Density Elimination and Standard Completeness for extensions of UL and MTL

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Standard Completeness

Completeness of a logic with respect
to algebras whose lattice reduct is the real interval $[0, 1]$.

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to algebras whose lattice reduct is the real interval $[0, 1]$.

- Intended semantics for *Fuzzy logic* (Hajek 1998)

Examples: standard complete logics

- HL : Logic of Continuous t-norms
- MTL: Logic of Left-continuous t-norms
- UL : Logic of Left-continuous uninorms

Our results

We prove standard completeness for

- Classes of axiomatic extensions of UL
- Classes of axiomatic extensions of MTL

The way to Standard Completeness

Given a logic L

1. General algebraic completeness, i.e. completeness w.r.t. L-algebras

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2. Completeness w.r.t. L-chains (linearly ordered L-algebras).

The way to Standard Completeness

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2. Completeness w.r.t. L-chains.
   - $UL \quad \Longleftrightarrow \quad UL$-chains
   - $MTL \quad \Longleftrightarrow \quad MTL$-chains

The way to Standard Completeness

Given a logic L

1. General algebraic completeness, i.e. completeness w.r.t. L-algebras.
2. Completeness w.r.t. L-chains.
   - $UL + \alpha \quad \Longleftrightarrow \quad UL$-chains satisfying $1 \leq \alpha$
   - $MTL + \alpha \quad \Longleftrightarrow \quad MTL$-chains satisfying $1 \leq \alpha$

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3. Completeness w.r.t countable dense L-chains (rational completeness).

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The way to Standard Completeness

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1. General algebraic completeness, i.e. completeness w.r.t. L-algebras.
2. Completeness w.r.t. L-chains.
3. Completeness w.r.t countable dense L-chains (rational completeness).
   - (Metcalfe, Montagna JSL 2007)
     Add the density rule to L

$$\frac{(\alpha \to p) \vee (p \to \beta) \vee \gamma}{(\alpha \to \beta) \vee \gamma}\ (density)$$

L + $(density)$ is rational complete.

Density Elimination and Rational completeness

$L + (density)$ is rational complete. We prove that L is rational complete as follows

- Find a suitable hypersequent calculus HL for L
- Show that the density rule is eliminable in HL

The way to Standard Completeness

Given a logic L

1. General algebraic completeness, i.e. completeness w.r.t. L-algebras.
2. Completeness w.r.t. L-chains.
3. Completeness of $L = L + (density)$ w.r.t countable dense L-chains (rational completeness).
4. Standard Completeness (via Dedekind-MacNeille completion)

Results on uninorm logic UL

$$UL = FL_e + ((\alpha \to \beta) \wedge t) \vee ((\beta \to \alpha) \wedge t)$$

We will show standard completeness for axiomatic extensions of UL with any *knotted axiom*, i.e.:

- $UL + (\alpha \to \alpha \cdot \alpha)$
- $UL + (\alpha \cdot \alpha \to \alpha)$
- $UL + (\alpha^k \to \alpha^n)$ (for $n, k > 1$)

Our basic calculus FL_e: sequent calculus

$$\frac{}{\alpha \Rightarrow \alpha}\ (init) \qquad \frac{}{\Rightarrow t}\ (tr) \qquad \frac{}{f \Rightarrow}\ (fl)$$

$$\frac{}{\Gamma \Rightarrow \top}\ (\top) \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ (\bot)$$

$$\frac{\Gamma \Rightarrow \Pi}{t, \Gamma \Rightarrow \Pi}\ (tl) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow f}\ (fr) \qquad \frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ (cut)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r) \qquad \frac{\alpha_i, \Gamma \Rightarrow \Pi}{\alpha_1 \wedge \alpha_2, \Gamma \Rightarrow \Pi}\ (\wedge l) \qquad \frac{\Gamma \Rightarrow \alpha_i}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ (\vee r)$$

$$\frac{\alpha, \Gamma \Rightarrow \Pi \quad \beta, \Gamma \Rightarrow \Pi}{\alpha \vee \beta, \Gamma \Rightarrow \Pi}\ (\vee l) \qquad \frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \Pi}{\Gamma, \alpha \rightarrow \beta, \Delta \Rightarrow \Pi}\ (\rightarrow l) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}\ (\rightarrow r)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\cdot r) \qquad \frac{\alpha, \beta, \Gamma \Rightarrow \Pi}{\alpha \cdot \beta, \Gamma \Rightarrow \Pi}\ (\cdot l)$$

Hypersequent calculus HUL for UL

(Avron '89): Hypersequent

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \Rightarrow \Pi_i$ is an ordinary sequent
$\mid$ is intended to denote a meta-level disjunction.

Hypersequent calculus HUL for UL

Embedd sequent rules for FL_e into hypersequents

$$\frac{}{G|\Rightarrow t}\ (tr) \qquad \frac{}{G|\alpha \Rightarrow \alpha}\ (init) \qquad \frac{}{G|f \Rightarrow}\ (fl)$$

$$\frac{}{G|\Gamma \Rightarrow \top}\ (\top) \qquad \frac{}{G|\Gamma, \bot \Rightarrow \Delta}\ (\bot)$$

$$\frac{G|\Gamma \Rightarrow \alpha \quad G|\alpha, \Delta \Rightarrow \Pi}{G|\Gamma, \Delta \Rightarrow \Pi}\ (cut) \qquad \frac{G|\Gamma \Rightarrow \Pi}{G|t, \Gamma \Rightarrow \Pi}\ (tl) \qquad \frac{G|\Gamma \Rightarrow}{G|\Gamma \Rightarrow f}\ (fr)$$

$$\frac{G|\Gamma \Rightarrow \alpha \quad G|\Gamma \Rightarrow \beta}{G|\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r) \qquad \frac{G|\alpha_i, \Gamma \Rightarrow \Pi}{G|\alpha_1 \wedge \alpha_2, \Gamma \Rightarrow \Pi}\ (\wedge l) \qquad \frac{G|\Gamma \Rightarrow \alpha_i}{G|\Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ (\vee r)$$

$$\frac{G|\alpha, \Gamma \Rightarrow \Pi \quad G|\beta, \Gamma \Rightarrow \Pi}{G|\alpha \vee \beta, \Gamma \Rightarrow \Pi}\ (\vee l) \qquad \frac{G|\Gamma \Rightarrow \alpha \quad G|\beta, \Delta \Rightarrow \Pi}{G|\Gamma, \alpha \to \beta, \Delta \Rightarrow \Pi}\ (\to l) \qquad \frac{G|\alpha, \Gamma \Rightarrow \beta}{G|\Gamma \Rightarrow \alpha \to \beta}\ (\to r)$$

$$\frac{G|\Gamma \Rightarrow \alpha \quad G|\Delta \Rightarrow \beta}{G|\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\cdot\, r) \qquad \frac{G|\alpha, \beta, \Gamma \Rightarrow \Pi}{G|\alpha \cdot \beta, \Gamma \Rightarrow \Pi}\ (\cdot\, l)$$

Hypersequent calculus HUL for UL

We add:

- Suitable rules to manipulate the additional layer of structure.

$$\frac{G}{G \mid \Gamma \Rightarrow \alpha}\ (\mathrm{ew}) \qquad \frac{G \mid \Gamma \Rightarrow \alpha \mid \Gamma \Rightarrow \alpha}{G \mid \Gamma \Rightarrow \alpha}\ (\mathrm{ec})$$

Hypersequent calculus HUL for UL

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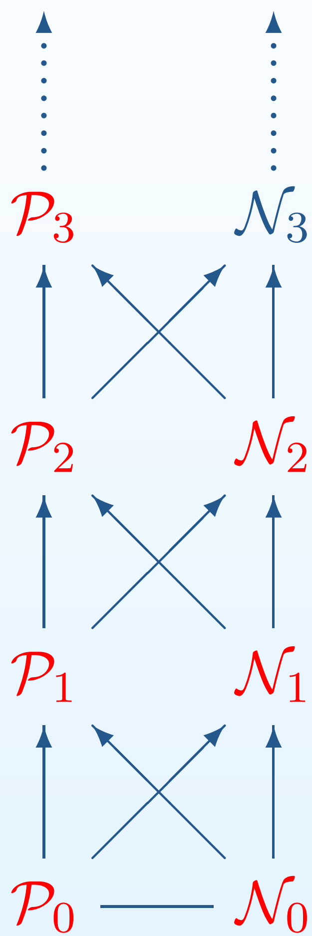
$$\frac{G}{G \mid \Gamma \Rightarrow \alpha}\ (\text{ew}) \qquad \frac{G \mid \Gamma \Rightarrow \alpha \mid \Gamma \Rightarrow \alpha}{G \mid \Gamma \Rightarrow \alpha}\ (\text{ec})$$

- A hypersequent structural rule corresponding to prelinearity :

$$((\alpha \to \beta) \wedge t) \vee ((\beta \to \alpha) \wedge t)$$

$$\frac{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Delta_1, \Delta_2 \Rightarrow \Pi_2}\ (\text{com})$$

Hypersequent calculi for extensions of UL?



(Ciabattoni, Galatos, Terui 2008).
Sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas defined by:

$\mathcal{P}_0$, $\mathcal{N}_0$:= Atomic formulas

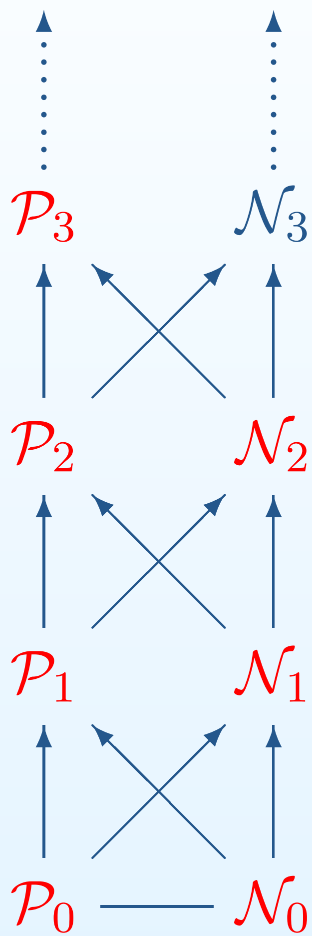
$$\mathcal{P}_{n+1} := \mathcal{N}_n \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1 \mid \bot$$

$$\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0 \mid \top$$

Examples:

- To the class $\mathcal{N}_2$ belong :
  $\alpha \rightarrow \alpha \cdot \alpha \qquad \alpha \cdot \alpha \rightarrow \alpha \qquad \alpha^k \rightarrow \alpha^n$
- To the class $\mathcal{P}_3$ belong :
  $\neg\alpha \vee \neg\neg\alpha \qquad \neg(\alpha \cdot \beta) \vee ((\alpha \wedge \beta) \rightarrow (\alpha \cdot \beta))$

Hypersequent calculi for extensions of UL?



Algorithm to convert axioms into “good” rules, preserving cut-elimination.

- Axioms in $\mathcal{N}_2 \Rightarrow$ Sequent structural rules
- Axioms in (subclass of) $\mathcal{P}_3 \Rightarrow$ Hypersequent structural rules

Correspondence axioms - rules

Class	Axiom	Rule
$\mathcal{N}_2$	$(\alpha \to t) \wedge (f \to \alpha)$	$\dfrac{G \mid \Pi \Rightarrow \Psi}{G \mid \Pi, \alpha \Rightarrow \Psi}\ (wl) \qquad \dfrac{G \mid \Pi \Rightarrow}{G \mid \Pi \Rightarrow \alpha}\ (wr)$
	$\alpha \to \alpha \cdot \alpha$	$\dfrac{G_1 \vert \Pi, \Gamma, \Gamma \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma \Rightarrow \Psi}\ (c)$
	$\alpha \cdot \alpha \to \alpha$	$\dfrac{G_1 \vert \Pi, \Gamma_1 \Rightarrow \Psi \quad G_1 \vert \Pi, \Gamma_2 \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma_1, \Gamma_2, \Rightarrow \Psi}\ (mgl)$
	$\alpha^k \to \alpha^n$	$\dfrac{G \vert \Pi, \Gamma_1^n \Rightarrow \Psi \quad \ldots \quad G_1 \vert \Pi, \Gamma_k^n \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma_1, \ldots \Gamma_k \Rightarrow \Psi}\ (knot_k^n)$

Density elimination

- Density rule in hypersequent calculus :

$$\frac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} \ (density)$$

where p does not occur in the conclusion (*eigenvariable*).

Density elimination

- Density rule in hypersequent calculus :

$$\frac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ (density)$$

  where p does not occur in the conclusion (*eigenvariable*).

- Similar to cut elimination

$$\frac{G \mid \Gamma \Rightarrow A \quad G \mid A, \Sigma \Rightarrow \Delta}{G \mid \Gamma, \Sigma \Rightarrow \Delta}\ (cut)$$

  Proof by induction on the length of derivations

Density elimination

(Ciabattoni, Metcalfe TCS 2008)
Given a density-free derivation, ending in

$$\dfrac{\begin{array}{c}\vdots\, d \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\ \text{(density)}$$

Density elimination

(Ciabattoni, Metcalfe TCS 2008)
Given a density-free derivation, ending in

$$\dfrac{\begin{array}{c}\vdots\, d \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\ (\text{density})$$

- Asymmetric substitution: p is replaced
  - With Δ when occurring on the right
  - With Γ when occurring on the left

Density elimination

$$\dfrac{\begin{matrix} \vdots\, d^* \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \end{matrix}}{G \mid \Gamma \Rightarrow \Delta}\ (\text{EC})$$

- Asymmetric substitution: p is replaced
  - With Δ when occurring on the right
  - With Γ when occurring on the left

A problem

$$\dfrac{\begin{array}{c} p \Rightarrow p \\ \vdots\, d \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta \end{array}}{G \mid \Gamma \Rightarrow \Delta}\ (\mathrm{EC})$$

- $p \Rightarrow p$ axiom

A problem

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta}{\vdots\, d}}{}$$

$$\Gamma \Rightarrow \Delta$$

$$\vdots\; d$$

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ (\text{EC})$$

- $p \Rightarrow p$ axiom
- $\Gamma \Rightarrow \Delta$ not an axiom

A possible solution

$$\dfrac{\begin{array}{c} p \Rightarrow p \\ \vdots\, d \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta \end{array}}{G \mid \Gamma \Rightarrow \Delta}\ (\mathrm{EC})$$

A possible solution

(Ciabattoni, Metcalfe 2008)

$$\dfrac{\begin{array}{c}\Rightarrow t\\ \vdots\ d^*\\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\ (\mathrm{EC})$$

- We substitute:
  - $p \Rightarrow p$ (axiom) with $\Rightarrow t$ (axiom).
  - p with Δ when occurring on the right.
  - p with Γ when occurring on the left.

Works for HUL. What about extensions?

Contraction and mingle

Consider UL extended with

- $$\alpha \to \alpha \cdot \alpha$$
- $$\alpha \cdot \alpha \to \alpha$$

Contraction and mingle

Hypersequent calculus: HUL plus

- $$\frac{G_1|\Pi, \Gamma, \Gamma \Rightarrow \Psi}{G_1|\Pi, \Gamma \Rightarrow \Psi}\ (c)$$
- $$\frac{G_1|\Pi, \Gamma_1 \Rightarrow \Psi \quad G_1|\Pi, \Gamma_2 \Rightarrow \Psi}{G_1|\Pi, \Gamma_1, \Gamma_2 \Rightarrow \Psi}\ (mgl)$$

Recall

(Ciabattoni, Metcalfe 2008)

$$\dfrac{\begin{matrix} \Rightarrow t \\ \vdots \; d^* \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \end{matrix}}{G \mid \Gamma \Rightarrow \Delta} \; (\text{EC})$$

- We substitute:
  - $p \Rightarrow p$ (axiom) with $\Rightarrow t$ (axiom).
  - p with Δ when occurring on the right.
  - p with Γ when occurring on the left.

Problematic case

Consider

$$\frac{\Pi, p, p \Rightarrow p}{\Pi, p \Rightarrow p}\ (c)$$

Can we get:

$$\frac{\Pi, \Gamma \Rightarrow t}{\Pi \Rightarrow t}\ ?$$

Proof by cases

Recall: the hypersequent symbol '|' is interpreted as disjunction.

- For any hypersequent calculus HL the following meta-rule holds:

$$\frac{G_1 \vdash_{HL} H \quad G_2 \vdash_{HL} H}{G_1 | G_2 \vdash_{HL} H}$$

Proof by cases and Density Elimination

Recall our derivation

$$\dfrac{\begin{array}{c}\vdots\, d\\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\,(density)$$

Proof by cases and Density Elimination

In d we instantiate p with t, obtaining

$$\begin{array}{c} \vdots\, d_t \\ G \mid \Gamma \Rightarrow t \mid t \Rightarrow \Delta \end{array}$$

Proof by cases and Density Elimination

We instantiate p with t, obtaining

$$\begin{array}{c} \vdots\, d_t \\ G \mid \Gamma \Rightarrow t \mid t \Rightarrow \Delta \end{array}$$

We find density free proofs of:

$$\begin{array}{c} G|\Gamma \Rightarrow t \\ \vdots\, d_1 \\ G|\Gamma \Rightarrow \Delta \end{array} \qquad \begin{array}{c} G|t \Rightarrow \Delta \\ \vdots\, d_2 \\ G|\Gamma \Rightarrow \Delta \end{array}$$

Proof by cases and Density Elimination

We find density free proofs of:

$$\dfrac{G|\Gamma \Rightarrow t}{\vdots\, d_1 \atop G|\Gamma \Rightarrow \Delta} \qquad \dfrac{G|t \Rightarrow \Delta}{\vdots\, d_2 \atop G|\Gamma \Rightarrow \Delta}$$

Applying the proof by cases property, we get:

$$\begin{array}{c} G|\Gamma \Rightarrow t|t \Rightarrow \Delta \\ \vdots \\ G|\Gamma \Rightarrow \Delta \end{array}$$

Proof by cases and Density Elimination

We find density free proofs of:

$$\dfrac{G|\Gamma \Rightarrow t}{G|\Gamma \Rightarrow \Delta}\ \vdots\, d_1 \qquad \dfrac{G|t \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta}\ \vdots\, d_2$$

Applying the proof by cases property, we get:

$$\begin{array}{c} \vdots\, d_t \\ G|\Gamma \Rightarrow t|t \Rightarrow \Delta \\ \vdots \\ G|\Gamma \Rightarrow \Delta \end{array}$$

Knotted rules

- Consider UL extended with

$$\alpha^k \rightarrow \alpha^n$$

for k, n>1

Knotted rules

- Consider UL extended with

$$\alpha^k \to \alpha^n$$

for k, n>1

- Hypersequent calculus: HUL plus

$$\frac{G|\Pi, \Gamma_1^n \Rightarrow \Psi \quad \ldots \quad G_1|\Pi, \Gamma_k^n \Rightarrow \Psi}{G_1|\Pi, \Gamma_1, \ldots \Gamma_k \Rightarrow \Psi} \; (knot_k^n)$$

Knotted rules: Problematic case

$$\frac{\Pi, p, p, p \Rightarrow p \quad \Pi, p, p, p \Rightarrow p}{\Pi, p, p \Rightarrow p} \ (knot_2^3)$$

We would like to get:

$$\frac{\Pi, \Gamma, \Gamma \Rightarrow t \quad \Pi, \Gamma, \Gamma \Rightarrow t}{\Pi, \Gamma \Rightarrow t} \ ?$$

Knotted rules: Problematic case

$$\frac{\Pi, p, p, p \Rightarrow p \quad \Pi, p, p, p \Rightarrow p}{\Pi, p, p \Rightarrow p} \ (knot_2^3)$$

Using some derivabilities in HUL + $(knot_2^3)$, we can show:

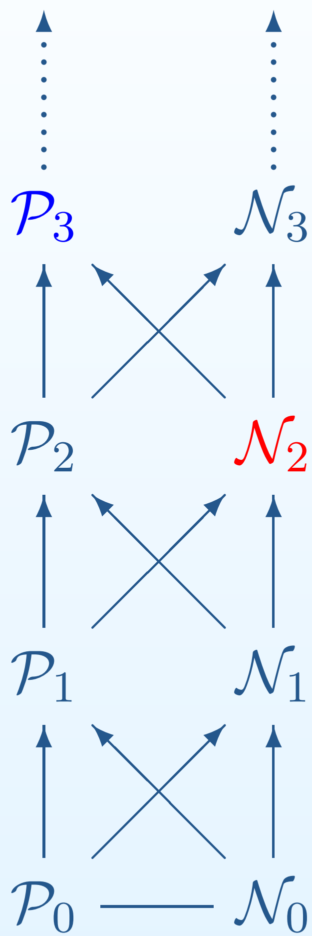
$$\begin{array}{c} \Pi, \Gamma, \Gamma \Rightarrow t \quad \Pi, \Gamma, \Gamma \Rightarrow t \\ \vdots \\ \Pi, \Gamma \Rightarrow t \end{array}$$

Standard completeness for extensions of UL

Let $L = UL + (\alpha^k \to \alpha^n)$

1. General algebraic completeness, i.e. completeness w.r.t. L-algebras.
2. Completeness w.r.t. L-chains.
3. $L = L$ + $(density)$ is Rational complete .
4. L is Standard complete (via Dedekind-MacNeille completion).

Closure under DM-completions



(Ciabattoni, Terui, Galatos 2011) Axioms on FL $\leftrightarrow$ equations over residuated lattices

- A subclass of equations in class $\mathcal{N}_2$ are preserved by Dedekind-MacNeille completion. All the axioms we considered are in this class.
- A subclass of equations in class $\mathcal{P}_3$ are preserved by Dedekind-MacNeille completion, when applied to subdirectly irreducible algebras

Standard completeness for extensions of MTL

- $MTL = UL + (f \to \alpha) \wedge (\alpha \to t)$
- Hypersequent calculus $HMTL = HUL + (wl) + (wr)$

$$\frac{G \mid \Pi \Rightarrow \Psi}{G \mid \Pi, \alpha \Rightarrow \Psi}\ (wl) \qquad \frac{G \mid \Pi \Rightarrow}{G \mid \Pi \Rightarrow \alpha}\ (wr)$$

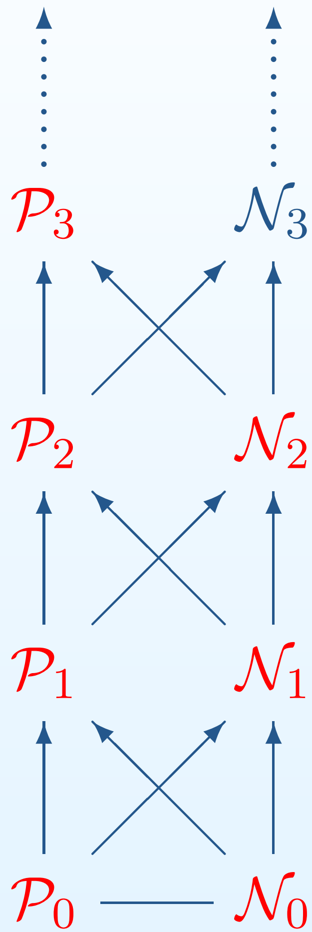
Standard completeness for extensions of MTL

- Density Elimination holds for $HMTL$ extended with any structural sequent rule
  - Any axiomatic extension of MTL with axioms within $\mathcal{N}_2$ is standard complete (2008 Ciabattoni, Metcalfe).

Standard completeness for extensions of MTL

- Density Elimination holds for $HMTL$ extended with any structural sequent rule
  - Any axiomatic extension of MTL with axioms within $\mathcal{N}_2$ is standard complete (2008 Ciabattoni, Metcalfe).
- Density elimination holds for extensions of $HMTL$ with structural hypersequent rules which do not “mix too much” the components (convergent rules).
  - Any axiomatic extension of MTL with axioms within a subclass of $\mathcal{P}_3$ is standard complete (2012 Baldi, Ciabattoni, Spendier)

Recall:Correspondence axioms-rules



(Ciabattoni, Galatos, Terui 2008).
Sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas defined by:

$$\mathcal{P}_0, \mathcal{N}_0 := \text{Atomic formulas}$$

$$\mathcal{P}_{n+1} := \mathcal{N}_n \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1 \mid \bot$$

$$\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0 \mid \top$$

Examples:

- To the class $\mathcal{N}_2$ belong :
  $\alpha \to \alpha \cdot \alpha \qquad \alpha \cdot \alpha \to \alpha$
- To the class $\mathcal{P}_3$ belong :
  $\neg\alpha \vee \neg\neg\alpha \qquad ((\alpha \to \beta) \wedge t) \vee ((\beta \to \alpha) \wedge t)$

Recall:Correspondence axioms - rules

Class	Axiom	Rule
$\mathcal{N}_2$	$(\alpha \to t) \wedge (f \to \alpha)$	$\dfrac{G \mid \Pi \Rightarrow \Psi}{G \mid \Pi, \alpha \Rightarrow \Psi}\ (wl) \quad \dfrac{G \mid \Pi \Rightarrow}{G \mid \Pi \Rightarrow \alpha}\ (wr)$
	$\alpha \to \alpha \cdot \alpha$	$\dfrac{G_1 \vert \Pi, \Gamma, \Gamma \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma \Rightarrow \Psi}\ (c)$
	$\alpha \cdot \alpha \to \alpha$	$\dfrac{G_1 \vert \Pi, \Gamma_1 \Rightarrow \Psi \quad G_1 \vert \Pi, \Gamma_2 \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma_1, \Gamma_2, \Rightarrow \Psi}\ (mgl)$
	$\alpha^k \to \alpha^n$	$\dfrac{G \vert \Pi, \Gamma_1^n \Rightarrow \Psi \quad \ldots \quad G_1 \vert \Pi, \Gamma_k^n \Rightarrow \Psi}{G_1 \vert \Pi, \Gamma_1, \ldots \Gamma_k \Rightarrow \Psi}\ (knot_k^n)$
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	$\dfrac{G \mid \Pi, \Gamma \Rightarrow \Psi}{G \mid \Gamma \Rightarrow \ \mid \Pi \Rightarrow \Psi}$ *(em)*
$\mathcal{P}_3$	$\neg\alpha \vee \neg\neg\alpha$	$\dfrac{G \mid \Gamma_1, \Gamma_2 \Rightarrow}{G \mid \Gamma_1 \Rightarrow \ \mid \Gamma_2 \Rightarrow}$ *(lq)*

Examples of convergent rules

- Axioms in $\mathcal{P}_3$ extending MTL
  - (wnm) :
    $$\neg(\alpha \cdot \beta) \vee ((\alpha \wedge \beta) \to (\alpha \cdot \beta))$$
  - (lq) :
    $$\neg\alpha \vee \neg\neg\alpha$$
- Corresponding convergent rules
  - $$\frac{\begin{array}{cc} G \,|\, \Gamma_2, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \,|\, \Gamma_1, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \\ G \,|\, \Gamma_1, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \,|\, \Gamma_2, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \end{array}}{G \,|\, \Gamma_2, \Gamma_3 \Rightarrow \,|\, \Gamma_1, \Delta_1 \Rightarrow \Pi_1} \; (wnm)$$
  - $$\frac{G \,|\, \Gamma_1, \Gamma_2 \Rightarrow}{G \,|\, \Gamma_1 \Rightarrow \,|\, \Gamma_2 \Rightarrow} \; (lq)$$

A non convergent rule

- Axiom in $\mathcal{P}_3$ extending MTL
  - $\alpha \vee \neg\alpha$
- Corresponding rule
  - $$\frac{G|\Gamma, \Sigma \Rightarrow \Delta}{G|\Gamma \Rightarrow \ |\Sigma \Rightarrow \Delta}\ (em)$$

Our results

- Standard completeness for extensions of UL:
  - $UL + \alpha^k \to \alpha^n$ (includes mingle and contraction axioms).
- Standard completeness for extensions of MTL:
  - Any axiomatic extension of MTL with axioms within a subclass of $\mathcal{P}_3$ is standard complete.

Open problem

- Conjecture: Let α be any formula in $\mathcal{N}_2$. $UL + \alpha$ is standard complete.
- Let (r) be any sequent rule obtained by an axiom in $\mathcal{N}_2$. Does $HUL + (r)$ admits Density Elimination?
  - Example, what about:

$$\frac{G \mid \Gamma_1 \Rightarrow \quad \Gamma_2^2, \Pi \Rightarrow \Psi}{G \mid \Gamma_1, \Gamma_2, \Pi \Rightarrow \Psi}$$

Work in progress

- A general characterization of density elimination, hence standard completeness, for:
  - Extensions of MTL with axioms up to the class $\mathcal{P}_3$ in the substructural hierarchy.
  - Extensions of UL with axioms up to the class $\mathcal{N}_2$ in the substructural hierarchy.
  - Noncommutative variants of MTL and UL.
  - Logics with involutive negation. Long standing open problem: IUL

Appendix A: A class of structural rules

Let HL be HUL extended with any structural sequent rule

$$\frac{G_1|\Pi_1, \Psi_1 \Rightarrow \Delta_1 \quad \ldots \Pi_1, \Psi_m \Rightarrow \Delta_1}{G_1|\Pi_1, \Gamma_1, \ldots \Gamma_k \Rightarrow \Delta_1} \ (r)$$

HL admits density elimination if (r) satisfies the following:

- Each Ψ_i is a multiset $\{\Gamma_{i_1}, \ldots, \Gamma_{i_{n_i}}\}$ with $i_1 \ldots i_{n_i}$ varying over $\{1, \ldots k\}$
- Either the minimum among the n_i is bigger than k or the maximum is smaller than k
- For any Γ_i there is at least one Ψ_j where Γ_i does not appear.
- For any Γ_i there is at least one Ψ_j where Γ_i appears more then once .

Appendix B: Convergent rules

Definition. Let (r) be a hypersequent structural rule with $G|S_i$, $i \in \{1, ..m\}$ premises, $C_1|...|C_q$ conclusion.

(0-pivot) $G|S_i$ is a 0-*pivot* if there is an $s \in \{1, \dots, q\}$ such that $R(S_i) = R(C_s)$ and metavariables in $L(S_i)$ are contained in $L(C_s)$.

(n-pivot) $G|S_j$ is an n-*pivot for* $G|S_i$ with respect to $[\Delta_k/\Gamma_k]_{k\in\{1,\dots,n\}}$, with $\Gamma_k \in L(S_i)$ and $\Delta_k \in L(S_j)$, if the following conditions hold:

- $G|S_j$ is a 0-*pivot*
- $R(S_i) = R(S_j)$,
- $L(S_j) = L(\ S_i[^{\Delta_k}/_{\Gamma_k}]^l_{k\in\{1,\dots,n\}}\ )$,
- If $n > 1$, $G|S_j$ is a $(n-1)$-*pivot* for n premises $G|S_{j_p}$, $p = 1, \dots, n$, with respect to $[^{\Delta_k}/_{\Gamma_k}]_{k\in\{1,\dots,n\}\setminus\{p\}}$.

Definition. A completed hypersequent rule (r) is *convergent* if for each premise $G|S_i$ one of the following conditions holds:

- $R(S_i) = \emptyset$,
- $G|S_i$ is a 0-*pivot*
- there is a premise $G|S_j$ which is an n-*pivot* for $G|S_i$, with $n > 0$.