Fuzzy Logic

2. Completeness of propositional Łukasiewicz and Gödel–Dummett logics

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Syntax

Primitive connectives: $\mathcal{L} = \{\to, \wedge, \vee, \overline{0}\}$.

Defined connectives: $\neg$, $\overline{1}$, $\leftrightarrow$:

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi).$$

Formulas: built using the connectives from a fixed countable set of atoms.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulas.

A Hilbert-style proof system

Axioms:

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
($\wedge$a)	$\varphi \wedge \psi \to \varphi$	
($\wedge$b)	$\varphi \wedge \psi \to \psi$	
($\wedge$c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	
($\vee$a)	$\varphi \to \varphi \vee \psi$	
($\vee$b)	$\psi \to \varphi \vee \psi$	
($\vee$c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$	
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Con)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction

Inference rule: from φ and $\varphi \to \psi$ infer ψ *modus ponens*

The relation of provability

Proof: a proof of a formula φ from a set of formulas (theory) Γ is a finite sequence of formulas $\langle \psi_1, \ldots, \psi_n \rangle$ such that:

- $\psi_n = \varphi$
- for every $i \leq n$, either
  - $\psi_i \in \Gamma$, or
  - ψ_i is an instance of an axiom, or
  - there are $j, k < i$ such that $\psi_k = \psi_j \to \psi_i$.

We write $\Gamma \vdash_{\mathrm{G}} \varphi$ if there is a proof of φ from Γ.

A formula φ is a theorem of Gödel–Dummett logic if $\vdash_{\mathrm{G}} \varphi$.

Proposition 2.1

The provability relation of Gödel–Dummett logic is finitary: if $\Gamma \vdash_{\mathrm{G}} \varphi$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{G}} \varphi$.

Algebraic semantics

A Gödel algebra (or just G-algebra) is a structure $\boldsymbol{B} = \langle B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \rightarrow^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}} \rangle$ such that:

(1) $\langle B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}} \rangle$ is a bounded lattice

(2) $z \leq x \rightarrow^{\boldsymbol{B}} y$ iff $x \wedge^{\boldsymbol{B}} z \leq y$ (residuation)

(3) $(x \rightarrow^{\boldsymbol{B}} y) \vee^{\boldsymbol{B}} (y \rightarrow^{\boldsymbol{B}} x) = \overline{1}^{\boldsymbol{B}}$ (prelinearity)

where $x \leq^{\boldsymbol{B}} y$ is defined as $x \wedge^{\boldsymbol{B}} y = x$ or (equivalently) as $x \rightarrow^{\boldsymbol{B}} y = \overline{1}^{\boldsymbol{B}}$.

A G-algebra $\boldsymbol{B}$ is linearly ordered (or G-chain) if $\leq^{\boldsymbol{B}}$ is a total order.

By $\mathbb{G}$ (or $\mathbb{G}_{\text{lin}}$ resp.) we denote the class of all G-algebras (G-chains resp.)

Standard semantics

Consider algebra $[0,1]_{\mathrm{G}} = \langle [0,1], \wedge^{[0,1]_{\mathrm{G}}}, \vee^{[0,1]_{\mathrm{G}}}, \rightarrow^{[0,1]_{\mathrm{G}}}, 0, 1\rangle$, where:

$$a \wedge^{[0,1]_{\mathrm{G}}} b = \min\{a,b\}$$

$$a \vee^{[0,1]_{\mathrm{G}}} b = \max\{a,b\}$$

$$a \rightarrow^{[0,1]_{\mathrm{G}}} b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Exercise 1

Prove that $[0,1]_{\mathrm{G}}$ is the unique G-chain with the lattice reduct $\langle [0,1], \min, \max, 0, 1\rangle$.

Semantical consequence

Definition 2.2

A ***B*-evaluation** is a mapping e from $Fm_{\mathcal{L}}$ to B such that:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi)$

Definition 2.3

A formula φ is a logical consequence of a set of formulas Γ w.r.t. a class $\mathbb{K}$ of G-algebras, $\Gamma \models_{\mathbb{K}} \varphi$, if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

$$\text{if } e(\gamma) = \overline{1} \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) = \overline{1}.$$

Completeness theorem

Theorem 2.4

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{\mathbb{G}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{\mathrm{lin}}} \varphi$
4. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Exercise 1

Prove the implications from top to bottom.

Some theorems and derivations in G

Proposition 2.5

(T1) $\vdash_{\mathrm{G}} \varphi \to \varphi$

(T2) $\vdash_{\mathrm{G}} \varphi \to (\psi \to \varphi \wedge \psi)$

(D1) $\overline{1} \leftrightarrow \varphi \vdash_{\mathrm{G}} \varphi$ *and* $\varphi \vdash_{\mathrm{G}} \overline{1} \leftrightarrow \varphi$

(D2) $\varphi \to \psi \vdash_{\mathrm{G}} \varphi \wedge \psi \leftrightarrow \varphi$ *and* $\varphi \wedge \psi \leftrightarrow \varphi \vdash_{\mathrm{G}} \varphi \to \psi$

(D3) $\varphi \to (\psi \to \chi) \vdash_{\mathrm{G}} \varphi \wedge \psi \to \chi$ *and* $\varphi \wedge \psi \to \chi \vdash_{\mathrm{G}} \varphi \to (\psi \to \chi)$

Proposition 2.6

$\vdash_{\mathrm{G}} \varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$

$\vdash_{\mathrm{G}} \varphi \wedge (\psi \wedge \chi) \leftrightarrow (\varphi \wedge \psi) \wedge \chi$

$\vdash_{\mathrm{G}} \varphi \wedge (\varphi \vee \psi) \leftrightarrow \varphi$

$\vdash_{\mathrm{G}} \overline{1} \wedge \varphi \leftrightarrow \varphi$

$\vdash_{\mathrm{G}} (\varphi \to \psi) \vee (\psi \to \varphi) \leftrightarrow \overline{1}$

$\vdash_{\mathrm{G}} \varphi \vee \psi \leftrightarrow \psi \vee \varphi$

$\vdash_{\mathrm{G}} \varphi \vee (\psi \vee \chi) \leftrightarrow (\varphi \vee \psi) \vee \chi$

$\vdash_{\mathrm{G}} \varphi \vee (\varphi \wedge \psi) \leftrightarrow \varphi$

$\vdash_{\mathrm{G}} \overline{0} \vee \varphi \leftrightarrow \varphi$

The rule of substitution

Proposition 2.7

$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\varphi \wedge \chi) \leftrightarrow (\psi \wedge \chi)$ $\quad$ $\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\varphi \vee \chi) \leftrightarrow (\psi \vee \chi)$

$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\chi \wedge \varphi) \leftrightarrow (\chi \wedge \psi)$ $\quad$ $\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\chi \vee \varphi) \leftrightarrow (\chi \vee \psi)$

$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\varphi \to \chi) \leftrightarrow (\psi \to \chi)$ $\quad$ $\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\chi \to \varphi) \leftrightarrow (\chi \to \psi)$

$\vdash_{\mathrm{G}} \varphi \leftrightarrow \varphi$ $\quad$ $\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} \psi \leftrightarrow \varphi$ $\quad$ $\varphi \leftrightarrow \psi, \psi \leftrightarrow \chi \vdash_{\mathrm{G}} \varphi \leftrightarrow \chi$

Corollary 2.8

$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} \chi \leftrightarrow \chi'$, *where χ' results from χ by replacing its subformula φ by ψ.*

Exercise 2

Prove this corollary and the two previous propositions (you can use Theorem 2.11 and Lemma 2.12).

Lindenbaum–Tarski algebra

Definition 2.9

Let Γ be a theory. We define

$$[\varphi]_\Gamma = \{\psi \mid \Gamma \vdash_{\mathrm{G}} \varphi \leftrightarrow \psi\} \qquad L_\Gamma = \{[\varphi]_\Gamma \mid \varphi \in Fm_{\mathcal{L}}\}.$$

The Lindenbaum–Tarski algebra of a theory Γ ($\mathbf{Lind}_\Gamma$) as an algebra with the domain L_Γ and operations:

$$\begin{aligned}
\overline{0}^{\mathbf{Lind}_\Gamma} &= [\overline{0}]_\Gamma \\
\overline{1}^{\mathbf{Lind}_\Gamma} &= [\overline{1}]_\Gamma \\
[\varphi]_\Gamma \rightarrow^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma &= [\varphi \rightarrow \psi]_\Gamma \\
[\varphi]_\Gamma \wedge^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma &= [\varphi \wedge \psi]_\Gamma \\
[\varphi]_\Gamma \vee^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma &= [\varphi \vee \psi]_\Gamma.
\end{aligned}$$

Lindenbaum–Tarski algebra: basic properties

Proposition 2.10

1. $[\varphi]_\Gamma = [\psi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi \leftrightarrow \psi$
2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$
3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi$
4. $\mathbf{Lind}_\Gamma$ *is an* G*-algebra*
5. $\mathbf{Lind}_\Gamma$ *is an* G*-chain iff* $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$ *or* $\Gamma \vdash_{\mathrm{G}} \psi \to \varphi$ *for each* φ, ψ

Proof.

1. Left-to-right is the just definition and 'reflexivity' of $\leftrightarrow$. Conversely, we use 'transitivity' and 'symmetry' of $\leftrightarrow$.

2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ iff $[\varphi]_\Gamma \wedge^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma = [\varphi]_\Gamma$ iff $[\varphi \wedge \psi]_\Gamma = [\varphi]_\Gamma$ iff (by 1.) $\Gamma \vdash_{\mathrm{G}} \varphi \wedge \psi \leftrightarrow \varphi$ iff (by (D2)) $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$.

3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ iff (by 1.) $\Gamma \vdash_{\mathrm{G}} \overline{1} \leftrightarrow \varphi$ iff (by (D1)) $\Gamma \vdash_{\mathrm{G}} \varphi$.

5. Trivial after we prove 4.

Lindenbaum–Tarski algebra: basic properties

Proposition 2.10

1. $[\varphi]_\Gamma = [\psi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi \leftrightarrow \psi$
2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$
3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi$
4. $\mathbf{Lind}_\Gamma$ *is an* G*-algebra*
5. $\mathbf{Lind}_\Gamma$ *is an* G*-chain iff* $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$ *or* $\Gamma \vdash_{\mathrm{G}} \psi \to \varphi$ *for each* φ, ψ

Proof.

4. The definition of $\mathbf{Lind}_\Gamma$ is sound due to 1. and Proposition 2.7.

The lattice identities hold due to 1. and Proposition 2.6; prelinearity due to 3. and axiom (Prl); finally, the residuation:

$[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma \to^{\mathbf{Lind}_\Gamma} [\chi]_\Gamma = [\psi \to \chi]_\Gamma$ iff (by 2.) $\Gamma \vdash_{\mathrm{G}} \varphi \to (\psi \to \chi)$ iff (by (D3)) $\Gamma \vdash_{\mathrm{G}} \varphi \wedge \psi \to \chi$ iff (by 2.) $[\varphi]_\Gamma \wedge^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\chi]_\Gamma$. □

General/linear/standard completeness theorem

Theorem 2.4

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{\mathbb{G}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{\mathrm{lin}}} \varphi$
4. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Proof.

2. implies 1.: contrapositively, assume that $\Gamma \nvdash_{\mathrm{G}} \varphi$.

We know that $\mathbf{Lind}_\Gamma \in \mathbb{G}$ and the function e defined as $e(\psi) = [\psi]_\Gamma$

- is a $\mathbf{Lind}_\Gamma$-evaluation and
- $e(\psi) = \overline{1}^{\mathbf{Lind}_\Gamma}$ iff $\Gamma \vdash_{\mathrm{G}} \psi$.

Thus clearly $e(\chi) = \overline{1}^{\mathbf{Lind}_\Gamma}$ for each $\chi \in \Gamma$ and $e(\varphi) \neq \overline{1}^{\mathbf{Lind}_\Gamma}$. □

Deduction Theorem

Theorem 2.11 (Deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{\mathrm{G}} \psi \textit{ iff } \Gamma \vdash_{\mathrm{G}} \varphi \to \psi$$

Proof.

$\Leftarrow$: follows from *modus ponens*

$\Rightarrow$: let $\alpha_1, \ldots, \alpha_n = \psi$ be the proof of ψ in Γ, φ. We show by induction that $\Gamma \vdash_{\mathrm{G}} \varphi \to \alpha_i$ for each $i \leq n$.

If $\alpha_i = \varphi$ we use (T1); if α_i is an axiom or $\alpha_i \in \Gamma$, then $\Gamma \vdash_{\mathrm{G}} \alpha_i$ and so we can use axiom (We) and MP.

□

Deduction Theorem

Theorem 2.11 (Deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$*,*

$$\Gamma, \varphi \vdash_{\mathrm{G}} \psi \textit{ iff } \Gamma \vdash_{\mathrm{G}} \varphi \to \psi$$

Proof.

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$\Rightarrow$: let $\alpha_1, \ldots, \alpha_n = \psi$ be the proof of ψ in Γ, φ. We show by induction that $\Gamma \vdash_{\mathrm{G}} \varphi \to \alpha_i$ for each $i \leq n$.

Otherwise there has to be $k, j < i$ such that $\alpha_k = \alpha_j \to \alpha_i$.

Induction assumption gives: $\Gamma \vdash_{\mathrm{G}} \varphi \to \alpha_j$ and $\Gamma \vdash \varphi \to (\alpha_j \to \alpha_i)$.

Using $\Gamma \vdash \varphi \to (\alpha_j \to \alpha_i)$, (Ex), and MP we get $\Gamma \vdash \alpha_j \to (\varphi \to \alpha_i)$,

using this, $\Gamma \vdash_{\mathrm{G}} \varphi \to \alpha_j$, (Tr), and MP twice we get $\Gamma \vdash \varphi \to (\varphi \to \alpha_i)$.

Finally we use (Con) and MP.

□

Semilinearity Property

Lemma 2.12 (Semilinearity Property)

If $\Gamma, \varphi \to \psi \vdash_{\mathrm{G}} \chi$ *and* $\Gamma, \psi \to \varphi \vdash_{\mathrm{G}} \chi$*, then* $\Gamma \vdash_{\mathrm{G}} \chi$*.*

Proof.

By the deduction theorem: $\Gamma \vdash_{\mathrm{G}} (\varphi \to \psi) \to \chi$ and $\Gamma \vdash_{\mathrm{G}} (\psi \to \varphi) \to \chi$.

Thus by ($\vee$c) we get $\Gamma \vdash_{\mathrm{G}} (\varphi \to \psi) \vee (\psi \to \varphi) \to \chi$.

Axiom (Prl) completes the proof. □

Linear Extension Property

Definition 2.13

A theory Γ is linear if $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$ or $\Gamma \vdash_{\mathrm{G}} \psi \to \varphi$ for each φ, ψ.

Lemma 2.14 (Linear Extension Property)

If $\Gamma \nvdash_{\mathrm{G}} \varphi$, then there is linear theory $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \nvdash_{\mathrm{G}} \varphi$.

Proof.

Enumerate all pairs of formulas: $\langle \varphi_0, \psi_0 \rangle, \langle \psi_1, \varphi_1 \rangle, \dots$
Construct theories $\Gamma_0, \Gamma_1, \dots$ s.t. $\Gamma_0 = \Gamma$; $\Gamma_i \subseteq \Gamma_{i+1}$, and $\Gamma_i \nvdash_{\mathrm{G}} \varphi$:

- *if* $\Gamma_i, \varphi_i \to \psi_i \nvdash_{\mathrm{G}} \varphi$, *then* $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i \to \psi_i\}$
- *if* $\Gamma_i, \varphi_i \to \psi_i \vdash_{\mathrm{G}} \varphi$, *then* $\Gamma_{i+1} = \Gamma_i \cup \{\psi_i \to \varphi_i\}$

Clearly $\Gamma_{i+1} \nvdash_{\mathrm{G}} \varphi$ (the 1st is obvious, in the 2nd would $\Gamma_{i+1} \vdash_{\mathrm{G}} \varphi$ entail $\Gamma_i \vdash_{\mathrm{G}} \varphi$ by the *Semilinearity Property*, a contradiction with the IH).
Define $\Gamma' = \bigcup \Gamma_i$. Clearly Γ' is linear, $\Gamma' \supseteq \Gamma$, and $\Gamma' \nvdash_{\mathrm{G}} \varphi$. □

General/linear/standard completeness theorem

Theorem 2.4

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{\mathbb{G}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{\mathrm{lin}}} \varphi$
4. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Proof.

3. implies 1.: contrapositively, assume that $\Gamma \nvdash_{\mathrm{G}} \varphi$. Due to the Linear Extension Property there is a linear theory $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \nvdash_{\mathrm{G}} \varphi$.

We know that $\mathbf{Lind}_{\Gamma'} \in \mathbb{G}_{\mathrm{lin}}$ and the function e defined as $e(\psi) = [\psi]_{\Gamma'}$

- is a $\mathbf{Lind}_{\Gamma'}$-evaluation and
- $e(\psi) = \overline{1}^{\mathbf{Lind}_{\Gamma'}}$ iff $\Gamma' \vdash_{\mathrm{G}} \psi$.

Thus $e(\chi) = \overline{1}^{\mathbf{Lind}_{\Gamma'}}$ for each $\chi \in \Gamma$ (as $\Gamma' \vdash_{\mathrm{G}} \chi$) and $e(\varphi) \neq \overline{1}^{\mathbf{Lind}_{\Gamma'}}$. □

The proof of the standard completeness theorem

Note that the algebra $\mathbf{Lind}_{\Gamma'}$ from the previous proof is countable.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f\colon L_{\Gamma'} \to [0,1]$ such that $f(\overline{0}^{\mathbf{Lind}_{\Gamma'}}) = 0, f(\overline{1}^{\mathbf{Lind}_{\Gamma'}}) = 1$, and for each $a, b \in L_{T'}$ we have:

$$a \leq b \quad \text{iff} \quad f(a) \leq f(b)$$

We define a mapping $\overline{e}\colon Fm_{\mathcal{L}} \to [0,1]$ as

$$\overline{e}(\psi) = f(e(\psi))$$

and prove (by induction) that it is an $[0,1]_{\mathrm{G}}$-evaluation.

Then $\overline{e}(\psi) = 1$ iff $e(\psi) = \overline{1}^{\mathbf{Lind}_{\Gamma'}}$ and so $\overline{e}[\Gamma] \subseteq \{1\}$ and $\overline{e}(\varphi) \neq 1$.

Syntax

Primitive connectives: $\mathcal{L} = \{\rightarrow, \wedge, \vee, \overline{0}\}$.

Defined connectives: $\neg$, $\overline{1}$, $\leftrightarrow$:

$$\neg\varphi = \varphi \rightarrow \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi).$$

Formulas: built using the connectives from a fixed countable set of atoms.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulas.

We also use additional connectives $\oplus$ and $\&$ defined as:

$$\varphi \oplus \psi = \neg\varphi \rightarrow \psi \qquad \varphi \,\&\, \psi = \neg(\varphi \rightarrow \neg\psi)$$

A Hilbert-style proof system

Axioms:

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
($\wedge$a)	$\varphi \wedge \psi \to \varphi$	
($\wedge$b)	$\varphi \wedge \psi \to \psi$	
($\wedge$c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	
($\vee$a)	$\varphi \to \varphi \vee \psi$	
($\vee$b)	$\psi \to \varphi \vee \psi$	
($\vee$c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$	
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom

Inference rule: from φ and $\varphi \to \psi$ infer ψ — *modus ponens*

The relation of provability

Proof: a proof of a formula φ from a set of formulas (theory) Γ is a finite sequence of formulas $\langle\psi_1, \dots, \psi_n\rangle$ such that:

- $\psi_n = \varphi$
- for every $i \leq n$, either
  - $\psi_i \in \Gamma$, or
  - ψ_i is an instance of an axiom, or
  - there are $j, k < i$ such that $\psi_k = \psi_j \to \psi_i$.

We write $\Gamma \vdash_{Ł} \varphi$ if there is a proof of φ from Γ.

A formula φ is a theorem of Łukasiewicz logic if $\vdash_{Ł} \varphi$.

Proposition 2.15

The provability relation of Łukasiewicz logic is finitary: if $\Gamma \vdash_{Ł} \varphi$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{Ł} \varphi$.

Algebraic semantics

An MV-*algebra* is a structure $\boldsymbol{B} = \langle B, \oplus, \neg, \overline{0} \rangle$ such that:

(1) $\langle B, \oplus, \overline{0} \rangle$ is a commutative monoid,

(2) $\neg\neg x = x$,

(3) $x \oplus \neg\overline{0} = \neg\overline{0}$,

(4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

In each MV-algebra we define additional operations:

$x \to y$	is	$\neg x \oplus y$	implication
$x \mathbin{\&} y$	is	$\neg(\neg x \oplus \neg y)$	strong conjunction
$x \vee y$	is	$\neg(\neg x \oplus y) \oplus y$	max-disjunction
$x \wedge y$	is	$\neg(\neg x \vee \neg y)$	min-conjunction
$\overline{1}$	is	$\neg\overline{0}$	top

Exercise 3

Prove that $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice.

Algebraic semantics cont. and standard semantics

We say that an MV-algebra $\boldsymbol{B}$ is linearly ordered (or MV-chain) if its lattice reduct is.

By $\mathbb{MV}$ (or $\mathbb{MV}_{\mathrm{lin}}$ resp.) we denote the class of all MV-algebras (MV-chains resp.)

Take the algebra $[0,1]_{Ł} = \langle [0,1], \oplus, \neg, 0\rangle$, with operations defined as:

$$\neg a = 1 - a \qquad\qquad a \oplus b = \min\{1, a + b\}.$$

Proposition 2.16

$[0,1]_{Ł}$ *is the unique (up to isomorphism)* MV*-chain with the lattice reduct* $\langle [0,1], \min, \max, 0, 1\rangle$.

Exercise 4

Check that $[0,1]_{Ł}$ is an MV-chain and find another MV-chain isomorphic to $[0,1]_{Ł}$ with the same lattice reduct.

Semantical consequence

Definition 2.17

A ***B*-evaluation** is a mapping e from $Fm_{\mathcal{L}}$ to B such that:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi) = \neg^{\boldsymbol{B}} e(\varphi) \oplus^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\boldsymbol{B}} e(\psi) = \cdots$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{B}} e(\psi) = \cdots$

Definition 2.18

A formula φ is a logical consequence of a set of formulas Γ w.r.t. a class $\mathbb{K}$ of MV-algebras, $\Gamma \models_{\mathbb{K}} \varphi$, if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

$$\text{if } e(\gamma) = \overline{1} \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) = \overline{1}.$$

General/linear/standard completeness theorem

Theorem 2.19

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{Ł} \varphi$
2. $\Gamma \models_{\mathbb{MV}} \varphi$
3. $\Gamma \models_{\mathbb{MV}_{lin}} \varphi$.

If Γ *is finite we can add:*

4. $\Gamma \models_{[0,1]_Ł} \varphi$.

Some theorems and derivations

Proposition 2.20

(T1)	$\vdash_{Ł} \varphi \to \varphi$
(T2)	$\vdash_{Ł} \varphi \to (\psi \to \varphi \wedge \psi)$
(T3)	$\vdash_{Ł} \varphi \vee \chi \to ((\varphi \to \psi) \vee \chi \to \psi \vee \chi)$
(T4)	$\vdash_{Ł} \varphi \vee \varphi \to \varphi$
(T5)	$\vdash_{Ł} \varphi \vee \psi \to \psi \vee \varphi$
(D1)	$\overline{1} \leftrightarrow \varphi \vdash_{Ł} \varphi$ *and* $\varphi \vdash_{Ł} \overline{1} \leftrightarrow \varphi$
(D2)	$\varphi \to \psi \vdash_{Ł} \varphi \wedge \psi \leftrightarrow \varphi$ *and* $\varphi \wedge \psi \leftrightarrow \varphi \vdash_{Ł} \varphi \to \psi$
(D3′)	$\varphi \to (\psi \to \chi) \vdash_{G} \varphi \,\&\, \psi \to \chi$ *and* $\varphi \,\&\, \psi \to \chi \vdash_{G} \varphi \to (\psi \to \chi)$

Proposition 2.21

$\vdash_{Ł} \varphi \oplus \psi \leftrightarrow \psi \oplus \varphi$

$\vdash_{Ł} \varphi \oplus (\psi \oplus \chi) \leftrightarrow (\varphi \oplus \psi) \oplus \chi$

$\vdash_{Ł} \overline{0} \oplus \varphi \leftrightarrow \varphi$

$\vdash_{Ł} \neg\neg\varphi \leftrightarrow \varphi$

$\vdash_{Ł} \varphi \oplus \neg\overline{0} \leftrightarrow \neg\overline{0}$

$\vdash_{Ł} \neg(\neg\varphi \oplus \psi) \oplus \psi \leftrightarrow \neg(\neg\psi \oplus \varphi) \oplus \varphi$

The rule of substitution

Proposition 2.22

$$\varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \wedge \chi) \leftrightarrow (\psi \wedge \chi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \vee \chi) \leftrightarrow (\psi \vee \chi)$$
$$\varphi \leftrightarrow \psi \vdash_{Ł} (\chi \wedge \varphi) \leftrightarrow (\chi \wedge \psi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\chi \vee \varphi) \leftrightarrow (\chi \vee \psi)$$
$$\varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \to \chi) \leftrightarrow (\psi \to \chi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\chi \to \varphi) \leftrightarrow (\chi \to \psi)$$

$$\vdash_{Ł} \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash_{Ł} \psi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi, \psi \leftrightarrow \chi \vdash_{Ł} \varphi \leftrightarrow \chi$$

Corollary 2.23

$\varphi \leftrightarrow \psi \vdash_{Ł} \chi \leftrightarrow \chi'$, *where χ' results from χ by replacing its subformula φ by ψ.*

Exercise 5

Prove this corollary and the two previous propositions (you can use Theorem 2.27 and Lemma 2.28).

Lindenbaum–Tarski algebra

Definition 2.24

Let Γ be a theory. We define

$$[\varphi]_\Gamma = \{\psi \mid \Gamma \vdash_{Ł} \varphi \leftrightarrow \psi\} \qquad L_\Gamma = \{[\varphi]_\Gamma \mid \varphi \in Fm_{\mathcal{L}}\}$$

The Lindenbaum–Tarski algebra of a theory Γ ($\mathbf{Lind}_\Gamma$) as an algebra with the domain L_Γ and operations:

$$\begin{aligned}
\bar{0}^{\mathbf{Lind}_\Gamma} &= [\bar{0}]_\Gamma \\
\neg^{\mathbf{Lind}_\Gamma}[\varphi]_\Gamma &= [\neg\varphi]_\Gamma \\
[\varphi]_\Gamma \oplus^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma &= [\varphi \oplus \psi]_\Gamma.
\end{aligned}$$

Lindenbaum–Tarski algebra: basic properties

Proposition 2.25

1. $[\varphi]_\Gamma = [\psi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi \leftrightarrow \psi$
2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi \to \psi$
3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi$
4. $\mathbf{Lind}_\Gamma$ *is an* MV*-algebra*
5. $\mathbf{Lind}_\Gamma$ *is an* MV*-chain iff* $\Gamma \vdash_{Ł} \varphi \to \psi$ *or* $\Gamma \vdash_{Ł} \psi \to \varphi$ *for each* φ, ψ*.*

Proof.

1. Left-to-right is the just definition and 'reflexivity' of $\leftrightarrow$. Conversely, we use 'transitivity' and 'symmetry' of $\leftrightarrow$.

2. $[\varphi]_\Gamma^{\mathbf{Lind}_\Gamma} \leq [\psi]_\Gamma$ iff $[\varphi]_\Gamma \wedge^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma = [\varphi]_\Gamma$ iff $[\varphi \wedge \psi]_\Gamma = [\varphi]_\Gamma$ iff (by 1.) $\Gamma \vdash_{Ł} \varphi \wedge \psi \leftrightarrow \varphi$ iff (by (D2)) $\Gamma \vdash_{Ł} \varphi \to \psi$.

3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ iff (by 2.) $\Gamma \vdash_{Ł} \overline{1} \to \varphi$ iff (by (D1)) $\Gamma \vdash_{Ł} \varphi$.

5. Trivial after we prove 4.

Lindenbaum–Tarski algebra: basic properties

Proposition 2.25

1. $[\varphi]_\Gamma = [\psi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi \leftrightarrow \psi$
2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi \to \psi$
3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi$
4. $\mathbf{Lind}_\Gamma$ *is an* MV*-algebra*
5. $\mathbf{Lind}_\Gamma$ *is an* MV*-chain iff* $\Gamma \vdash_{Ł} \varphi \to \psi$ *or* $\Gamma \vdash_{Ł} \psi \to \varphi$ *for each* φ, ψ*.*

Proof.

4. The definition of $\mathbf{Lind}_\Gamma$ is sound due to 1. and Proposition 2.7.

The identities defining MV-algebras hold due to 1. and Proposition 2.21.

□

Łukasiewicz logic vs. Gödel–Dummett

Some things are the same, not only (T1), (T2), (D1), and (D2), but also:

$\varphi \wedge \psi \to \chi \vdash_{Ł} \varphi \to (\psi \to \chi)$ $\qquad$ $\varphi \wedge \psi \to \chi \vdash_{\mathrm{G}} \varphi \to (\psi \to \chi)$

$\vdash_{Ł} \varphi \to \neg\neg\varphi$ $\qquad$ $\vdash_{\mathrm{G}} \varphi \to \neg\neg\varphi$

$\vdash_{Ł} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ $\qquad$ $\vdash_{\mathrm{G}} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

Some are different:

$\varphi \to (\psi \to \chi) \nvdash_{Ł} \varphi \wedge \psi \to \chi$ $\qquad$ $\varphi \to (\psi \to \chi) \vdash_{\mathrm{G}} \varphi \wedge \psi \to \chi$

$\vdash_{Ł} \neg\neg\varphi \to \varphi$ $\qquad$ $\nvdash_{\mathrm{G}} \neg\neg\varphi \to \varphi$

$\vdash_{Ł} (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$ $\qquad$ $\nvdash_{\mathrm{G}} (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

Contrast this with known derivation (D3′):

$$\varphi \to (\psi \to \chi) \vdash_{Ł} \varphi \,\&\, \psi \to \chi \qquad \varphi \,\&\, \psi \to \chi \vdash_{Ł} \varphi \to (\psi \to \chi)$$

Failure of the Deduction Theorem

Assume that we would have that for every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{Ł} \psi \text{ iff } \Gamma \vdash_{Ł} \varphi \to \psi$$

Clearly (MP twice): $\varphi, \varphi \to (\varphi \to \psi) \vdash_{Ł} \psi$.

Thus by the deduction theorem we would get

$$\vdash_{Ł} (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi).$$

This is the axiom of contraction known to fail in Łukasiewicz logic.

A possible solution

We can prove that:

$$\vdash_{Ł} \varphi \,\&\, \psi \leftrightarrow \psi \,\&\, \varphi \qquad \vdash_{Ł} \varphi \,\&\, \overline{1} \leftrightarrow \varphi \qquad \vdash_{Ł} (\varphi \,\&\, \psi) \,\&\, \chi \leftrightarrow \psi \,\&\, (\varphi \,\&\, \chi)$$

Thus it makes sense to define $\varphi^0 = \overline{1}$ and $\varphi^{n+1} = \varphi^n \,\&\, \varphi$

Exercise 6

Let χ be a $\&$-conjunction of n formulas φ with arbitrary bracketing. Prove that $\vdash_{Ł} \chi \leftrightarrow \varphi^n$. Furthermore prove that $\varphi \vdash_{Ł} \varphi^n$.

Local Deduction Theorem

Theorem 2.26 (Local deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{Ł} \psi \text{ iff there is } n \text{ such that } \Gamma \vdash_{Ł} \varphi^n \to \psi$$

Proof.

$\Leftarrow$: follows from *modus ponens* and the previous exercise

$\Rightarrow$: let $\alpha_1, \dots, \alpha_n = \psi$ be the proof of ψ in Γ, φ. We show by induction that for each $i \leq n$ there is n_i such that $\Gamma \vdash_{Ł} \varphi^{n_i} \to \alpha_i$

If $\alpha_i = \varphi$ we set $n_i = 1$ and use (T1); if α_i is an axiom or $\alpha_i \in \Gamma$, then $\Gamma \vdash_{Ł} \alpha_i$ and so we can set $n_i = 1$ and use axiom (We) and MP.

□

Local Deduction Theorem

Theorem 2.26 (Local deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{Ł} \psi \text{ iff there is } n \text{ such that } \Gamma \vdash_{Ł} \varphi^n \to \psi$$

Proof.

$\Leftarrow$: follows from *modus ponens* and the previous exercise

$\Rightarrow$: let $\alpha_1, \dots, \alpha_n = \psi$ be the proof of ψ in Γ, φ. We show by induction that for each $i \leq n$ there is n_i such that $\Gamma \vdash_{Ł} \varphi^{n_i} \to \alpha_i$

Otherwise there has to be $k, j < i$ such that $\alpha_k = \alpha_j \to \alpha_i$.

Induction assumption gives: $\Gamma \vdash_{Ł} \varphi^{n_j} \to \alpha_j$ and $\Gamma \vdash \varphi^{n_k} \to (\alpha_j \to \alpha_i)$.

Using $\Gamma \vdash \varphi^{n_k} \to (\alpha_j \to \alpha_i)$, (Ex), and MP we get $\Gamma \vdash \alpha_j \to (\varphi^{n_k} \to \alpha_i)$,

using this, $\Gamma \vdash_{Ł} \varphi^{n_j} \to \alpha_j$, (Tr), and MP we get $\Gamma \vdash \varphi^{n_j} \to (\varphi^{n_k} \to \alpha_i)$.

Finally we use (D3′) and the previous exercise to get $\Gamma \vdash \varphi^{n_j + n_k} \to \alpha_i$. $\square$

Proof by Cases Property

Theorem 2.27 (Proof by Cases Property)

If $\Gamma, \varphi \vdash_{Ł} \chi$ *and* $\Gamma, \psi \vdash_{Ł} \chi$*, then* $\Gamma, \varphi \vee \psi \vdash_{Ł} \chi$*.*

Proof.

Claim If $\Gamma \vdash_{Ł} \varphi$, then $\Gamma \vee \chi \vdash_{Ł} \delta \vee \chi$ for each formula χ and each δ appearing in the proof of φ from Γ.

Proof of the claim: trivial for $\delta \in \Gamma$ or δ an axiom; if we used MP, then by IH there has to be η st.

$\Gamma \vee \chi \vdash_{Ł} \eta \vee \chi \qquad \Gamma \vee \chi \vdash_{Ł} (\eta \to \delta) \vee \chi$ thus (T3) completes the proof.

Now using the claim: $\Gamma \vee \psi, \varphi \vee \psi \vdash_{Ł} \chi \vee \psi$ and $\Gamma \vee \chi, \psi \vee \chi \vdash_{Ł} \chi \vee \chi$. Using ($\vee$a), (T4), and (T5) we get $\Gamma, \varphi \vee \psi \vdash_{Ł} \psi \vee \chi$ and $\Gamma, \psi \vee \chi \vdash_{Ł} \chi$ and the rest is trivial. □

Semilinearity Property

Lemma 2.28 (Semilinearity Property)

If $\Gamma, \varphi \to \psi \vdash_{Ł} \chi$ *and* $\Gamma, \psi \to \varphi \vdash_{Ł} \chi$, *then* $\Gamma \vdash_{Ł} \chi$.

Proof.

By the Proof by Cases Property and axiom (Prl). □

Linear Extensions Property

Definition 2.29

A theory Γ is linear if $\Gamma \vdash_{\text{Ł}} \varphi \to \psi$ or $\Gamma \vdash_{\text{Ł}} \psi \to \varphi$ for each φ, ψ.

Lemma 2.30 (Linear Extension Property)

If $\Gamma \nvdash_{\text{Ł}} \varphi$, then there is linear theory $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \nvdash_{\text{Ł}} \varphi$.

Proof.

The same as in the case of Gödel–Dummett logic. □

Linear Extensions Property

Definition 2.29

A theory Γ is linear if $\Gamma \vdash_{Ł} \varphi \to \psi$ or $\Gamma \vdash_{Ł} \psi \to \varphi$ for each φ, ψ.

Lemma 2.30 (Linear Extension Property)

If $\Gamma \nvdash_{Ł} \varphi$, then there is linear theory $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \nvdash_{Ł} \varphi$.

Proof.

Enumerate all pairs of formulas: $\langle \varphi_0, \psi_0 \rangle, \langle \psi_1, \varphi_1 \rangle, \ldots$
Construct theories $\Gamma_0, \Gamma_1, \ldots$ s.t. $\Gamma_0 = \Gamma$; $\Gamma_i \subseteq \Gamma_{i+1}$, and $\Gamma_i \nvdash_{Ł} \varphi$:

- *if* $\Gamma_i, \varphi_i \to \psi_i \nvdash_{Ł} \varphi$, *then* $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i \to \psi_i\}$
- *if* $\Gamma_i, \varphi_i \to \psi_i \vdash_{Ł} \varphi$, *then* $\Gamma_{i+1} = \Gamma_i \cup \{\psi_i \to \varphi_i\}$

Clearly $\Gamma_{i+1} \nvdash_{Ł} \varphi$ (the 1st is obvious, in the 2nd would $\Gamma_{i+1} \vdash_{Ł} \varphi$ entail $\Gamma_i \vdash_{Ł} \varphi$ by the Semilinearity Property, a contradiction with the IH.
Define $\Gamma' = \bigcup \Gamma_i$. Clearly Γ' is linear, $\Gamma' \supseteq \Gamma$, and $\Gamma' \nvdash_{Ł} \varphi$. □

General/linear/standard completeness theorem

Theorem 2.19

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{Ł} \varphi$
2. $\Gamma \models_{\mathbb{MV}} \varphi$
3. $\Gamma \models_{\mathbb{MV}_{lin}} \varphi$.

If Γ *is finite we can add:*

4. $\Gamma \models_{[0,1]_Ł} \varphi$.

Exercise 7

Finish the proof of the equivalence of the first three items and that they imply the last one.

We give a proof of 4. implies 1. but first . . .

Failure of standard completeness for infinite theories

Non-theorem

For every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ we have:

$$\Gamma \vdash_{Ł} \varphi \text{ if, and only if, } \Gamma \models_{[0,1]_{Ł}} \varphi.$$

- Consider theory $\Gamma = \{(p \oplus \overset{n}{\ldots} \oplus p) \to q \mid n \geq 1\} \cup \{\neg p \to q\}$.
- Note that for any $[0,1]_{Ł}$-evaluation e s.t. $e[\Gamma] = \{1\}$ we have $e(q) = 1$ and so $\Gamma \models_{[0,1]_{Ł}} q$.
- Thus by our *Non-theorem* $\Gamma \vdash_{Ł} q$ and as proofs are finite, there must be a finite $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \vdash_{Ł} q$.
- Thus by our *Non-theorem* $\Gamma_0 \models_{[0,1]_{Ł}} q$
- Let n be the maximal n s.t. $(p \oplus \overset{n}{\ldots} \oplus p) \to q \in \Gamma_0$.
- $[0,1]_{Ł}$-evaluation $e(p) = \frac{1}{n+1}$ and $e(q) = \frac{n}{n+1}$ yields a contradiction.

MV-algebras and LOAGs

A lattice ordered Abelian group (*LOAG* for short) is a structure $\langle G, +, 0, -, \leq \rangle$ s.t. $\langle G, +, 0, - \rangle$ is an Abelian group and:

(i) $\langle G, \leq \rangle$ *is a lattice,*
(ii) *if* $x \leq y$*, then* $x + z \leq y + z$ *for all* $z \in G$*.*

A strong unit u is an element s.t.

$$(\forall x \in G)(\exists n \in N)(x \leq nu)$$

For LOAG $\boldsymbol{G} = \langle G, +, 0, -, \leq \rangle$ and strong unit u we define algebra $\Gamma(\boldsymbol{G}, u) = \langle [0, u], \oplus, \neg, \overline{0} \rangle$, where $x \oplus y = \min\{u, x + y\}$, $\neg x = u - x$, $\overline{0} = 0$.

By $\boldsymbol{R}$ we denote the additive LOAG of reals.

Proposition 2.31

$\Gamma(\boldsymbol{G}, u)$ *is an* MV*-algebra and for each* $u > 0$, $\Gamma(\boldsymbol{R}, u)$ *is isomorphic to the standard* MV*-algebra* $[0, 1]_{Ł}$*.*

The proof of the standard completeness theorem

If $\Gamma \nvdash_{Ł} \varphi$ we know that there is a countable MV-chain $\boldsymbol{B}$ s.t. $\Gamma \not\models_{\boldsymbol{B}} \varphi$. Let $x_1, \ldots, x_n$ be variables occurring in $\Gamma \cup \{\varphi\}$. Then:

$$\not\models_{\boldsymbol{B}} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma} (\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1})$$

Let us define an algebra $\boldsymbol{B'} = \langle Z \times B, +, -, 0 \rangle$ as:

$$\langle i, x \rangle + \langle j, y \rangle = \begin{cases} \langle i + j, x \oplus y \rangle & \text{if } x \,\&\, y = 0 \\ \langle i + j + 1, x \,\&\, y \rangle & \text{otherwise} \end{cases}$$

$$-\langle i, x \rangle = \langle -i - 1, \neg x \rangle \text{ and } 0 = \langle 0, \overline{0} \rangle$$

Proposition 2.32

$\boldsymbol{B'}$ is a LOAG and $\boldsymbol{B} = \boldsymbol{\Gamma}(\boldsymbol{B'}, \langle 1, \overline{0} \rangle)$.

The proof of the standard completeness theorem

Let us fix an extra variable u, we define a translation of MV-terms into LOAG-terms:

$$x' = x \qquad \overline{0}' = 0 \qquad (\neg t)' = u - t' \qquad (t_1 \oplus t_2)' = (t_1' + t_2') \wedge u.$$

Recall that we have:

$$\not\models_{\boldsymbol{B}} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma} (\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1}),$$

Thus also:

$$\not\models_{\boldsymbol{B'}} (\forall u)(\forall x_1, \ldots, x_n)[(0 < u) \wedge \bigwedge_{i \leq n} (x_i \leq u) \wedge (0 \leq x_i) \wedge \bigwedge_{\psi \in \Gamma} (\psi' \approx u) \Rightarrow (\varphi' \approx u)]$$

The proof of the standard completeness theorem

Gurevich–Kokorin theorem: each $\forall_1$-sentence of LOAGs is true in additive LOAG of reals iff it is true in all linearly ordered LOAGs.
Thus

$$\not\models_{\boldsymbol{R}} (\forall u)(\forall x_1, \ldots, x_n)[(0 < u) \wedge \bigwedge_{i \leq n}(x_i \leq u) \wedge (0 \leq x_i) \wedge \bigwedge_{\psi \in \Gamma}(\psi' \approx u) \Rightarrow (\varphi' \approx u)]$$

And so

$$\not\models_{\boldsymbol{\Gamma}(\boldsymbol{R},u)} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma}(\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1})$$

And so

$$\not\models_{[0,1]_{Ł}} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma}(\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1})$$

i.e., $\Gamma \not\models_{[0,1]_{Ł}} \varphi$