The quest for the basic fuzzy logic

Petr Cintula[1] Rostislav Horčík[1] Carles Noguera[1,2]

[1]Institute of Computer Science, Academy of Sciences of the Czech Republic
Pod vodárenskou věží 2, 182 07 Prague, Czech Republic

[2]Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic
Pod vodárenskou věží 4, 182 08 Prague, Czech Republic

The beginnings

- The original three fuzzy logics ($Ł$, G, and Π) are complete w.r.t. a standard semantics on $[0, 1]$ of a particular (continuous) residuated t-norm, and w.r.t. algebraic semantics (MV-, G-, and Π-algebras).

Hájek logic

- Hájek logic BL (1998): complete w.r.t. standard semantics given by all continuous t-norms, and w.r.t. BL-algebras (semilinear divisible integral commutative lattice-ordered residuated monoids).

A BL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1} \rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a commutative monoid,

(3) $z \leq x \rightarrow y$ iff $x \& z \leq y$, (residuation)

(4) $x \& (x \rightarrow y) = x \wedge y$ (divisibility)

(5) $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$ (prelinearity)

Basic fuzzy logic?

BL was *basic* in the following two senses:

1. *it could not be made weaker without losing essential properties and*
2. *it provided a base for the study of all fuzzy logics.*

Basic fuzzy logic?

BL was *basic* in the following two senses:

1. *it could not be made weaker without losing essential properties and*
2. *it provided a base for the study of all fuzzy logics.*

Because:

- BL is complete w.r.t. the semantics given by *all* continuous t-norms
- $\mathrm{Ł}$, G, and Π are axiomatic extensions of BL. The methods to introduce, algebraize, and study BL could be utilized for any other logic based on continuous t-norms. Hájek developed a uniform mathematical theory for MFL

fuzzy logics = axiomatic extensions of BL

Monoidal t-norm logic MTL

Left-continuity of the t-norm is sufficient for residuation (i.e. so we can define $x \Rightarrow y = \max\{z \in [0,1] \mid z * x \leq y\}$).

Monoidal t-norm logic MTL

Left-continuity of the t-norm is sufficient for residuation (i.e. so we can define $x \Rightarrow y = \max\{z \in [0,1] \mid z * x \leq y\}$).

- MTL (2001): complete w.r.t. standard semantics given by all left-continuous t-norms, and w.r.t. MTL-algebras (semilinear integral commutative lattice-ordered residuated monoids).

Monoidal t-norm logic MTL

Left-continuity of the t-norm is sufficient for residuation (i.e. so we can define $x \Rightarrow y = \max\{z \in [0,1] \mid z * x \leq y\}$).

- MTL (2001): complete w.r.t. standard semantics given by all left-continuous t-norms, and w.r.t. MTL-algebras (semilinear integral commutative lattice-ordered residuated monoids).

An MTL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1} \rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a commutative monoid,

(3) $z \leq x \rightarrow y$ iff $x \,\&\, z \leq y$, (residuation)

(4) $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$ (prelinearity)

Monoidal t-norm logic MTL

Left-continuity of the t-norm is sufficient for residuation (i.e. so we can define $x \Rightarrow y = \max\{z \in [0,1] \mid z * x \leq y\}$).

- MTL (2001): complete w.r.t. standard semantics given by all left-continuous t-norms, and w.r.t. MTL-algebras (semilinear integral commutative lattice-ordered residuated monoids).

An MTL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1}\rangle$ such that:

$$
\begin{array}{lll}
(1) & \langle B, \wedge, \vee, \overline{0}, \overline{1}\rangle \text{ is a bounded lattice,} & \\
(2) & \langle B, \&, \overline{1}\rangle \text{ is a commutative monoid,} & \\
(3) & z \leq x \rightarrow y \text{ iff } x \,\&\, z \leq y, & \text{(residuation)} \\
(4) & (x \rightarrow y) \vee (y \rightarrow x) = \overline{1} & \text{(prelinearity)}
\end{array}
$$

fuzzy logics = axiomatic expansions of MTL

Monoidal t-norm logic MTL

Left-continuity of the t-norm is sufficient for residuation (i.e. so we can define $x \Rightarrow y = \max\{z \in [0,1] \mid z * x \leq y\}$).

- MTL (2001): complete w.r.t. standard semantics given by all left-continuous t-norms, and w.r.t. MTL-algebras (semilinear integral commutative lattice-ordered residuated monoids).

An MTL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \overline{0}, \overline{1}\rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1}\rangle$ is a commutative monoid,

(3) $z \leq x \to y$ iff $x \,\&\, z \leq y$, (residuation)

(4) $(x \to y) \vee (y \to x) = \overline{1}$ (prelinearity)

fuzzy logics = axiomatic expansions of MTL

$$\mathrm{MTL} = \mathrm{FL}^{\ell}_{ew}.$$

“Pulling legs from the flea”

- $\mathrm{psMTL}^r = \mathrm{FL}_w^\ell$ (2003): logic of semilinear integral lattice-ordered residuated monoids. It is standard complete.

“Pulling legs from the flea”

- $\mathrm{psMTL}^r = \mathrm{FL}_w^\ell$ (2003): logic of semilinear integral lattice-ordered residuated monoids. It is standard complete.
- $\mathrm{UL} = \mathrm{FL}_e^\ell$ (2007): logic of semilinear commutative lattice-ordered residuated monoids. It is standard complete.

“Pulling legs from the flea”

- $\mathrm{psMTL}^r = \mathrm{FL}_w^\ell$ (2003): logic of semilinear integral lattice-ordered residuated monoids. It is standard complete.
- $\mathrm{UL} = \mathrm{FL}_e^\ell$ (2007): logic of semilinear commutative lattice-ordered residuated monoids. It is standard complete.
- FL^ℓ (2009): logic of semilinear lattice-ordered residuated monoids. It is NOT standard complete.

“Pulling legs from the flea”

- $\mathrm{psMTL}^r = \mathrm{FL}_w^\ell$ (2003): logic of semilinear integral lattice-ordered residuated monoids. It is standard complete.
- $\mathrm{UL} = \mathrm{FL}_e^\ell$ (2007): logic of semilinear commutative lattice-ordered residuated monoids. It is standard complete.
- FL^ℓ (2009): logic of semilinear lattice-ordered residuated monoids. It is NOT standard complete.

What is the basic fuzzy logic?

The hidden thing

Associativity is always assumed.

The hidden thing

Associativity is always assumed.

What if we pull this final leg? Will the flea jump again?

The hidden thing

Associativity is always assumed.
What if we pull this final leg? Will the flea jump again?

Some works on non-associative substructural logics:

- Lambek (1961)
- Buszkowski and Farulewski (2009)
- Galatos and Ono. Cut elimination and strong separation for substructural logics: An algebraic approach, *Annals of Pure and Applied Logic*, 161(9):1097–1133, 2010.
- Botur (2011)

A basic substructural logic

SL: Galatos-Ono logic

Non-associative full Lambek logic

A basic substructural logic

SL: Galatos-Ono logic

Non-associative full Lambek logic

Aims

1. Find an algebraic semantics for SL.
2. Axiomatize its semilinear extension SL^{ℓ}.
3. Proof standard completeness for SL^{ℓ}.

Lattice-ordered residuated unital groupoid or SL-algebra is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a doubly pointed lattice satisfying $x = 1 \cdot x = x \cdot 1$ and for all $a, b, c \in A$ we have

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c/b\,.$$

SL-chain: linearly ordered SL-algebra.

Variety of all SL-algebras: $\mathbb{SL}$.

Given a class $\mathbb{K} \subseteq \mathbb{SL}$, a set of formulae Γ and a formula φ, $\Gamma \models_{\mathbb{K}} \varphi$ if for every $\boldsymbol{A} \in \mathbb{K}$ and every $\boldsymbol{A}$-evaluation e, if $e(\psi) \geq 1$ for every $\psi \in \Gamma$, then $e(\varphi) \geq 1$.

Algebraic semantics – 2

Theorem

For every set of formulae Γ *and every formula* φ *we have:*

$$\Gamma \vdash_{\mathrm{SL}} \varphi \text{ if, and only if, } \Gamma \models_{\mathbb{SL}} \varphi.$$

SL is an algebraizable logic and $\mathbb{SL}$ is its equivalent algebraic semantics with translations:

$$E(p,q) = \{p \to q, q \to p\} \text{ and } \mathcal{E}(p) = \{p \wedge \overline{1} \approx \overline{1}\}.$$

Finitary extensions of SL correspond to quasivarieties of SL-algebras.

Almost (MP)-based logics: definition

Definition

Let bDT be a set of $\star$-formulae. A substructural logic L is almost (MP)-based w.r.t. the set of basic deduction terms bDT if:

- L has a presentation where the only deduction rules are *modus ponens* and $\{\varphi \vdash \gamma(\varphi) \mid \varphi \in Fm_{\mathcal{L}_{\mathrm{SL}}}, \gamma \in \mathrm{bDT}\}$,

Almost (MP)-based logics: definition

Definition

Let bDT be a set of $\star$-formulae. A substructural logic L is almost (MP)-based w.r.t. the set of basic deduction terms bDT if:

- L has a presentation where the only deduction rules are *modus ponens* and $\{\varphi \vdash \gamma(\varphi) \mid \varphi \in Fm_{\mathcal{L}_{\mathrm{SL}}}, \gamma \in \mathrm{bDT}\}$,
- the set bDT is closed under all $\star$-substitutions σ such that $\sigma(\star) = \star$, and
- for each $\beta \in \mathrm{bDT}$ and each formulae φ, ψ, there exist $\beta_1, \beta_2 \in \mathrm{bDT}^*$ such that:

$$\vdash_{\mathrm{L}} \beta_1(\varphi \to \psi) \to (\beta_2(\varphi) \to \beta(\psi)).$$

Almost (MP)-based logics: definition

Definition

Let bDT be a set of $\star$-formulae. A substructural logic L is almost (MP)-based w.r.t. the set of basic deduction terms bDT if:

- L has a presentation where the only deduction rules are *modus ponens* and $\{\varphi \vdash \gamma(\varphi) \mid \varphi \in Fm_{\mathcal{L}_{\mathrm{SL}}}, \gamma \in \mathrm{bDT}\}$,
- the set bDT is closed under all $\star$-substitutions σ such that $\sigma(\star) = \star$, and
- for each $\beta \in \mathrm{bDT}$ and each formulae φ, ψ, there exist $\beta_1, \beta_2 \in \mathrm{bDT}^*$ such that:

$$\vdash_{\mathrm{L}} \beta_1(\varphi \to \psi) \to (\beta_2(\varphi) \to \beta(\psi)).$$

L is called (MP)-based if it admits the empty set as a set of basic deduction terms.

New Hilbert-system $\mathcal{AS}$ for SL – axioms

$$
\begin{array}{rl}
(\mathrm{Adj}_{\&}) & \varphi \to (\psi \to \psi \,\&\, \varphi) \\
(\mathrm{Adj}_{\&\rightsquigarrow}) & \varphi \to (\psi \rightsquigarrow \varphi \,\&\, \psi) \\
(\&\wedge) & (\varphi \wedge \overline{1}) \,\&\, (\psi \wedge \overline{1}) \to \varphi \wedge \psi \\
(\wedge 1) & \varphi \wedge \psi \to \varphi \\
(\wedge 2) & \varphi \wedge \psi \to \psi \\
(\wedge 3) & (\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi) \\
(\vee 1) & \varphi \to \varphi \vee \psi \\
(\vee 2) & \psi \to \varphi \vee \psi \\
(\vee 3) & (\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi) \\
(\mathrm{Push}) & \varphi \to (\overline{1} \to \varphi) \\
(\mathrm{Pop}) & (\overline{1} \to \varphi) \to \varphi \\
(\mathrm{Res}') & \psi \,\&\, (\varphi \,\&\, (\varphi \to (\psi \to \chi))) \to \chi \\
(\mathrm{Res}'_{\rightsquigarrow}) & (\varphi \,\&\, (\varphi \to (\psi \rightsquigarrow \chi))) \,\&\, \psi \to \chi \\
(\mathrm{T}') & (\varphi \to (\varphi \,\&\, (\varphi \to \psi)) \,\&\, (\psi \to \chi)) \to (\varphi \to \chi) \\
(\mathrm{T}'_{\rightsquigarrow}) & (\varphi \rightsquigarrow ((\varphi \rightsquigarrow \psi) \,\&\, \varphi) \,\&\, (\psi \to \chi)) \to (\varphi \rightsquigarrow \chi)
\end{array}
$$

New Hilbert-system $\mathcal{AS}$ for SL – rules

$$
\begin{array}{rl}
(\mathbf{MP}) & \varphi, \varphi \to \psi \vdash \psi \\
(\mathrm{Adj_u}) & \varphi \vdash \varphi \wedge \overline{1} \\
(\alpha) & \varphi \vdash \delta \mathbin{\&} \varepsilon \to \delta \mathbin{\&} (\varepsilon \mathbin{\&} \varphi) \\
(\alpha') & \varphi \vdash \delta \mathbin{\&} \varepsilon \to (\delta \mathbin{\&} \varphi) \mathbin{\&} \varepsilon \\
(\beta) & \varphi \vdash \delta \to (\varepsilon \to (\varepsilon \mathbin{\&} \delta) \mathbin{\&} \varphi) \\
(\beta') & \varphi \vdash \delta \to (\varepsilon \rightsquigarrow (\delta \mathbin{\&} \varepsilon) \mathbin{\&} \varphi)
\end{array}
$$

SL is almost (MP)-based

Theorem

$\mathcal{AS}$ is an axiomatic system for SL.

SL is almost (MP)-based

Theorem

$\mathcal{AS}$ is an axiomatic system for SL.

Definition

Given arbitrary formulae δ, ε, we define the following $\star$-formulae:

$$\alpha_{\delta,\varepsilon} = (\delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \star))$$
$$\alpha'_{\delta,\varepsilon} = (\delta \,\&\, \varepsilon \to (\delta \,\&\, \star) \,\&\, \varepsilon)$$
$$\beta_{\delta,\varepsilon} = (\delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \star)$$
$$\beta'_{\delta,\varepsilon} = (\delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \star)$$

SL is almost (MP)-based

Theorem

$\mathcal{AS}$ is an axiomatic system for SL.

Definition

Given arbitrary formulae δ, ε, we define the following $\star$-formulae:

$$\alpha_{\delta,\varepsilon} = (\delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \star))$$
$$\alpha'_{\delta,\varepsilon} = (\delta \,\&\, \varepsilon \to (\delta \,\&\, \star) \,\&\, \varepsilon)$$
$$\beta_{\delta,\varepsilon} = (\delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \star)$$
$$\beta'_{\delta,\varepsilon} = (\delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \star)$$

Theorem

SL is almost (MP)-based with respect to the set

$$\mathrm{bDT}_{\mathrm{SL}} = \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}, \star \wedge \overline{1}, \mid \delta, \varepsilon \textit{ formulae}\}.$$

Simplifications in extensions

Logic L	$\mathrm{bDT}_{\mathrm{L}}$
SL	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon$ formulae$\}$
SL_{w}	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon} \mid \delta, \varepsilon$ formulae$\}$
SL_{e}	$\{\alpha_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon$ formulae$\}$
$\mathrm{SL}_{\mathrm{ew}}$	$\{\alpha_{\delta,\varepsilon}, \beta_{\delta,\varepsilon} \mid \delta, \varepsilon$ formulae$\}$
SL_{a}	$\{\lambda_{\varepsilon}, \rho_{\varepsilon}, \star \wedge \overline{1} \mid \varepsilon$ a formula$\}$
$\mathrm{SL}_{\mathrm{ae}}$	$\{\star \wedge \overline{1}\}$
$\mathrm{SL}_{\mathrm{aew}}$	$\emptyset$

Recall the conjugates in FL: $\lambda_{\varepsilon} = \varepsilon \to \star \,\&\, \varepsilon$ and $\rho_{\varepsilon} = \varepsilon \rightsquigarrow \varepsilon \,\&\, \star$.

Definition

Let L be a expansion of SL and let $\mathbb{K}$ be the class of all L-chains. We say that L is *semilinear* if *one of the following equivalent conditions* is met:

- For every set of formulae $\Gamma \cup \{\varphi\}$ we have:

  $$\Gamma \vdash_{\mathrm{L}} \varphi \quad \text{if, and only if,} \quad \Gamma \models_{\mathbb{K}} \varphi.$$

- For every set of formulae $\Gamma \cup \{\varphi, \psi, \chi\}$ we have:

  $$\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \quad \text{and} \quad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi \quad \text{imply} \quad \Gamma \vdash_{\mathrm{L}} \chi.$$

- $\mathbb{K}$ is the class of all relatively finitely subdirectly irreducible L-algebras.

Axiomatization of semilinear extensions

Given L, we define L^{ℓ} as the least semilinear logic extending L (i.e. the logic of L-chains).

Theorem

Let L be an almost (MP)-based logic with the set bDT of basic deductive terms. Then L^{ℓ} is axiomatized, relatively to L, by any of the following four sets of axioms/rules:

A $\gamma_1(\varphi \to \psi) \vee \gamma_2(\psi \to \varphi)$, *for every* $\gamma_1, \gamma_2 \in (\mathrm{bDT} \cup \{\star \wedge \overline{1}\})^*$

B $(\varphi \to \psi) \vee (\psi \to \varphi)$

$(\varphi \to \psi) \vee \chi, \varphi \vee \chi \vdash \psi \vee \chi$

$\varphi \vee \psi \vdash \gamma(\varphi) \vee \psi$, *for every* $\gamma \in \mathrm{bDT}$

C $((\varphi \to \psi) \wedge \overline{1}) \vee \gamma((\psi \to \varphi) \wedge \overline{1})$, *for every* $\gamma \in \mathrm{bDT} \cup \{\star\}$

D $(\varphi \vee \psi \to \psi) \vee \gamma(\varphi \vee \psi \to \psi)$, *for every* $\gamma \in \mathrm{bDT} \cup \{\star \wedge \overline{1}\}$

Proof of strong completeness w.r.t. standard chains

dp-chain

Doubly pointed chain: $\boldsymbol{A} = \langle A, \wedge, \vee, 0, 1\rangle$ a chain endowed with additional constants $0, 1$.

rt-groupoid

Semiunital residuated totally ordered groupoid:
$\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a *dp*-chain satisfying $x \leq (1 \cdot x) \wedge (x \cdot 1)$ and for all $a, b, c \in A$ we have

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c/b\,.$$

SL-chain

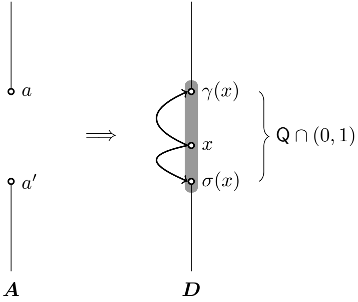
Unital residuated totally ordered groupoid: *rt*-groupoid satisfying $1 \cdot x = x = x \cdot 1$

The proof – 1

- Suppose that we have a countable nontrivial $\mathcal{SL}$-chain
$$\boldsymbol{A} = \langle A, \wedge, \vee, \circ^{\boldsymbol{A}}, \backslash^{\boldsymbol{A}}, /^{\boldsymbol{A}}, 0, 1\rangle$$

The proof – 1

- Suppose that we have a countable nontrivial SL-chain $\boldsymbol{A} = \langle A, \wedge, \vee, \circ^{\boldsymbol{A}}, \backslash^{\boldsymbol{A}}, /^{\boldsymbol{A}}, 0, 1\rangle$
- We extend its reduct $\langle A, \wedge, \vee, 0, 1\rangle$ to a bounded countably infinite dense dp-chain $\langle D, \wedge, \vee, 0, 1\rangle$ and get closure and interior operators γ and σ s.t. $\gamma[D] = \sigma[D] = A$.



The proof – 2

- We build a bounded rt-groupoid

$$\boldsymbol{D} = \langle D, \wedge, \vee, \circ^{\boldsymbol{D}}, \backslash^{\boldsymbol{D}}, /^{\boldsymbol{D}}, 0, 1\rangle$$

$$x \circ^{\boldsymbol{D}} y = \gamma(x) \circ^{\boldsymbol{A}} \gamma(y) \qquad x /^{\boldsymbol{D}} y = \sigma(x) /^{\boldsymbol{A}} \gamma(y) \qquad x \backslash^{\boldsymbol{D}} y = \gamma(x) \backslash^{\boldsymbol{A}} \sigma(y)$$

The proof – 2

- We build a bounded rt-groupoid

$$\boldsymbol{D} = \langle D, \wedge, \vee, \circ^{\boldsymbol{D}}, \backslash^{\boldsymbol{D}}, /^{\boldsymbol{D}}, 0, 1\rangle$$

$$x \circ^{\boldsymbol{D}} y = \gamma(x) \circ^{\boldsymbol{A}} \gamma(y) \qquad x /^{\boldsymbol{D}} y = \sigma(x) /^{\boldsymbol{A}} \gamma(y) \qquad x \backslash^{\boldsymbol{D}} y = \gamma(x) \backslash^{\boldsymbol{A}} \sigma(y)$$

- We build a bounded SL-chain

$$\boldsymbol{M}(\boldsymbol{D}) = \langle D, \wedge, \vee, \odot, \rightarrow, 0, 1\rangle$$

$$x \odot y = \begin{cases} \top & \text{if } x, y > 1, \\ \bot & \text{if } x = \bot \text{ or } y = \bot, \\ x \wedge y & \text{if } x, y \leq 1, \\ x \vee y & \text{otherwise.} \end{cases}$$

The proof – 3

- Then we we build

$$\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D}) = \langle D, \wedge, \vee, \circ, \backslash, /, 0, 1 \rangle$$

$$a \circ b = (a \circ^{\boldsymbol{D}} b) \wedge (a \circ^{\boldsymbol{M}(\boldsymbol{D})} b),$$

$$a \backslash b = (a \backslash^{\boldsymbol{D}} b) \vee (a \backslash^{\boldsymbol{M}(\boldsymbol{D})} b), \qquad a/b = (a/^{\boldsymbol{D}} b) \vee (a/^{\boldsymbol{M}(\boldsymbol{D})} b).$$

which is a bounded countably infinite dense SL-chain.

The proof – 3

- Then we we build
$$\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D}) = \langle D, \wedge, \vee, \circ, \backslash, /, 0, 1 \rangle$$
$$a \circ b = (a \circ^{\boldsymbol{D}} b) \wedge (a \circ^{\boldsymbol{M}(\boldsymbol{D})} b) \,,$$
$$a \backslash b = (a \backslash^{\boldsymbol{D}} b) \vee (a \backslash^{\boldsymbol{M}(\boldsymbol{D})} b) \,, \qquad a / b = (a /^{\boldsymbol{D}} b) \vee (a /^{\boldsymbol{M}(\boldsymbol{D})} b) \,.$$
which is a bounded countably infinite dense SL-chain.
- The identity map is an embedding of $\boldsymbol{A}$ into $\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D})$.

The proof – 3

- Then we we build

$$\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D}) = \langle D, \wedge, \vee, \circ, \backslash, /, 0, 1 \rangle$$

$$a \circ b = (a \circ^{\boldsymbol{D}} b) \wedge (a \circ^{\boldsymbol{M}(\boldsymbol{D})} b)\,,$$

$$a \backslash b = (a \backslash^{\boldsymbol{D}} b) \vee (a \backslash^{\boldsymbol{M}(\boldsymbol{D})} b)\,, \qquad a / b = (a /^{\boldsymbol{D}} b) \vee (a /^{\boldsymbol{M}(\boldsymbol{D})} b)\,.$$

  which is a bounded countably infinite dense SL-chain.
- The identity map is an embedding of $\boldsymbol{A}$ into $\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D})$.
- Finally we embed [Galatos-Jipsen] $\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D})$ into a complete SL-chain which has to be isomorphic with some standard one.

The proof – 3

- Then we we build

$$\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D}) = \langle D, \wedge, \vee, \circ, \backslash, /, 0, 1\rangle$$

$$a \circ b = (a \circ^{\boldsymbol{D}} b) \wedge (a \circ^{\boldsymbol{M}(\boldsymbol{D})} b)\,,$$

$$a \backslash b = (a \backslash^{\boldsymbol{D}} b) \vee (a \backslash^{\boldsymbol{M}(\boldsymbol{D})} b)\,, \qquad a / b = (a /^{\boldsymbol{D}} b) \vee (a /^{\boldsymbol{M}(\boldsymbol{D})} b)\,.$$

  which is a bounded countably infinite dense SL-chain.
- The identity map is an embedding of $\boldsymbol{A}$ into $\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D})$.
- Finally we embed [Galatos-Jipsen] $\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D})$ into a complete SL-chain which has to be isomorphic with some standard one.
- Morever, the embedding preserves existing suprema and infima.

Definition

A logic L is a *core semilinear logic* if it expands SL^ℓ by some sets of axioms Ax and rules R such that for each $\langle\Gamma, \varphi\rangle \in R$ and every formula ψ we have:

$$\Gamma \vee \psi \vdash_{\mathrm{L}} \varphi \vee \psi,$$

where by $\Gamma \vee \psi$ we denote the set $\{\chi \vee \psi \mid \chi \in \Gamma\}$.

Conclusions

- SL^{ℓ} is a very weak logic (it does not even satisfy associativity)

Conclusions

- SL^{ℓ} is a very weak logic (it does not even satisfy associativity)
- SL^{ℓ} is a basic fuzzy logic:

Conclusions

- SL^{ℓ} is a very weak logic (it does not even satisfy associativity)
- SL^{ℓ} is a basic fuzzy logic:
  1. SL^{ℓ} has standard completeness (even at first-order level).

Conclusions

- SL^{ℓ} is a very weak logic (it does not even satisfy associativity)
- SL^{ℓ} is a basic fuzzy logic:
  1. SL^{ℓ} has standard completeness (even at first-order level).
  2. Core semilinear logics are a framework (based on SL^{ℓ}) encompassing virtually all fuzzy logics.

Conclusions

- SL^{ℓ} is a very weak logic (it does not even satisfy associativity)
- SL^{ℓ} is a basic fuzzy logic:
  1. SL^{ℓ} has standard completeness (even at first-order level).
  2. Core semilinear logics are a framework (based on SL^{ℓ}) encompassing virtually all fuzzy logics.
- BL should be renamed to HL (Hájek Logic).

Bibliography

- P. Cintula, R. Horčík, and C. Noguera. Non-associative substructural logis and their semilinear extensions: axiomatization and completeness properties, *The Review of Symbolic Logic* 6 (2013) 794-423.
- P. Cintula, R. Horčík, and C. Noguera. The quest for the basic fuzzy logic, to appear in *Petr Hájek on Mathematical Fuzzy Logic*, Trends in Logic, Springer.