Models of set theory in Łukasiewicz logic

Zuzana Haniková

Institute of Computer Science
Academy of Sciences of the Czech Republic

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(joint work with Petr Hájek)
Why fuzzy set theory?

- try to capture a mathematical world: develop fuzzy mathematics (indicate a direction)
- study the notion of a set, and rudimentary notions of set theory (some properties may be available on a limited scale; classically equivalent notions need not be available in a weak setting)
- wider set-theoretic universe: recast the classical universe of sets as a subuniverse of the universe of fuzzy sets
- Explore the limits of (relative) consistency. (Which logics allow for an interpretation of classical ZF? Which logics give a consistent system?)
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Programme

Work with classical metamathematics.

Consider a logic L, magenta weaker than classical logic. (Also, with well-developed algebraic semantics.)

Consider an axiomatic set theory $T$, governed by L.

The theory $T$ should:
- generate a cumulative universe of sets
- be provably distinct from the classical set theory
- be reasonably strong
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Between classical and non-classical:
classical set-theoretic universe is a sub-universe of the non-classical one
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Plan for talk

1. Logics without the contraction rule
2. Łukasiewicz logic
3. A set theory can strengthen its logic
4. A-valued universes
5. the theory FST (over Ł)
6. generalizations
Consider propositional language $\mathcal{F}$.
($\text{FL}_{\text{ew}}$-language: $\{\cdot, \rightarrow, \land, \lor, 0, 1\}$.)

A logic in a language $\mathcal{F}$ is a set of formulas closed under substitution and deduction.

“Substructural” — absence of some structural rules (of the Gentzen calculus for INT).
In particular, $\text{FL}_{\text{ew}}$ is contraction free.

Structural rules:

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\begin{array}{c}
\frac{\Gamma, \phi, \psi, \Delta \Rightarrow \chi}{\Gamma, \psi, \phi, \Delta \Rightarrow \chi} \quad (e) \\
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Removal of these rules calls for some changes:

- splitting of connectives
- changes to interpretation of a sequent

NB: $\text{FL}_{\text{ew}}$ is equivalent to Höhle’s monoidal logic (ML).
A family of substructural logics: FL_{ew} and extensions

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A $\text{FL}_{\text{ew}}$-algebra is an algebra $A = \langle A, \cdot, \rightarrow, \wedge, \vee, 0, 1 \rangle$ such that:

1. $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice, $1$ is the greatest and $0$ the least element
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3. for all $x, y, z \in A$, $z \leq (x \rightarrow y)$ iff $x \cdot z \leq y$

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(More precisely, Łukasiewicz’s infinite-valued logic, ca. 1920. Denoted \( L \).)

Usually conceived in a narrower language, such as:

- \( \{+, \neg\} \)
- \( \{\rightarrow, \neg\} \) or \( \{\rightarrow, 0\} \)
- \( \{\cdot, \rightarrow, 0\} \)
- \( \ldots \)

Propositionally, the logic is given by the algebra

\[ [0, 1]_L = \langle [0, 1], \cdot_L, \rightarrow_L, \min, \max, 0, 1 \rangle \]

with the natural order of the reals on \([0, 1]\), and

\[ x \cdot_L y = \max(x + y - 1, 0) \]
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NB: all operations of \([0, 1]_L\) are continuous.
Hence, no two-valued operator is term-definable.
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Łukasiewicz logic — propositional axioms, completeness

Axioms:

- (Ł1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
- (Ł2) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
- (Ł3) $(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)$
- (Ł4) $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$

Deduction rule: modus ponens.

General algebraic semantics: MV-algebras.

Propositional Łukasiewicz logic is
- strongly complete w.r.t. MV-algebras
- finitely strongly complete w.r.t. $[0, 1]_L$
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Łukasiewicz logic with the $\Delta$-projection

Semantics of $\Delta$ in a linearly ordered algebra $A$:

- $\Delta(x) = 1$ if $x = 1$
- $\Delta(x) = 0$ otherwise

Axioms:

- $(\Delta 1)$ $\Delta \varphi \lor \neg \Delta \varphi$
- $(\Delta 2)$ $\Delta(\varphi \lor \psi) \rightarrow (\Delta \varphi \lor \Delta \psi)$
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Assume the language \( \{\in, =\} \).

Let \( A \) be an MV-chain.

Tarski-style definition of the value \( \|\varphi\|_{A, M, v} \) of a formula \( \varphi \) in an \( A \)-structure \( M \) and evaluation \( v \) in \( M \); in particular,

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An \( A \)-structure \( M \) is safe if \( \|\varphi\|_{A, M, v} \) is defined for each \( \varphi \) and \( v \).

The truth value of a formula \( \varphi \) of a predicate language \( L \) in a safe \( A \)-structure \( M \) for \( L \) is

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Łukasiewicz logic — first-order axioms

Axioms for quantifiers $\forall$, $\exists$:

$(\forall 1)$ $\forall x \varphi(x) \to \varphi(t)$ ($t$ substitutable for $x$ in $\varphi$)

$(\exists 1)$ $\varphi(t) \to \exists x \varphi(x)$ ($t$ substitutable for $x$ in $\varphi$)

$(\forall 2)$ $\forall x (\chi \to \varphi) \to (\chi \to \forall x \varphi)$ ($x$ not free in $\chi$)

$(\exists 2)$ $\forall x (\varphi \to \chi) \to (\exists x \varphi \to \chi)$ ($x$ not free in $\chi$)

$(\forall 3)$ $\forall x (\varphi \lor \chi) \to (\forall x \varphi \lor \chi)$ ($x$ not free in $\chi$)

The rule of generalization: from $\varphi$ entail $\forall x \varphi$.

NB: the two quantifiers are interdefinable in $\mathcal{L}$. 
Equality axioms for set-theoretic language:
- reflexivity
- symmetry
- transitivity
- congruence \( \forall x, y, z (x = y \land z \in x \rightarrow z \in y) \)
- congruence \( \forall x, y, z (x = y \land y \in z \rightarrow x \in z) \)

Moreover (for reasons given below), we postulate the law of the excluded middle for equality:
- \( \forall x, y (x = y \lor \neg(x = y)) \)
Theorem

Let \( T \cup \{\varphi\} \) be a set of sentences. Then \( T \vdash_L \varphi \) iff for each MV-chain \( A \) and each safe \( A \)-model \( M \) of \( T \), \( \varphi \) holds in \( M \).

NB: for a general language \( \mathcal{L} \), the truths of \([0,1]_\mathcal{L}\) are not recursively axiomatizable (in fact, they are \( \Pi_2 \)-complete).

Analogous completeness for the expansion with \( \Delta \).
Łukasiewicz logic

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Let $T \cup \{\varphi\}$ be a set of sentences. Then $T \vdash_L \varphi$ iff for each MV-chain $A$ and each safe $A$-model $M$ of $T$, $\varphi$ holds in $M$.

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Analogous completeness for the expansion with $\Delta$. 
Let \( L \) be a consistent \( FL_{ew} \)-extension.
Let \( T \) be a theory over \( L \).
If \( T \) proves \( \varphi \lor \neg\varphi \) for an arbitrary \( \varphi \), then
\( T \) is a theory over classical logic.

In other words,

adding the law of excluded middle (LEM): \( \varphi \lor \neg\varphi \) to \( FL_{ew} \) yields classical logic.

Example: Grayson’s proof of LEM from axiom of regularity:
Let \( \{\emptyset \upharpoonright \varphi\} \) stand for \( \{x \mid x = \emptyset \land \varphi\} \).
Consider \( z = \{\emptyset \upharpoonright \varphi, 1\} \) (where \( 1 = \{\emptyset\} \))
Then \( z \) is nonempty, and consequently has a \( \in \)-minimal element.
If \( \emptyset \) is minimal then \( \varphi \) holds,
while if \( 1 \) is minimal then \( \varphi \) fails.

Thus, from regularity, one proves LEM for any formula.
Let $L$ be a consistent $\text{FL}_{ew}$-extension.
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Strengthening the logic

Let $L$ be a consistent $FL_{ew}$-extension.
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Thus, from regularity, one proves LEM for any formula.
Lemma (Hájek ca. 2000)

Let $L$ be such that it proves the propositional formula $(p \rightarrow p \& p) \rightarrow (p \lor \neg p)$. Then, a set theory with

- separation (for open formulas),
- pairing (or singletons),
- congruence axiom for $\in$

proves $\forall xy(x = y \lor \neg(x = y))$ over $L$.

Proof: take $x, y$.
Let $z = \{ u \in \{ x \} \mid u = x \}$, whence $u \in z \equiv (u = x)^2$.
Since $(x = x)^2$, we have $x \in z$.
If $y = x$ then $y \in z$ by congruence. Then $(y = x)^2$.
We proved $y = x \rightarrow (y = x)^2$, thus (by assumption on the logic) $x = y \lor \neg(x = y)$. 
Strengthening the logic

Lemma (Grishin 1999)

In a theory with

- extensionality,
- successors,
- congruence,

LEM for $=$ implies LEM for $\in$. 

Zuzana Haniková | Models of set theory in Łukasiewicz logic
Axioms of FST

- (ext.) $\forall xy(x = y \equiv (\Delta(x \subseteq y) \& \Delta(y \subseteq x)))$
- (empty) $\exists x \Delta \forall y \neg(y \in x)$
- (pair) $\forall x \forall y \exists z \Delta \forall u(u \in z \equiv (u = x \lor u = y))$
- (union) $\forall x \exists z \Delta \forall u(u \in z \equiv \exists y (u \in y \& y \in x))$
- (weak power) $\forall x \exists z \Delta \forall u(u \in z \equiv \Delta(u \subseteq x))$
- (inf.) $\exists z \Delta(\emptyset \in z \& \forall x \in z(x \cup \{x\} \in z)$
- (sep.) $\forall x \exists z \Delta \forall u(u \in z \equiv (u \in x \& \varphi(u, x)))$
  for any $\varphi$ not containing free $z$
- (coll.) $\forall x \exists z \Delta[\forall u \in x \exists v \varphi(u, v) \rightarrow \forall u \in x \exists v \in z \varphi(u, v)]$
  for any $\varphi$ not containing free $z$
- ($\in$-ind.) $\Delta \forall x(\Delta \forall y (y \in x \rightarrow \varphi(y)) \rightarrow \varphi(x)) \rightarrow \Delta \forall x \varphi(x)$
  for any $\varphi$
- (support) $\forall x \exists z (\text{Crisp}(z) \& \Delta(x \subseteq z)))$
An $A$-valued universe

Work in classical ZFC. Assume $A$ is a complete (MV-)algebra.

Define $V^A$ by ordinal induction.

$$A^+ = A \setminus \{0^A\}.$$  

- $V^A_0 = \{\emptyset\}$
- $V^A_{\alpha+1} = \{f : \text{Fnc}(f) \& \text{Dom}(f) \subseteq V^A_\alpha \& \text{Rng}(f) \subseteq A^+\}$ for any ordinal $\alpha$
- $V^A_\lambda = \bigcup_{\alpha < \lambda} V^A_\alpha$ for limit ordinals $\lambda$

$$V^A = \bigcup_{\alpha \in \text{Ord}} V^A_\alpha$$

Define two binary functions from $V^A$ into $L$, assigning to any $u, v \in V^A$ the values $\|u \in v\|$ and $\|u = v\|$.

$$\|u \in v\| = v(u) \text{ if } u \in D(v), \text{ otherwise } 0$$

$$\|u = v\| = 1 \text{ if } u = v, \text{ otherwise } 0$$

By induction on the complexity of formulas, define for any $\varphi(x_1, \ldots, x_n)$ an $n$-ary function from $(V^A)^n$ into $L$, assigning to an $n$-tuple $u_1, \ldots, u_n$ the value $\|\varphi(u_1, \ldots, u_n)\|$.
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An $\mathbf{A}$-valued universe

Work in classical ZFC.
Assume $\mathbf{A}$ is a complete (MV-)algebra.

Define $V^\mathbf{A}$ by ordinal induction.

$A^+ = A \setminus \{0^\mathbf{A}\}$. 

- $V_0^\mathbf{A} = \{\emptyset\}$
- $V_{\alpha+1}^\mathbf{A} = \{f : \text{Fnc}(f) \& \text{Dom}(f) \subseteq V_x^\mathbf{A} \& \text{Rng}(f) \subseteq A^+\}$ for any ordinal $\alpha$
- $V_\lambda^\mathbf{A} = \bigcup_{\alpha < \lambda} V_\alpha^\mathbf{A}$ for limit ordinals $\lambda$

$V^\mathbf{A} = \bigcup_{\alpha \in \text{Ord}} V_\alpha^\mathbf{A}$

Define two binary functions from $V^\mathbf{A}$ into $L$, assigning to any $u, v \in V^\mathbf{A}$ the values $\|u \in v\|$ and $\|u = v\|$.

\[
\|u \in v\| = v(u) \text{ if } u \in D(v), \text{ otherwise } 0
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\[
\|u = v\| = 1 \text{ if } u = v, \text{ otherwise } 0
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By induction on the complexity of formulas, define for any $\varphi(x_1, \ldots, x_n)$ an $n$-ary function from $(V^\mathbf{A})^n$ into $L$, assigning to an $n$-tuple $u_1, \ldots, u_n$ the value $\|\varphi(u_1, \ldots, u_n)\|$.
Theorem

Let $\varphi$ be a closed formula provable in FST. Then $\varphi$ is valid in $V^A$, i.e., ZF proves $\|\varphi\| = 1$.

We have obtained an interpretation of FST in ZFC. FST is distinct from ZFC unless $A$ is a Boolean algebra.
Theorem

Let $\varphi$ be a closed formula provable in FST. Then $\varphi$ is valid in $V^A$, i.e., ZF proves $\|\varphi\| = 1$.

We have obtained an interpretation of FST in ZFC. *FST is distinct from ZFC* unless $A$ is a Boolean algebra.
An Inner Model of ZF in FST

Definition

(i) In a theory \( T \), we say that a formula \( \varphi(x_1, \ldots, x_n) \) in the language of \( T \) is crisp iff \( T \vdash \forall x_1, \ldots, x_n \varphi(x_1, \ldots, x_n) \).

(ii) In a (set) theory with language containing \( \in \) we define \( \text{Crisp}(x) \equiv \forall u \varphi(u \in x) \).

(Hereditarily crisp transitive set)

\[
HCT(x) \equiv \text{Crisp}(x) \& \forall u \in x(\text{Crisp}(u) \& u \subseteq x)
\]

(Hereditarily crisp set)

\[
H(x) \equiv \text{Crisp}(x) \& \exists x' \in HCT(x \subseteq x')
\]

Lemma

The class \( H \) is both crisp and transitive in FST:

- \( FST \vdash \forall x(x \in H \lor \neg(x \in H)) \)
- \( FST \vdash \forall x, y(y \in x \& x \in H \rightarrow y \in H) \)
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An Inner Model of ZF in FST

**Definition**

(i) In a theory $T$, we say that a formula $\varphi(x_1, \ldots, x_n)$ in the language of $T$ is crisp iff $T \vdash \forall x_1, \ldots, x_n \models \varphi(x_1, \ldots, x_n)$.

(ii) In a (set) theory with language containing $\in$ we define $\text{Crisp}(x) \equiv \forall u \models (u \in x)$.

**Lemma**

*The class $H$ is both crisp and transitive in FST:*

- $FST \vdash \forall x(x \in H \lor \neg(x \in H))$
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An Inner Model of ZF in FST

For \( \varphi \) a formula in the language of ZF, define \( \varphi^H \) inductively:

- \( \varphi^H = \varphi \) for \( \varphi \) atomic;
- \( \varphi^H = \varphi \) for \( \varphi = 0 \);
- \( \varphi^H = \psi^H \& \chi^H \) for \( \varphi = \psi \& \chi \);
- \( \varphi^H = \psi^H \rightarrow \chi^H \) for \( \varphi = \psi \rightarrow \chi \);
- \( \varphi^H = (\forall x \in H)\psi^H \) for \( \varphi = (\forall x)\psi \).

**Theorem**

*Let \( \varphi \) be a theorem of ZF. Then FST \( \vdash \varphi^H \).*

So \( H \) is an inner model of ZF in FST and ZF is consistent relative to FST.

Moreover, the interpretation is faithful: if FST \( \vdash \varphi^H \), then ZF \( \vdash \varphi^H \), but then ZF \( \vdash \varphi \).
For \( \varphi \) a formula in the language of ZF, define \( \varphi^H \) inductively:

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\begin{align*}
\varphi^H &= \varphi \text{ for } \varphi \text{ atomic;} \\
\varphi^H &= \varphi \text{ for } \varphi = 0; \\
\varphi^H &= \psi^H \& \chi^H \text{ for } \varphi = \psi \& \chi; \\
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**Theorem**

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Moreover, the interpretation is faithful: if FST $\vdash \varphi^H$, then ZF $\vdash \varphi^H$, but then ZF $\vdash \varphi$. 
Ordinals and rank in FST

Let \( \text{Ord}_0(x) \) define ordinal numbers in classical ZFC.

The inner model \( H \) provides a suitable notion of ordinal numbers in FST: if \( x \in H \), then

- \( \text{Ord}_0(x) \equiv \text{Ord}_0^H(x) \),
- \( \text{Ord}_0(x) \) is crisp.

Define ordinal numbers in FST:

\[
\text{Ord}(x) \equiv x \in H \& \text{Ord}_0(x)
\]

Define:

\[
\begin{align*}
V_0 &= \emptyset \\
V_{\alpha+1} &= WP(V_{\alpha}) \text{ for } \alpha \in \text{Ord} \\
V_\alpha &= \bigcup_{\beta \in \alpha} V_\beta \text{ for a limit } \alpha \in \text{Ord} \\
V &= \bigcup_{\alpha \in \text{Ord}} V_\alpha
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Then \( \forall x \exists \alpha(x \in V_\alpha) \).
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Define:

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$$V_\alpha = \bigcup \{V_\beta \text{ for a limit } \alpha \in \text{Ord} \}$$

Then $\forall x \exists \alpha (x \in V_\alpha)$. 

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Extensions and further work

- Work with an arbitrary MV-algebra (Chang’s algebra). Can one get “nearly classical”?

  Lemma. Let $A$ be an algebra, and let $M$ be a model over $A$. Let $\sim$ be a congruence on $A$. Then $M$ is a model over $A/\sim$.

- Work without $\Delta$.

- A completeness theorem?
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