ a_i = a_i = a_i \circ e \quad \text{and} \quad a_i \circ a_j = \begin{cases} e^- & \text{if } i \neq j \\ a_i^- & \text{if } i = j \end{cases}$$

Maddux uses $\mathfrak{E}_{n+1}^{\{2,3\}}$ for what we call **M**$_n$. Hirsch, Hodkinson use $\mathcal{M}_{n+1}$ and suggest the generic name *Monk* or *Maddux algebras*.

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- For $n = 2$ (folklore).

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- For $n = 3$ (folklore + Maddux's student).

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- For $n = 2$ (folklore).
- For $n = 3$ (folklore + Maddux's student).
- For $n = 4, 5$ (Comer).

EASY ANSWERS: TWO COLOURS

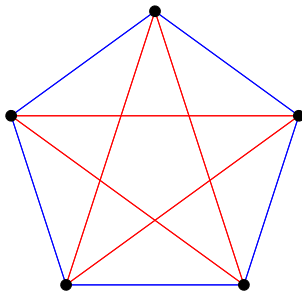Consider $\mathbf{M}_2$. We have atoms $e, a_1 = b, a_2 = r$, and the table is

○	e	r	b
e	e	r	b
r	a	e, b	r, b
b	b	r, b	e, r

Colourings
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Experimental results
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Easy answers: two colours

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Representation:

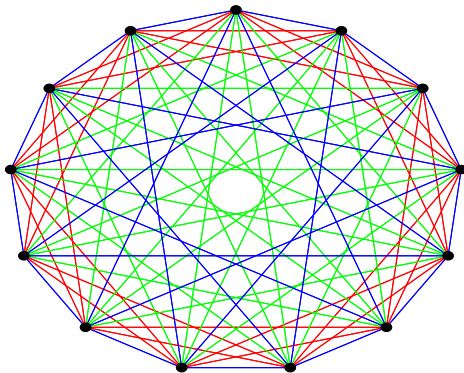## EASY ANSWERS: THREE COLOURS I

Essentially two representations of $\mathbf{M}_3$. One is on K_{16}, and uses the Clebsch graph

Colourings
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EASY ANSWERS: THEREE COLOURS II

The other is on K_{13}. It can be shown that nothing smaller will do, and neither will K_{14} or K_{15}.

A CLOSER LOOK AT THE PENTAGON

Consider $\mathbb{Z}_5$ as a finite field. Let g be a generator of its
multiplicative group $\mathbb{Z}_5^*$. Order of $\mathbb{Z}_5^*$ happens to be divisible by
the number of colours, so we build a rectangular matrix

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \cong \begin{pmatrix} g & g^3 \\ g^2 & g^4 \end{pmatrix} \cong \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$$

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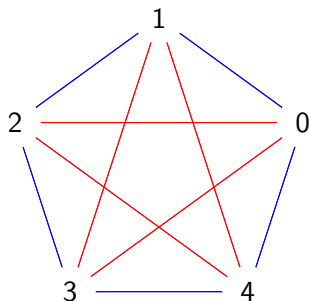
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Now, we assign colours to rows, and we get:

A CLOSER LOOK AT K_{13}

The same happens with $\mathbb{Z}_{13}$ and 3 colours. We get the matrix

$$\begin{pmatrix} g & g^4 & g^7 & g^{10} \\ g^2 & g^5 & g^8 & g^{11} \\ g^3 & g^6 & g^9 & g^{12} \end{pmatrix} \cong \begin{pmatrix} 2 & 3 & 11 & 10 \\ 4 & 6 & 9 & 7 \\ 8 & 12 & 5 & 1 \end{pmatrix}$$

Colourings
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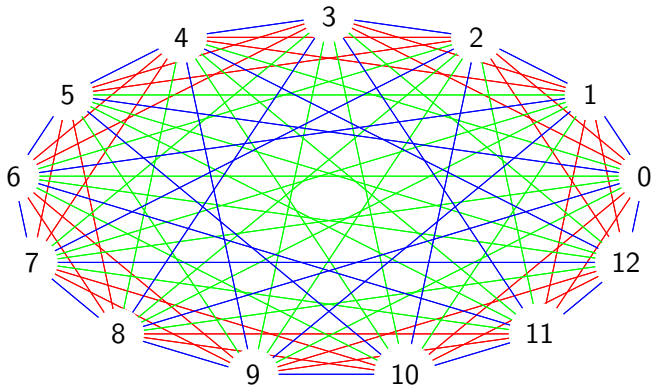○○○○○

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For *n* colours

Suppose we have found $GF(p^K)$, such that n divides $p^K - 1$. Put $(p^K - 1)/n = m$. Let g be a generator of the multiplicative group of $GF(p^K)$, and M be the $n \times m$ matrix

$$\begin{pmatrix} g & g^{n+1} & \cdots & g^{(m-1)n+1} \\ g^2 & g^{n+2} & & g^{(m-1)n+2} \\ \vdots & \vdots & & \vdots \\ g^n & g^{n+n} & \cdots & g^{(m-1)n+n} \end{pmatrix}$$

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where $g^{(m-1)n+n} = g^{mn} = 1$. We will write R_i for the i-th row of M, considered as a set. The complex operations on the rows have their usual meaning, that is

$$-R_i = \{-g^i, -g^{n+i}, \ldots, -g^{(m-1)n+i}\}$$

and

$$R_i + R_j = \{a + b \colon a \in R_i, \ b \in R_j\}.$$

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(iv) for every $k, \ell \in \{1, \dots, n-1\}$ with $k \neq \ell$, there are $i, j \in \{0, \dots, m-1\}$, such that $g^{in+\ell} + 1 = g^{jn+k}$.

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Lemma

If (i)–(iv) above hold, then:

1. $-R_i = R_i$,

2. $R_i + R_i = \bigcup_{j \neq i} R_j$,

3. $R_i + R_j = M$, if $i \neq j$,

for every $i, j \in \{1, \ldots, m\}$.

Representations/Colourings

Theorem

Let $\mathbf{M}_n$ be a Ramsey algebra, and $GF(p^K)$ is such that n divides $p^K - 1$. Put $m = (p^K - 1)/n$ and let M be an $n \times m$ matrix over $GF(p^K)$ constructed as before. Suppose M satisfies the representability conditions (i)–(iv). Then

- ▶ *$\mathbf{M}_n$ is representable over $GF(p^K)$ — more precisely, over the additive group of $GF(p^K)$.*
- ▶ *The representation of $\mathbf{M}_n$ is the subalgebra of the complex algebra of the additive group of $GF(p^K)$, whose atoms are the sets $\{0\}$ and R_i, for $i \in \{1, \ldots, n\}$.*

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That is, if for a given n we can find a suitable finite field, then all is well. But can we?

THREE ODDITIES

Clrs	Repres. over p^K	Upper bound	Comment
2	5	5	unique
3	13, $2^4 = 16$	16	Ramsey bound attained
4	41	65	exhaustive
5	71, 101	326	exhaustive
6	97, 157, 277	1957	exhaustive
7	491	13700	exhaustive
8	none	109601	exhaustive
9	$19^2 = 361$	986410	exh., no prime field repr.
10	1181	9864101	exhaustive
11	947, 1409	108505112	not exhaustive
12	769, 1201	1032061345	not exhaustive
13	???	13416797486	not exhaustive

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Colourings (representations) over prime fields

n	repr.		*n*	repr.		*n*	repr.		*n*	repr.		*n*	repr.
			25	3701		49	22541		73	44511		97	96419
2	5		26	4889		50	22901		74	58313		98	105449
3	13		27	5563		51	19687		75	48751		99	87517
4	41		28	8849		52	29537		76	39521		100	95801
5	71		29	6323		53	26501		77	70379		101	154127
6	97		30	5521		54	21493		78	53197		102	95881
7	491		31	6263		55	23321		79	64781		103	119687
8			32	5441		56	23297		80	53441		104	131249
9			33	8779		57	21319		81	65287		105	89671
10	1181		34	7481		58	30509		82	64781		106	144161
11	947		35	7841		59	28439		83	113213		107	88811
12	769		36	10657		60	26041		84	76777		108	122041
13			37	13469		61	45263		85	91121		109	128621
14	1709		38	12161		62	27281		86	80153		110	122321
15	1291		39	8971		63	30367		87	70123		111	95461
16	1217		40	14561		64	39041		88	67409		112	122753
17	4013		41	13367		65	37181		89	131543		113	120233
18	2521		42	19993		66	29569		90	74161		114	98953
19	1901		43	14621		67	38459		91	81173		115	115001
20	2801		44	12497		68	64601		92	80777		116	159617
21	1933		45	14401		69	31741		93	78307		117	118873
22	3257		46	14537		70	45641		94	70877		118	159773
23	3221		47	20117		71	36353		95	100511		119	166601
24	4129		48	18913		72	37441		96	136897		120	120721

WE ARE DOING SCIENCE
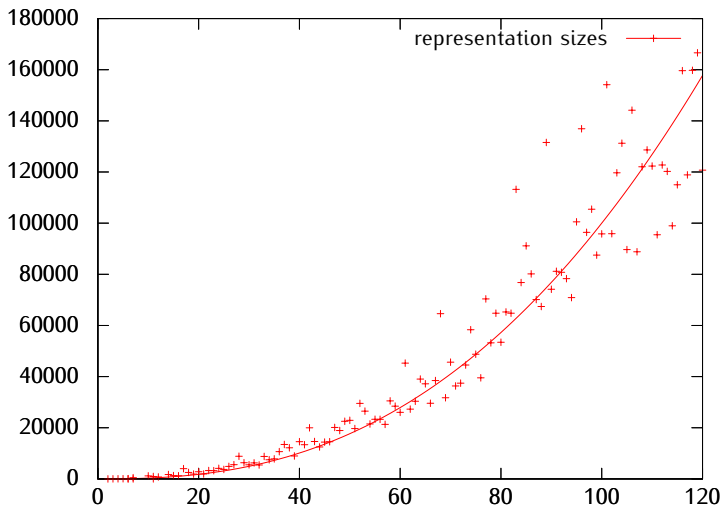
WE ARE TRYING TO DO MATHS

Conjecture

*Let $n > 13$. Then there exist a prime p such that n divides $p - 1$
and the $n \times m$ matrix (with $m = (p - 1)/n$)*

$$
\begin{pmatrix}
g & g^{n+1} & \cdots & g^{(m-1)n+1} \\
g^2 & g^{n+2} & & g^{(m-1)n+2} \\
\vdots & \vdots & & \vdots \\
g^n & g^{n+n} & \cdots & g^{(m-1)n+n}
\end{pmatrix}
$$

over $GF(p)$ satisfies representability conditions for n.

A POSTSCRIPT

Recently, Alm and Manske announced the following result:

Theorem

The algebras $\mathbf{M}_n$ are representable for $2 \leq n \leq 300$.

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Their construction is essentially different from mine, but still 8 and 13 are exceptions. One wonders…