Complete MV-algebra valued Pavelka logic

Esko Turunen
MC IEF Fellow, TU Wien
TU Tampere, Finland

14.12.2013

• Zadeh introduced his Fuzzy Sets in 1965.

• In 1968–9 Goguen outlined some characteristic features fuzzy logic should obey; in his article The logic of inexact concepts he game to a conclusion that complete residuated lattices should have a similar role to fuzzy logic than Boolean algebras have to Classical Logic.

• In 1979 Pavelka published a series of articles On Fuzzy Logic I, II, III, in which he discussed the matter in depth. This meant a generalization of Classical Logic in such a way that axioms, theories, theorems, and tautologies need not be only fully true or fully false, but may be also true to a degree and, therefore, giving rise to such concepts as fuzzy theories, fuzzy set of axioms, many-valued rules of inference, provability degree, truth degree, fuzzy consequence operation etc.

Pavelka's definitions and concepts are meaningful in any fixed complete residuated lattice L. Given L-valued (fuzzy sub-)sets X, Y, a fuzzy consequence operation $\mathcal{C}$ satisfies

- $X \leq \mathcal{C}(X)$,
- if $X \leq Y$ then $\mathcal{C}(X) \leq \mathcal{C}(Y)$,
- $\mathcal{C}(X) = \mathcal{C}(\mathcal{C}(X))$.

The main question is: how to define a semantic consequence operation $\mathcal{C}^{sem}$ and a syntactic consequence operation $\mathcal{C}^{syn}$ and when do they coincide, i.e.

$$\mathcal{C}^{sem}(X)(\alpha) = \mathcal{C}^{syn}(X)(\alpha) \text{ for all } X \text{ and all } \alpha \in X.$$

Pavelka 1979: If $L = [0,1]$ the answer is affirmative iff L is an MV-algebra.
Turunen 1995: affirmative if L is an injective MV-algebra.
New: the answer is affirmative iff L is a complete MV-algebra.

The set of atomic formulas $\mathcal{F}_0$ is composed of propositional variables $\mathrm{p}, \mathrm{q}, \mathrm{r}, \mathrm{s}, \cdots$ and truth constants $\mathbf{a}$ corresponding to elements $a \in L$; they generalize the classical truth constants $\perp$ and $\top$. The set $\mathcal{F}$ of all formulas is then constructed in the usual way. Any mapping $v : \mathcal{F}_0 \to L$ such that $v(\mathbf{a}) = a$ for all truth constants $\mathbf{a}$ can be extended recursively into the whole $\mathcal{F}$ by setting

$$v(\alpha \,\mathtt{imp}\, \beta) = v(\alpha) \to v(\beta) \qquad \text{and}$$
$$v(\alpha \,\mathtt{and}\, \beta) = v(\alpha) \odot v(\beta).$$

Such mappings v are called valuations. The truth degree of a wff α is the infimum of all values $v(\alpha)$, that is

$$\mathcal{C}^{\mathrm{sem}}(\alpha) = \bigwedge \{ v(\alpha) \mid v \text{ is a valuation } \}.$$

We may also fix some set $\mathcal{T} \subseteq \mathcal{F}$ of wffs and associate to each $\alpha \in \mathcal{T}$ a value $\mathcal{T}(\alpha)$ determining its degree of truth. We consider valuations v such that $\mathcal{T}(\alpha) \leq v(\alpha)$ for all wffs α. If such a valuation exists, then $\mathcal{T}$ is called satisfiable and v satisfies $\mathcal{T}$. We say that $\mathcal{T}$ is a fuzzy theory and the corresponding formulae α are the special axioms Then we consider values

$$\mathcal{C}^{\rm sem}(\mathcal{T})(\alpha) = \bigwedge\{v(\alpha) \mid v \text{ is a valuation, } v \text{ satisfies } \mathcal{T}\}.$$

The set of logical axioms in Pavelka's Fuzzy Logic, denoted by $\mathtt{A}$, is composed by the following eleven forms of formulae; they receive the value $\mathbf{1}$ in any valuation v (except (Ax. 7))

(Ax. 1)	$\alpha\ \mathtt{imp}\ \alpha$,
(Ax. 2)	$(\alpha\ \mathtt{imp}\ \beta)\ \mathtt{imp}\ [(\beta\ \mathtt{imp}\ \gamma)\ \mathtt{imp}\ (\alpha\ \mathtt{imp}\ \gamma)]$,
(Ax. 3)	$(\alpha_1\ \mathtt{imp}\ \beta_1)\ \mathtt{imp}\ \{(\beta_2\ \mathtt{imp}\ \alpha_2)\ \mathtt{imp}\ [(\beta_1\ \mathtt{imp}\ \beta_2)\ \mathtt{imp}\ (\alpha_1\ \mathtt{imp}\ \alpha_2)]\}$,
(Ax. 4)	$\alpha\ \mathtt{imp}\ \mathbf{1}$,
(Ax. 5)	$\mathbf{0}\ \mathtt{imp}\ \alpha$,
(Ax. 6)	$(\alpha\ \mathtt{and}\ \mathtt{not}\ \alpha)\ \mathtt{imp}\ \beta$,
(Ax. 7)	$\mathbf{a}$,
(Ax. 8)	$\alpha\ \mathtt{imp}\ (\beta\ \mathtt{imp}\ \alpha)$,
(Ax. 9)	$(\mathbf{1}\ \mathtt{imp}\ \alpha)\ \mathtt{imp}\ \alpha$,
(Ax. 10)	$[(\alpha\ \mathtt{imp}\ \beta)\ \mathtt{imp}\ \beta]\ \mathtt{imp}\ [(\beta\ \mathtt{imp}\ \alpha)\ \mathtt{imp}\ \alpha]$,
(Ax. 11)	$(\mathtt{not}\ \alpha\ \mathtt{imp}\ \mathtt{not}\ \beta)\ \mathtt{imp}\ (\beta\ \mathtt{imp}\ \alpha)$.

A fuzzy rule of inference is a scheme

$$\frac{\alpha_1, \cdots, \alpha_n}{r^{\mathrm{syn}}(\alpha_1, \cdots, \alpha_n)} \quad , \quad \frac{a_1, \cdots, a_n}{r^{\mathrm{sem}}(a_1, \cdots, a_n)}$$

where the wffs $\alpha_1, \cdots, \alpha_n$ are premises and the wff $r^{\mathrm{syn}}(\alpha_1, \cdots, \alpha_n)$ is the conclusion. The values $a_1, \cdots, a_n$ and $r^{\mathrm{sem}}(a_1, \cdots, a_n) \in L$ are the corresponding truth values. The mappings $r^{\mathrm{sem}} : L^n \to L$ are semi-continuous, i.e.

$$r^{\mathrm{sem}}(a_1, \cdots, \bigvee_{j \in \Gamma} a_{k_j}, \cdots, a_n) = \bigvee_{j \in \Gamma} r^{\mathrm{sem}}(a_1, \cdots, a_{k_j}, \cdots, a_n) \quad (1)$$

holds for all $1 \leq k \leq n$. Moreover, the fuzzy rules are required to be sound in the sense that

$$r^{\mathrm{sem}}(v(\alpha_1), \cdots, v(\alpha_n)) \leq v(r^{\mathrm{syn}}(\alpha_1, \cdots, \alpha_n))$$

holds for all valuations v.

REMARK 1 *The semi-continuity condition (1) can be replaced without any dramatic consequences by isotonicity condition (which is a weaker condition): if* $a_k \leq b_k$*, then*

$$r^{\text{sem}}(a_1, \cdots, a_k, \cdots, a_n) \leq r^{\text{sem}}(a_1, \cdots, b_k, \cdots, a_n) \qquad (2)$$

for each index $1 \leq k \leq n$.

The following Pavelka's fuzzy rules of inference, a set $\mathtt{R}$.

Generalized Modus Ponens:

$$\frac{\alpha, \alpha \ \mathtt{imp}\ \beta}{\beta} \quad , \quad \frac{a, b}{a \odot b}$$

a-Consistency testing rules:

$$\frac{\mathbf{a}}{\mathbf{0}} \quad , \quad \frac{b}{c}$$

where **a** is a truth constant and $c = \mathbf{0}$ if $b \leq a$ and $c = \mathbf{1}$ otherwise.

a-Lifting rules:

$$\frac{\alpha}{\mathbf{a}\ \mathtt{imp}\ \alpha} \quad , \quad \frac{b}{a \rightarrow b}$$

where **a** is a truth constant.

Rule of Bold Conjunction:

$$\frac{\alpha, \beta}{\alpha \ \mathtt{and}\ \beta} \quad , \quad \frac{a, b}{a \odot b}$$

It is easy to see that also a Rule of Bold Disjunction (not included in the list of Pavelka)

$$\frac{\alpha, \beta}{\alpha \; \texttt{or} \; \beta} \quad , \quad \frac{a, b}{a \oplus b}$$

is a rule of inference in Pavelka's sense. Indeed, isotonicity of r^{sem} follows by the isotonicity of the MV-operation $\oplus$ and soundness can be verified by taking a valuation v and observing that

$$\begin{aligned} r^{\mathrm{sem}}(v(\alpha), v(\beta)) &= v(\alpha) \oplus v(\beta) \\ &= v(\alpha \; \texttt{or} \; \beta) \\ &= v(r^{\mathrm{syn}}(\alpha, \beta)). \end{aligned}$$

This rule will be essential in Perfect Pavelka Logic.

A meta proof (called R-proof by Pavelka) w of a wff α in a fuzzy theory $\mathcal{T}$ is a finite sequence

$$\begin{array}{ccl} \alpha_1 & , & a_1 \\ \vdots & & \vdots \\ \alpha_m & , & a_m, \text{ the degree of the meta proof } w \end{array}$$

(i) $\alpha_m = \alpha$,
(ii) for each i, $1 \leq i \leq m$, α_i is a logical axiom, or is a special axiom of a fuzzy theory $\mathcal{T}$, or there is a fuzzy rule of inference and well formed formulae $\alpha_{i_1}, \cdots, \alpha_{i_n}$ with $i_1, \cdots, i_n < i$ such that $\alpha_i = r^{\rm syn}(\alpha_{i_1}, \cdots, \alpha_{i_n})$,
(iii) for each i, $1 \leq i \leq m$, the value $a_i \in L$ is given by

$$a_i = \begin{cases} a & \text{if } \alpha_i \text{ is the truth constant axiom } \mathbf{a}, \\ \mathbf{1} & \text{if } \alpha_i \text{ is some other logical axiom in the set } \mathtt{A}, \\ \mathcal{T}(\alpha_i) & \text{if } \alpha_i \text{ is a special axiom of a fuzzy theory } \mathcal{T}, \\ r^{\rm sem}(a_{i_1}, \cdots, a_{i_n}) & \text{if } \alpha_i = r^{\rm syn}(\alpha_{i_1}, \cdots, \alpha_{i_n}). \end{cases}$$

Since a wff α may have various meta proofs with different degrees, we define the provability degree of a formula α to be the supremum of all such values, i.e.,

$$\mathcal{C}^{\mathrm{syn}}(\mathcal{T})(\alpha) = \bigvee\{a_m \mid w \text{ is a meta proof for } \alpha \text{ in } \mathcal{T}\}.$$

In particular, $\mathcal{C}^{\mathrm{syn}}(\mathcal{T})(\alpha) = \mathbf{0}$ means that either α does not have any meta proof or that for any meta proof *w* of α the value $a_m = \mathbf{0}$. A fuzzy theory $\mathcal{T}$ is consistent if $\mathcal{C}^{\mathrm{sem}}(\mathcal{T})(\mathbf{a}) = a$ for all truth constants **a**. Any satisfiable fuzzy theory is consistent.
Completeness of Pavelka's Sentential Logic:

If $\mathcal{T}$ is consistent, then $\mathcal{C}^{\mathrm{sem}}(\mathcal{T})(\alpha) = \mathcal{C}^{\mathrm{syn}}(\mathcal{T})(\alpha)$ for any wff α.

Thus, in Pavelka's Fuzzy Sentential Logic we may talk about theorems of a degree *a* and tautologies of a degree *b* for $a, b \in L$, and these two values coincide for any formula α.

Let us now modify Pavelka approach such that L is a complete MV-algebra.

Axioms and rules of inference are the schemas (Ax.1) – (Ax.11) and the following

$$\text{(Ax.12)} \qquad [\alpha \texttt{ or } (\texttt{not}\alpha \texttt{ and } \beta)] \texttt{ imp } [(\alpha \texttt{ imp } \beta) \texttt{ imp } \beta],$$
$$\text{(Ax.13)} \qquad \mathbf{a} \texttt{ imp } \mathbf{b},$$

where α, β are wffs and **a**, **b** are truth constants.

The axioms (Ax.12) obtain value **1** in all valuations, and axioms (Ax.13), called book–keeping axioms, obtain a value $a \to b$.

Rules of inference are those of the original Pavelka logic and the Rule of Bold Disjunction

We need the following definitions and results to obtain Completeness of Complete MV–algebra valued Pavelka logic.

A fuzzy theory $\mathcal{T}$ is consistent if $\mathcal{C}^{sem}_{\mathcal{T}}(\mathbf{a}) = a$ for all truth constants $\mathbf{a}$, otherwise it is inconsistent.

PROPOSITION 2 *A fuzzy theory $\mathcal{T}$ is inconsistent iff $\mathcal{T} \vdash_1 \alpha$ holds for any wff α.*

PROPOSITION 3 *A fuzzy theory $\mathcal{T}$ is inconsistent iff the following condition holds:*

(C) There is a wff α and meta proofs w, w' with degrees $a_m, b_{m'}$ for α and $\texttt{not}\alpha$, respectively, such that $\mathbf{0} < a_m \odot b_{m'}$.

PROPOSITION 4 *A satisfiable fuzzy theory $\mathcal{T}$ is consistent.*

PROPOSITION 5 *If* $\mathcal{T} \vdash_a \alpha$ *then* $\mathcal{T} \vdash_1 (\mathbf{a}\ \mathtt{imp}\ \alpha)$.

PROPOSITION 6 $\mathcal{T} \vdash_1 [(\alpha\ \mathtt{and}\ \beta)\ \mathtt{imp}\ \alpha]$ *holds for any fuzzy theory* $\mathcal{T}$.

PROPOSITION 7 *If* $\mathcal{T}$ *is a consistent fuzzy theory and* $\mathcal{T} \vdash_a \alpha$, *then it holds that* $\mathcal{T} \vdash_0 (\mathtt{not}\mathbf{a}\ \mathtt{and}\ \alpha)$.

Assume $\mathcal{T}$ is a consistent fuzzy theory. Define

$$\alpha \equiv \beta \text{ if, and only if } \mathcal{T} \vdash_1 (\alpha \texttt{ imp } \beta) \text{ and } \mathcal{T} \vdash_1 (\beta \texttt{ imp } \alpha).$$

We obtain a congruence relation; denote the equivalence classes by $|\alpha|$ and by $\mathcal{F}/\equiv$ the set of all equivalence classes. Then we have

PROPOSITION 8 *Define* $|\alpha| \to |\beta| = |\alpha \texttt{ imp } \beta|$ *and* $|\alpha|^* = |\texttt{not}\alpha|$.
Then $\langle \mathcal{F}/\equiv, \to, ^*, |\mathbf{1}| \rangle$ *is a Wajsberg algebra and, hence, an MV–algebra.*

Even more can be proved:

PROPOSITION 9 *Assume* $\mathcal{T}$ *is a consistent fuzzy theory. If* $\mathcal{T} \vdash_{\mathsf{a}} \alpha$ *then* $|\alpha| = |\mathbf{a}|$ *in* $\mathcal{F}/\equiv$.

Thus $\mathcal{F}/\equiv$ is completely determined by the truth constants, which in turn are in one–to–one correspondence with the elements of L. Therefore there is an MV–isomorphism $\kappa : (\mathcal{F}/\equiv) \to L$ given by $\kappa(|\mathbf{a}|) = a$, in particular $\kappa(|\mathbf{1}|) = \mathbf{1}$.

Let π be the canonical mapping $\pi : \mathcal{F} \to \mathcal{F}/\equiv$. Then $\kappa \circ \pi$ is the valuation in demand; if $\mathcal{T} \vdash_a \alpha$ then $\kappa \circ \pi(\alpha) = \kappa(|\mathbf{a}|) = a$. In conclusion, we write

Completeness Theorem 1

Consider complete MV–algebra valued Pavelka style fuzzy sentential logic. If a formula α is provable at a degree $a \in L$ in a consistent fuzzy theory $\mathcal{T}$, then α is also a tautology at a degree a i.e. its truth degree is a.

As well known, a necessary condition for Pavelka style completeness is that the truth value set is a complete MV–algebra. By Completeness Theorem 1 we have that it is also a sufficient condition, i.e. we have

Completeness Theorem 2
Pavelka style fuzzy sentential logic is semantically complete if, and only if the set of truth values constitutes a complete MV–algebra.

We have studied Pavelka's fuzzy sentential logic and proved that it is semantically complete if, and only if the set of truth values constitutes a complete MV–algebra. However, a number of issues are still open, first of them concerns the completeness of first order logic. We conjecture that a similar result also applies to the first order Pavelka style fuzzy logic. Also the simplification of the presentation of is an open question; the set of inference rules can probably be reduced and the set of logical axioms is not a minimal one; the new axiom (Ax 12.) is redundant. From an application point of view it is also important that the set of truth constants could be reduced to a countable set; in this study the language under consideration is uncountable. All these issues are topics for a future work.