Complete MV-algebra valued Pavelka logic

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• Zadeh introduced his **Fuzzy Sets** in 1965.

• In 1968–9 Goguen outlined some characteristic features fuzzy logic should obey; in his article *The logic of inexact concepts* he game to a conclusion that complete residuated lattices should have a similar role to fuzzy logic than Boolean algebras have to Classical Logic.

• In 1979 Pavelka published a series of articles *On Fuzzy Logic I, II, III*, in which he discussed the matter in depth. This meant a generalization of Classical Logic in such a way that axioms, theories, theorems, and tautologies need not be only fully true or fully false, but may be also true to a degree and, therefore, giving rise to such concepts as fuzzy theories, fuzzy set of axioms, many-valued rules of inference, provability degree, truth degree, fuzzy consequence operation etc.
Pavelka’s definitions and concepts are meaningful in any fixed complete residuated lattice $L$. Given $L$-valued (fuzzy sub-)sets $X, Y$, a fuzzy consequence operation $C$ satisfies

- $X \leq C(X)$,
- if $X \leq Y$ then $C(X) \leq C(Y)$,
- $C(X) = C(C(X))$.

The main question is: how to define a semantic consequence operation $C^{sem}$ and a syntactic consequence operation $C^{syn}$ and when do they coincide, i.e.

$$C^{sem}(X)(\alpha) = C^{syn}(X)(\alpha) \text{ for all } X \text{ and all } \alpha \in X.$$ 

Pavelka 1979: If $L = [0, 1]$ the answer is affirmative iff $L$ is an MV-algebra.

Turunen 1995: affirmative if $L$ is an injective MV-algebra.

New: the answer is affirmative iff $L$ is a complete MV-algebra.
The set of atomic formulas $\mathcal{F}_0$ is composed of propositional variables $p, q, r, s, \cdots$ and truth constants $a$ corresponding to elements $a \in L$; they generalize the classical truth constants $\bot$ and $\top$. The set $\mathcal{F}$ of all formulas is then constructed in the usual way. Any mapping $\nu : \mathcal{F}_0 \to L$ such that $\nu(a) = a$ for all truth constants $a$ can be extended recursively into the whole $\mathcal{F}$ by setting

$$
\nu(\alpha \text{ imp } \beta) = \nu(\alpha) \to \nu(\beta) \quad \text{and} \quad \nu(\alpha \text{ and } \beta) = \nu(\alpha) \odot \nu(\beta).
$$

Such mappings $\nu$ are called valuations. The truth degree of a wff $\alpha$ is the infimum of all values $\nu(\alpha)$, that is

$$
\mathcal{C}^{sem}(\alpha) = \bigwedge \{ \nu(\alpha) \mid \nu \text{ is a valuation} \}.
$$
We may also fix some set $\mathcal{T} \subseteq \mathcal{F}$ of wffs and associate to each $\alpha \in \mathcal{T}$ a value $\mathcal{T}(\alpha)$ determining its degree of truth. We consider valuations $\nu$ such that $\mathcal{T}(\alpha) \leq \nu(\alpha)$ for all wffs $\alpha$. If such a valuation exists, then $\mathcal{T}$ is called satisfiable and $\nu$ satisfies $\mathcal{T}$. We say that $\mathcal{T}$ is a fuzzy theory and the corresponding formulae $\alpha$ are the special axioms. Then we consider values

$$C^{\text{sem}}(\mathcal{T})(\alpha) = \bigwedge \{\nu(\alpha) \mid \nu \text{ is a valuation, } \nu \text{ satisfies } \mathcal{T}\}.$$
The set of **logical axioms** in Pavelka’s Fuzzy Logic, denoted by \( A \), is composed by the following eleven forms of formulae; they receive the value 1 in any valuation \( v \) (except (Ax. 7))

(Ax. 1) \( \alpha \ imp \alpha \),

(Ax. 2) \( (\alpha \ imp \beta) \ imp [(\beta \ imp \gamma) \ imp (\alpha \ imp \gamma)] \),

(Ax. 3) \( (\alpha_1 \ imp \beta_1) \ imp \{(\beta_2 \ imp \alpha_2) \ imp [(\beta_1 \ imp \beta_2) \ imp (\alpha_1 \ imp \alpha_2)]\} \),

(Ax. 4) \( \alpha \ imp \ 1 \),

(Ax. 5) \( 0 \ imp \alpha \),

(Ax. 6) \( (\alpha \ and \ not \alpha) \ imp \beta \),

(Ax. 7) \( \alpha \),

(Ax. 8) \( \alpha \ imp (\beta \ imp \alpha) \),

(Ax. 9) \( 1 \ imp \alpha \) \ imp \( \alpha \),

(Ax. 10) \( [(\alpha \ imp \beta) \ imp \beta] \ imp [(\beta \ imp \alpha) \ imp \alpha] \),

(Ax. 11) \( (\text{not} \ \alpha \ imp \text{not} \ \beta) \ imp (\beta \ imp \alpha) \).
A fuzzy rule of inference is a scheme

\[
\frac{\alpha_1, \cdots, \alpha_n}{r^{\text{syn}}(\alpha_1, \cdots, \alpha_n)}, \quad \frac{a_1, \cdots, a_n}{r^{\text{sem}}(a_1, \cdots, a_n)}
\]

where the wffs \( \alpha_1, \cdots, \alpha_n \) are premises and the wff \( r^{\text{syn}}(\alpha_1, \cdots, \alpha_n) \) is the conclusion. The values \( a_1, \cdots, a_n \) and \( r^{\text{sem}}(a_1, \cdots, a_n) \in L \) are the corresponding truth values. The mappings \( r^{\text{sem}}: L^n \to L \) are semi-continuous, i.e.

\[
r^{\text{sem}}(a_1, \cdots, \bigvee_{j \in \Gamma} a_{k_j}, \cdots, a_n) = \bigvee_{j \in \Gamma} r^{\text{sem}}(a_1, \cdots, a_{k_j}, \cdots, a_n) \quad (1)
\]

holds for all \( 1 \leq k \leq n \). Moreover, the fuzzy rules are required to be sound in the sense that

\[
r^{\text{sem}}(v(\alpha_1), \cdots, v(\alpha_n)) \leq v(r^{\text{syn}}(\alpha_1, \cdots, \alpha_n))
\]

holds for all valuations \( v \).
Remark 1 The semi-continuity condition (1) can be replaced without any dramatic consequences by isotonicity condition (which is a weaker condition): if \( a_k \leq b_k \), then

\[
r_{sem}(a_1, \ldots, a_k, \ldots, a_n) \leq r_{sem}(a_1, \ldots, b_k, \ldots, a_n)
\]  

(2)

for each index \( 1 \leq k \leq n \).
The following Pavelka’s fuzzy rules of inference, a set R.

**Generalized Modus Ponens:**

\[
\frac{\alpha, \alpha \text{ imp } \beta}{\beta}, \frac{a, b}{a \odot b}
\]

**a-Consistency testing rules:**

\[
\frac{a}{0}, \frac{b}{c}
\]

where \(a\) is a truth constant and \(c = 0\) if \(b \leq a\) and \(c = 1\) otherwise.

**a-Lifting rules:**

\[
\frac{\alpha}{a \text{ imp } \alpha}, \frac{b}{a \rightarrow b}
\]

where \(a\) is a truth constant.

**Rule of Bold Conjunction:**

\[
\frac{\alpha, \beta}{\alpha \text{ and } \beta}, \frac{a, b}{a \odot b}
\]
It is easy to see that also a Rule of Bold Disjunction (not included in the list of Pavelka)

\[
\frac{\alpha, \beta}{\alpha \text{ or } \beta}, \quad \frac{a, b}{a \oplus b}
\]

is a rule of inference in Pavelka’s sense. Indeed, isotonicity of \( r^{\text{sem}} \) follows by the isotonicity of the MV-operation \( \oplus \) and soundness can be verified by taking a valuation \( v \) and observing that

\[
\begin{align*}
    r^{\text{sem}}(v(\alpha), v(\beta)) &= v(\alpha) \oplus v(\beta) \\
    &= v(\alpha \text{ or } \beta) \\
    &= v(\alpha \text{ or } \beta) \\
    &= v(r^{\text{syn}}(\alpha, \beta)).
\end{align*}
\]

This rule will be essential in Perfect Pavelka Logic.
A meta proof (called $\mathcal{R}$-proof by Pavelka) $\mathcal{w}$ of a wff $\alpha$ in a fuzzy theory $\mathcal{T}$ is a finite sequence

$$
\alpha_1, \ a_1, \\
\vdots \quad \vdots \\
\alpha_m, \ a_m,
$$

the degree of the meta proof $\mathcal{w}$

(i) $\alpha_m = \alpha$,
(ii) for each $i$, $1 \leq i \leq m$, $\alpha_i$ is a logical axiom, or is a special axiom of a fuzzy theory $\mathcal{T}$, or there is a fuzzy rule of inference and well formed formulae $\alpha_{i_1}, \ldots, \alpha_{i_n}$ with $i_1, \ldots, i_n < i$ such that $\alpha_i = r^{\text{syn}}(\alpha_{i_1}, \ldots, \alpha_{i_n})$,
(iii) for each $i$, $1 \leq i \leq m$, the value $a_i \in L$ is given by

$$
a_i = \begin{cases} 
a & \text{if } \alpha_i \text{ is the truth constant axiom } a, \\ 
1 & \text{if } \alpha_i \text{ is some other logical axiom in the set } A, \\ 
\mathcal{T}(\alpha_i) & \text{if } \alpha_i \text{ is a special axiom of a fuzzy theory } \mathcal{T}, \\ 
\mathcal{r}^{\text{sem}}(a_{i_1}, \ldots, a_{i_n}) & \text{if } \alpha_i = r^{\text{syn}}(\alpha_{i_1}, \ldots, \alpha_{i_n}). \\ 
\end{cases}
$$
Since a wff $\alpha$ may have various meta proofs with different degrees, we define the provability degree of a formula $\alpha$ to be the supremum of all such values, i.e.,

$$C^{\text{syn}}(\mathcal{T})(\alpha) = \bigvee \{a_m \mid w \text{ is a meta proof for } \alpha \text{ in } \mathcal{T}\}.$$
In particular, $C^{\text{syn}}(\mathcal{T})(\alpha) = 0$ means that either $\alpha$ does not have any meta proof or that for any meta proof $w$ of $\alpha$ the value $a_m = 0$. A fuzzy theory $\mathcal{T}$ is consistent if $C^{\text{sem}}(\mathcal{T})(a) = a$ for all truth constants $a$. Any satisfiable fuzzy theory is consistent.

Completeness of Pavelka’s Sentential Logic:

If $\mathcal{T}$ is consistent, then $C^{\text{sem}}(\mathcal{T})(\alpha) = C^{\text{syn}}(\mathcal{T})(\alpha)$ for any wff $\alpha$.

Thus, in Pavelka’s Fuzzy Sentential Logic we may talk about theorems of a degree $a$ and tautologies of a degree $b$ for $a, b \in L$, and these two values coincide for any formula $\alpha$. 
Let us now modify Pavelka approach such that $L$ is a complete MV-algebra.

Axioms and rules of inference are the schemas (Ax.1) – (Ax.11) and the following:

(Ax.12) \[ [\alpha \lor (\neg \alpha \land \beta)] \imp [(\alpha \imp \beta) \imp \beta], \]

(Ax.13) \[ a \imp b, \]

where $\alpha, \beta$ are wffs and $a, b$ are truth constants.

The axioms (Ax.12) obtain value 1 in all valuations, and axioms (Ax.13), called book–keeping axioms, obtain a value $a \rightarrow b$.

Rules of inference are those of the original Pavelka logic and the Rule of Bold Disjunction.
We need the following definitions and results to obtain **Completeness** of Complete MV–algebra valued Pavelka logic.

A fuzzy theory $\mathcal{T}$ is **consistent** if $C^\text{sem}_\mathcal{T}(a) = a$ for all truth constants $a$, otherwise it is **inconsistent**.

**Proposition 2**  A fuzzy theory $\mathcal{T}$ is inconsistent iff $\mathcal{T} \vdash_1 \alpha$ holds for any wff $\alpha$.

**Proposition 3**  A fuzzy theory $\mathcal{T}$ is inconsistent iff the following condition holds:

(C) There is a wff $\alpha$ and meta proofs $w, w'$ with degrees $a_m, b_{m'}$ for $\alpha$ and $\neg \alpha$, respectively, such that $0 < a_m \odot b_{m'}$. 
**Proposition 4**  A satisfiable fuzzy theory $\mathcal{T}$ is consistent.

**Proposition 5**  If $\mathcal{T} \vdash_a \alpha$ then $\mathcal{T} \vdash_1 (a \text{ imp } \alpha)$.

**Proposition 6**  $\mathcal{T} \vdash_1 [(\alpha \text{ and } \beta) \text{ imp } \alpha]$ holds for any fuzzy theory $\mathcal{T}$.

**Proposition 7**  If $\mathcal{T}$ is a consistent fuzzy theory and $\mathcal{T} \vdash_a \alpha$, then it holds that $\mathcal{T} \vdash_0 (\text{not} a \text{ and } \alpha)$.
Assume $\mathcal{T}$ is a consistent fuzzy theory. Define

$$\alpha \equiv \beta \text{ if, and only if } \mathcal{T} \vdash_1 (\alpha \text{ imp } \beta) \text{ and } \mathcal{T} \vdash_1 (\beta \text{ imp } \alpha).$$

We obtain a congruence relation; denote the equivalence classes by $|\alpha|$ and by $\mathcal{F}/\equiv$ the set of all equivalence classes. Then we have

**Proposition 8** Define $|\alpha| \to |\beta| = |\alpha \text{ imp } \beta|$ and $|\alpha|^* = |\text{not } \alpha|$. Then $\langle \mathcal{F}/\equiv, \to, ^*, |1| \rangle$ is a Wajsberg algebra and, hence, an MV-algebra.

Even more can be proved:

**Proposition 9** Assume $\mathcal{T}$ is a consistent fuzzy theory. If $\mathcal{T} \vdash_a \alpha$ then $|\alpha| = |a|$ in $\mathcal{F}/\equiv$. 
Thus $\mathcal{F}/\equiv$ is completely determined by the truth constants, which in turn are in one–to–one correspondence with the elements of $L$. Therefore there is an MV–isomorphism $\kappa : (\mathcal{F}/\equiv) \rightarrow L$ given by $\kappa(|a|) = a$, in particular $\kappa(|1|) = 1$.

Let $\pi$ be the canonical mapping $\pi : \mathcal{F} \rightarrow \mathcal{F}/\equiv$. Then $\kappa \circ \pi$ is the valuation in demand; if $T \vdash_a \alpha$ then $\kappa \circ \pi(\alpha) = \kappa(|a|) = a$. In conclusion, we write

**Completeness Theorem 1**

Consider complete MV–algebra valued Pavelka style fuzzy sentential logic. If a formula $\alpha$ is provable at a degree $a \in L$ in a consistent fuzzy theory $T$, then $\alpha$ is also a tautology at a degree $a$ i.e. its truth degree is $a$. 

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As well known, a necessary condition for Pavelka style completeness is that the truth value set is a complete MV–algebra. By Completeness Theorem 1 we have that it is also a sufficient condition, i.e. we have

**Completeness Theorem 2**

Pavelka style fuzzy sentential logic is semantically complete if, and only if the set of truth values constitutes a complete MV–algebra.
We have studied Pavelka’s fuzzy sentential logic and proved that it is semantically complete if, and only if the set of truth values constitutes a complete MV–algebra. However, a number of issues are still open, first of them concerns the completeness of first order logic. We conjecture that a similar result also applies to the first order Pavelka style fuzzy logic. Also the simplification of the presentation of is an open question; the set of inference rules can probably be reduced and the set of logical axioms is not a minimal one; the new axiom (Ax 12.) is redundant. From an application point of view it is also important that the set of truth constants could be reduced to a countable set; in this study the language under consideration is uncountable. All these issues are topics for a future work.