A Gentle Introduction to Mathematical Fuzzy Logic

2. Basic properties of Łukasiewicz and Gödel–Dummett logic

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Syntax

We consider primitive connectives $\mathcal{L} = \{\to, \wedge, \vee, \overline{0}\}$ and defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$:

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

Formulas are built from a fixed countable set of atoms using the connectives.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulas.

A Hilbert-style proof system

Axioms:

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
(∧a)	$\varphi \wedge \psi \to \varphi$	
(∧b)	$\varphi \wedge \psi \to \psi$	
(∧c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	
(∨a)	$\varphi \to \varphi \vee \psi$	
(∨b)	$\psi \to \varphi \vee \psi$	
(∨c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$	
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Con)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction

Inference rule: from φ and $\varphi \to \psi$ infer ψ — *modus ponens*

The relation of provability

Proof: a proof of a formula φ from a set of formulas (theory) Γ is a finite sequence of formulas $\langle \psi_1, \ldots, \psi_n \rangle$ such that:

- $\psi_n = \varphi$
- for every $i \leq n$, either $\psi_i \in \Gamma$, or ψ_i is an instance of an axiom, or there are $j, k < i$ such that $\psi_k = \psi_j \to \psi_i$.

We write $\Gamma \vdash_{\mathrm{G}} \varphi$ if there is a proof of φ from Γ.

A formula φ is a theorem of Gödel–Dummett logic if $\vdash_{\mathrm{G}} \varphi$.

Proposition 2.1

The provability relation of Gödel–Dummett logic is finitary: if $\Gamma \vdash_{\mathrm{G}} \varphi$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{G}} \varphi$.

Algebraic semantics

A Gödel algebra (or just G-algebra) is a structure
$\boldsymbol{B} = \langle B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \rightarrow^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}} \rangle$ such that:

(1) $\langle B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}} \rangle$ is a bounded lattice

(2) $z \leq x \rightarrow^{\boldsymbol{B}} y$ iff $x \wedge^{\boldsymbol{B}} z \leq y$ (residuation)

(3) $(x \rightarrow^{\boldsymbol{B}} y) \vee^{\boldsymbol{B}} (y \rightarrow^{\boldsymbol{B}} x) = \overline{1}^{\boldsymbol{B}}$ (prelinearity)

where $x \leq^{\boldsymbol{B}} y$ is defined as $x \wedge^{\boldsymbol{B}} y = x$ or (equivalently) as $x \rightarrow^{\boldsymbol{B}} y = \overline{1}^{\boldsymbol{B}}$.

A G-algebra $\boldsymbol{B}$ is linearly ordered (or G-chain) if $\leq^{\boldsymbol{B}}$ is a total order.

By $\mathbb{G}$ (or $\mathbb{G}_{\text{lin}}$ resp.) we denote the class of all G-algebras (G-chains resp.)

Standard semantics

Consider algebra $[0,1]_{\mathrm{G}} = \langle [0,1], \wedge^{[0,1]_{\mathrm{G}}}, \vee^{[0,1]_{\mathrm{G}}}, \to^{[0,1]_{\mathrm{G}}}, 0, 1 \rangle$, where:

$$a \wedge^{[0,1]_{\mathrm{G}}} b = \min\{a, b\}$$

$$a \vee^{[0,1]_{\mathrm{G}}} b = \max\{a, b\}$$

$$a \to^{[0,1]_{\mathrm{G}}} b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Exercise 1

(a) Prove that $[0,1]_{\mathrm{G}}$ is the unique G-chain with the lattice reduct $\langle [0,1], \min, \max, 0, 1 \rangle$.

Semantical consequence

Definition 2.2

A ***B*-evaluation** is a mapping e from $Fm_{\mathcal{L}}$ to B such that:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi)$

Definition 2.3

A formula φ is a logical consequence of a set of formulas Γ w.r.t. a class $\mathbb{K}$ of G-algebras, $\Gamma \models_{\mathbb{K}} \varphi$, if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

$$\text{if } e(\gamma) = \overline{1} \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) = \overline{1}.$$

Completeness theorem

Theorem 2.4

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{\mathbb{G}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{\mathrm{lin}}} \varphi$
4. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Exercise 1

(a) Prove the implications from top to bottom.

Some theorems and derivations in G

Proposition 2.5

(T1) $\vdash_{\mathrm{G}} \varphi \to \varphi$

(T2) $\vdash_{\mathrm{G}} \varphi \to (\psi \to \varphi \wedge \psi)$

(D1) $\overline{1} \leftrightarrow \varphi \vdash_{\mathrm{G}} \varphi$ *and* $\varphi \vdash_{\mathrm{G}} \overline{1} \leftrightarrow \varphi$

(D2) $\varphi \to \psi \vdash_{\mathrm{G}} \varphi \wedge \psi \leftrightarrow \varphi$ *and* $\varphi \wedge \psi \leftrightarrow \varphi \vdash_{\mathrm{G}} \varphi \to \psi$

(D3) $\varphi \to (\psi \to \chi) \vdash_{\mathrm{G}} \varphi \wedge \psi \to \chi$ *and* $\varphi \wedge \psi \to \chi \vdash_{\mathrm{G}} \varphi \to (\psi \to \chi)$

Proposition 2.6

$\vdash_{\mathrm{G}} \varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$

$\vdash_{\mathrm{G}} \varphi \wedge (\psi \wedge \chi) \leftrightarrow (\varphi \wedge \psi) \wedge \chi$

$\vdash_{\mathrm{G}} \varphi \wedge (\varphi \vee \psi) \leftrightarrow \varphi$

$\vdash_{\mathrm{G}} \overline{1} \wedge \varphi \leftrightarrow \varphi$

$\vdash_{\mathrm{G}} (\varphi \to \psi) \vee (\psi \to \varphi) \leftrightarrow \overline{1}$

$\vdash_{\mathrm{G}} \varphi \vee \psi \leftrightarrow \psi \vee \varphi$

$\vdash_{\mathrm{G}} \varphi \vee (\psi \vee \chi) \leftrightarrow (\varphi \vee \psi) \vee \chi$

$\vdash_{\mathrm{G}} \varphi \vee (\varphi \wedge \psi) \leftrightarrow \varphi$

$\vdash_{\mathrm{G}} \overline{0} \vee \varphi \leftrightarrow \varphi$

The rule of substitution

Proposition 2.7

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\varphi \wedge \chi) \leftrightarrow (\psi \wedge \chi) \qquad \varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\varphi \vee \chi) \leftrightarrow (\psi \vee \chi)$$
$$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\chi \wedge \varphi) \leftrightarrow (\chi \wedge \psi) \qquad \varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\chi \vee \varphi) \leftrightarrow (\chi \vee \psi)$$
$$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\varphi \to \chi) \leftrightarrow (\psi \to \chi) \qquad \varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\chi \to \varphi) \leftrightarrow (\chi \to \psi)$$

$$\vdash_{\mathrm{G}} \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash_{\mathrm{G}} \psi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi, \psi \leftrightarrow \chi \vdash_{\mathrm{G}} \varphi \leftrightarrow \chi$$

Corollary 2.8

$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} \chi \leftrightarrow \chi'$, *where χ' results from χ by replacing its subformula φ by ψ.*

Exercise 2

(a) Prove this corollary and the two previous propositions.

Lindenbaum–Tarski algebra

Definition 2.9

Let Γ be a theory. We define

$$[\varphi]_\Gamma = \{\psi \mid \Gamma \vdash_{\mathrm{G}} \varphi \leftrightarrow \psi\} \qquad L_\Gamma = \{[\varphi]_\Gamma \mid \varphi \in Fm_{\mathcal{L}}\}$$

The Lindenbaum–Tarski algebra of a theory Γ ($\mathbf{Lind}_\Gamma$) as an algebra with the domain L_Γ and operations:

$$\begin{aligned}
\overline{0}^{\mathbf{Lind}_\Gamma} &= [\overline{0}]_\Gamma \\
\overline{1}^{\mathbf{Lind}_\Gamma} &= [\overline{1}]_\Gamma \\
[\varphi]_\Gamma \to^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma &= [\varphi \to \psi]_\Gamma \\
[\varphi]_\Gamma \wedge^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma &= [\varphi \wedge \psi]_\Gamma \\
[\varphi]_\Gamma \vee^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma &= [\varphi \vee \psi]_\Gamma
\end{aligned}$$

Lindenbaum–Tarski algebra: basic properties

Proposition 2.10

1. $[\varphi]_\Gamma = [\psi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi \leftrightarrow \psi$
2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$
3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi$
4. $\mathbf{Lind}_\Gamma$ *is a* G*-algebra*
5. $\mathbf{Lind}_\Gamma$ *is a* G*-chain iff* $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$ *or* $\Gamma \vdash_{\mathrm{G}} \psi \to \varphi$ *for each* φ, ψ

Proof.

1. Left-to-right is the just definition and 'reflexivity' of $\leftrightarrow$. Conversely, we use 'transitivity' and 'symmetry' of $\leftrightarrow$.

2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ iff $[\varphi]_\Gamma \wedge^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma = [\varphi]_\Gamma$ iff $[\varphi \wedge \psi]_\Gamma = [\varphi]_\Gamma$ iff (by 1.) $\Gamma \vdash_{\mathrm{G}} \varphi \wedge \psi \leftrightarrow \varphi$ iff (by (D2)) $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$.

3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ iff (by 1.) $\Gamma \vdash_{\mathrm{G}} \overline{1} \leftrightarrow \varphi$ iff (by (D1)) $\Gamma \vdash_{\mathrm{G}} \varphi$.

5. Trivial after we prove 4.

Lindenbaum–Tarski algebra: basic properties

Proposition 2.10

1. $[\varphi]_\Gamma = [\psi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi \leftrightarrow \psi$
2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$
3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ *iff* $\Gamma \vdash_{\mathrm{G}} \varphi$
4. $\mathbf{Lind}_\Gamma$ *is a* G*-algebra*
5. $\mathbf{Lind}_\Gamma$ *is a* G*-chain iff* $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$ *or* $\Gamma \vdash_{\mathrm{G}} \psi \to \varphi$ *for each* φ, ψ

Proof.

4. First we note that the definition of $\mathbf{Lind}_\Gamma$ is sound due to 1. and Proposition 2.7.

The lattice identities hold due to 1. and Proposition 2.6, prelinearity due to 3. and axiom (Prl).

Finally, the residuation: $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma \to^{\mathbf{Lind}_\Gamma} [\chi]_\Gamma = [\psi \to \chi]_\Gamma$ iff $\Gamma \vdash_{\mathrm{G}} \varphi \to (\psi \to \chi)$ iff (by (D3)) $\Gamma \vdash_{\mathrm{G}} \varphi \wedge \psi \to \chi$ iff $[\varphi]_\Gamma \wedge^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\chi]_\Gamma$. □

General/linear/standard completeness theorem

Theorem 2.4

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{\mathbb{G}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{\mathrm{lin}}} \varphi$
4. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Proof.

2. implies 1.: contrapositively, assume that $\Gamma \nvdash_{\mathrm{G}} \varphi$.

We know that $\mathbf{Lind}_\Gamma \in \mathbb{G}$ and the function e defined as $e(\psi) = [\psi]_\Gamma$

- is a $\mathbf{Lind}_\Gamma$-evaluation and
- $e(\psi) = \bar{1}^{\mathbf{Lind}_\Gamma}$ iff $\Gamma \vdash_{\mathrm{G}} \psi$.

Thus clearly $e(\chi) = \bar{1}^{\mathbf{Lind}_\Gamma}$ for each $\chi \in \Gamma$ and $e(\varphi) \neq \bar{1}^{\mathbf{Lind}_\Gamma}$. □

Deduction Theorem

Theorem 2.11 (Deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$*,*

$$\Gamma, \varphi \vdash_{\mathrm{G}} \psi \text{ iff } \Gamma \vdash_{\mathrm{G}} \varphi \to \psi$$

Proof.

$\Leftarrow$: follows from *modus ponens*

$\Rightarrow$: let $\alpha_1, \dots, \alpha_n = \psi$ be the proof of ψ in Γ, φ. We show by induction that $\Gamma \vdash_{\mathrm{G}} \varphi \to \alpha_i$ for each $i \leq n$.

If $\alpha_i = \varphi$ we use (T1); if α_i is an axiom or $\alpha_i \in \Gamma$ then $\Gamma \vdash_{\mathrm{G}} \alpha_i$ and so we can use axiom (We) and MP.

□

Deduction Theorem

Theorem 2.11 (Deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{\mathrm{G}} \psi \text{ iff } \Gamma \vdash_{\mathrm{G}} \varphi \to \psi$$

Proof.

$\Leftarrow$: follows from *modus ponens*

$\Rightarrow$: let $\alpha_1, \ldots, \alpha_n = \psi$ be the proof of ψ in Γ, φ. We show by induction that $\Gamma \vdash_{\mathrm{G}} \varphi \to \alpha_i$ for each $i \leq n$.

Otherwise there has to be $k, j < i$ such that $\alpha_k = \alpha_j \to \alpha_i$.

Induction assumption gives: $\Gamma \vdash_{\mathrm{G}} \varphi \to \alpha_j$ and $\Gamma \vdash \varphi \to (\alpha_j \to \alpha_i)$.

Using $\Gamma \vdash \varphi \to (\alpha_j \to \alpha_i)$, (Ex), and MP we get $\Gamma \vdash \alpha_j \to (\varphi \to \alpha_i)$,

using this, $\Gamma \vdash_{\mathrm{G}} \varphi \to \alpha_j$, (Tr), and MP twice we get $\Gamma \vdash \varphi \to (\varphi \to \alpha_i)$.

Finally we use (Con) and MP. □

Semilinearity Property

Lemma 2.12 (Semilinearity Property)

If $\Gamma, \varphi \to \psi \vdash_{\mathrm{G}} \chi$ *and* $\Gamma, \psi \to \varphi \vdash_{\mathrm{G}} \chi$, *then* $\Gamma \vdash_{\mathrm{G}} \chi$.

Proof.

By the deduction theorem: $\Gamma \vdash_{\mathrm{G}} (\varphi \to \psi) \to \chi$ and $\Gamma \vdash_{\mathrm{G}} (\psi \to \varphi) \to \chi$.

Thus by ($\vee$c) we get $\Gamma \vdash_{\mathrm{G}} (\varphi \to \psi) \vee (\psi \to \varphi) \to \chi$.

Axiom (Prl) completes the proof. □

Linear Extension Property

Definition 2.13

A theory Γ is linear if $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$ or $\Gamma \vdash_{\mathrm{G}} \psi \to \varphi$ for each φ, ψ.

Lemma 2.14 (Linear Extension Property)

If $\Gamma \nvdash_{\mathrm{G}} \varphi$, then there is a linear theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash_{\mathrm{G}} \varphi$.

Proof.

Enumerate all pairs of formulas: $\langle \varphi_0, \psi_0 \rangle, \langle \psi_1, \varphi_1 \rangle, \ldots$
Construct theories $\Gamma_0, \Gamma_1, \ldots$ such that $\Gamma_0 = \Gamma$; $\Gamma_i \subseteq \Gamma_{i+1}$, and $\Gamma_i \nvdash_{\mathrm{G}} \varphi$:

- *if* $\Gamma_i, \varphi_i \to \psi_i \nvdash_{\mathrm{G}} \varphi$, *then* $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i \to \psi_i\}$
- *if* $\Gamma_i, \varphi_i \to \psi_i \vdash_{\mathrm{G}} \varphi$, *then* $\Gamma_{i+1} = \Gamma_i \cup \{\psi_i \to \varphi_i\}$

Clearly $\Gamma_{i+1} \nvdash_{\mathrm{G}} \varphi$ (the 1st case is obvious; in the 2nd $\Gamma_{i+1} \vdash_{\mathrm{G}} \varphi$ would entail $\Gamma_i \vdash_{\mathrm{G}} \varphi$ by the *Semilinearity Property*, a contradiction with the IH.
Define $\Gamma' = \bigcup \Gamma_i$. Clearly Γ' is linear, $\Gamma' \supseteq \Gamma$, and $\Gamma' \nvdash_{\mathrm{G}} \varphi$. □

General/linear/standard completeness theorem

Theorem 2.4

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{\mathbb{G}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{\mathrm{lin}}} \varphi$
4. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Proof.

3. implies 1.: contrapositively, assume that $\Gamma \nvdash_{\mathrm{G}} \varphi$. Due to the Linear Extension Property there is a linear theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash_{\mathrm{G}} \varphi$.

We know that $\mathbf{Lind}_{\Gamma'} \in \mathbb{G}_{\mathrm{lin}}$ and the function e defined as $e(\psi) = [\psi]_{\Gamma'}$

- is a $\mathbf{Lind}_{\Gamma'}$-evaluation and
- $e(\psi) = \overline{1}^{\mathbf{Lind}_{\Gamma'}}$ iff $\Gamma' \vdash_{\mathrm{G}} \psi$

Thus $e(\chi) = \overline{1}^{\mathbf{Lind}_{\Gamma'}}$ for each $\chi \in \Gamma$ (as $\Gamma' \vdash_{\mathrm{G}} \chi$) and $e(\varphi) \neq \overline{1}^{\mathbf{Lind}_{\Gamma'}}$. □

The proof of the standard completeness theorem

We continue the previous proof: note that the algebra $\mathbf{Lind}_{\Gamma'}$ is countable.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f\colon L_{\Gamma'} \to [0,1]$ such that $f(\overline{0}^{\mathbf{Lind}_{\Gamma'}}) = 0$, $f(\overline{1}^{\mathbf{Lind}_{\Gamma'}}) = 1$, and for each $a, b \in L_{T'}$ we have:

$$a \leq b \quad \text{iff} \quad f(a) \leq f(b)$$

We define a mapping $\bar{e}\colon Fm_{\mathcal{L}} \to [0,1]$ as

$$\bar{e}(\psi) = f(e(\psi))$$

and prove (by induction) that it is an $[0,1]_{\mathrm{G}}$-evaluation.

Then $\bar{e}(\psi) = 1$ iff $e(\psi) = \overline{1}^{\mathbf{Lind}_{\Gamma'}}$ and so $\bar{e}[\Gamma] \subseteq \{1\}$ and $\bar{e}(\varphi) \neq 1$.

Syntax

We consider primitive connectives $\mathcal{L} = \{\to, \wedge, \vee, \overline{0}\}$ and defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$:

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

Formulas are built from a fixed countable set of atoms using the connectives.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulas.

We also use additional connectives $\oplus$ and $\&$ defined as:

$$\varphi \oplus \psi = \neg\varphi \to \psi \qquad \varphi \mathbin{\&} \psi = \neg(\varphi \to \neg\psi)$$

A Hilbert-style proof system

Axioms:

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
(∧a)	$\varphi \wedge \psi \to \varphi$	
(∧b)	$\varphi \wedge \psi \to \psi$	
(∧c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	
(∨a)	$\varphi \to \varphi \vee \psi$	
(∨b)	$\psi \to \varphi \vee \psi$	
(∨c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$	
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom

Inference rule: from φ and $\varphi \to \psi$ infer ψ *modus ponens*

The relation of provability

Proof: a proof of a formula φ from a set of formulas (theory) Γ is a finite sequence of formulas $\langle\psi_1, \ldots, \psi_n\rangle$ such that:

- $\psi_n = \varphi$
- for every $i \leq n$, either $\psi_i \in \Gamma$, or ψ_i is an instance of an axiom, or there are $j, k < i$ such that $\psi_k = \psi_j \to \psi_i$.

We write $\Gamma \vdash_{Ł} \varphi$ if there is a proof of φ from Γ.

A formula φ is a theorem of Łukasiewicz logic if $\vdash_{Ł} \varphi$.

Proposition 2.15

The provability relation of Łukasiewicz logic is finitary: if $\Gamma \vdash_{Ł} \varphi$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{Ł} \varphi$.

Algebraic semantics

An MV-*algebra* is a structure $\boldsymbol{B} = \langle B, \oplus, \neg, \overline{0} \rangle$ such that:

(1) $\langle B, \oplus, \overline{0} \rangle$ is a commutative monoid,

(2) $\neg\neg x = x$,

(3) $x \oplus \neg\overline{0} = \neg\overline{0}$,

(4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

In each MV-algebra we define additional operations:

$x \to y$	is	$\neg x \oplus y$	implication
$x \,\&\, y$	is	$\neg(\neg x \oplus \neg y)$	strong conjunction
$x \vee y$	is	$\neg(\neg x \oplus y) \oplus y$	max-disjunction
$x \wedge y$	is	$\neg(\neg x \vee \neg y)$	min-conjunction
$\overline{1}$	is	$\neg\overline{0}$	top

Exercise 3

Prove that $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice.

Algebraic semantics cont. and standard semantics

We say that an MV-algebra $\boldsymbol{B}$ is linearly ordered (or MV-chain) if its lattice reduct is.

By $\mathbb{MV}$ (or $\mathbb{MV}_{\mathrm{lin}}$ resp.) we denote the class of all MV-algebras (MV-chains resp.)

Take the algebra $[0,1]_Ł = \langle [0,1], \oplus, \neg, 0 \rangle$, with operations defined as:

$$\neg a = 1 - a \qquad\qquad a \oplus b = \min\{1, a+b\}.$$

Proposition 2.16

$[0,1]_Ł$ *is the unique (up to isomorphism)* MV*-chain with the lattice reduct* $\langle [0,1], \min, \max, 0, 1 \rangle$.

Exercise 1

(b) Check that $[0,1]_Ł$ is an MV-chain and find another MV-chain isomorphic to $[0,1]_Ł$ with the same lattice reduct.

Semantical consequence

Definition 2.17

A $\boldsymbol{B}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to B such that:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi) = \neg^{\boldsymbol{B}} e(\varphi) \oplus^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\boldsymbol{B}} e(\psi) = \cdots$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{B}} e(\psi) = \cdots$

Definition 2.18

A formula φ is a logical consequence of a set of formulas Γ w.r.t. a class $\mathbb{K}$ of MV-algebras, $\Gamma \models_{\mathbb{K}} \varphi$, if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

$$\text{if } e(\gamma) = \overline{1} \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) = \overline{1}.$$

General/linear/standard completeness theorem

Theorem 2.19

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{Ł} \varphi$
2. $\Gamma \models_{\mathbb{MV}} \varphi$
3. $\Gamma \models_{\mathbb{MV}_{lin}} \varphi$

If Γ *is finite we can add:*

4. $\Gamma \models_{[0,1]_{Ł}} \varphi$

Exercise 1

(b) Prove the implications from top to bottom.

Some theorems and derivations

Proposition 2.20

(T1) $\vdash_{Ł} \varphi \to \varphi$

(T2) $\vdash_{Ł} \varphi \to (\psi \to \varphi \wedge \psi)$

(T3) $\vdash_{Ł} \varphi \vee \chi \to ((\varphi \to \psi) \vee \chi \to \psi \vee \chi)$

(T4) $\vdash_{Ł} \varphi \vee \varphi \to \varphi$

(T5) $\vdash_{Ł} \varphi \vee \psi \to \psi \vee \varphi$

(D1) $\overline{1} \leftrightarrow \varphi \vdash_{Ł} \varphi$ *and* $\varphi \vdash_{Ł} \overline{1} \leftrightarrow \varphi$

(D2) $\varphi \to \psi \vdash_{Ł} \varphi \wedge \psi \leftrightarrow \varphi$ *and* $\varphi \wedge \psi \leftrightarrow \varphi \vdash_{Ł} \varphi \to \psi$

(D3′) $\varphi \to (\psi \to \chi) \vdash_{G} \varphi \,\&\, \psi \to \chi$ *and* $\varphi \,\&\, \psi \to \chi \vdash_{G} \varphi \to (\psi \to \chi)$

Proposition 2.21

$\vdash_{Ł} \varphi \oplus \psi \leftrightarrow \psi \oplus \varphi$

$\vdash_{Ł} \varphi \oplus (\psi \oplus \chi) \leftrightarrow (\varphi \oplus \psi) \oplus \chi$

$\vdash_{Ł} \overline{0} \oplus \varphi \leftrightarrow \varphi$

$\vdash_{Ł} \neg\neg\varphi \leftrightarrow \varphi$

$\vdash_{Ł} \varphi \oplus \neg\overline{0} \leftrightarrow \neg\overline{0}$

$\vdash_{Ł} \neg(\neg\varphi \oplus \psi) \oplus \psi \leftrightarrow \neg(\neg\psi \oplus \varphi) \oplus \varphi$

The rule of substitution

Proposition 2.22

$$\varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \wedge \chi) \leftrightarrow (\psi \wedge \chi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \vee \chi) \leftrightarrow (\psi \vee \chi)$$
$$\varphi \leftrightarrow \psi \vdash_{Ł} (\chi \wedge \varphi) \leftrightarrow (\chi \wedge \psi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\chi \vee \varphi) \leftrightarrow (\chi \vee \psi)$$
$$\varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \to \chi) \leftrightarrow (\psi \to \chi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\chi \to \varphi) \leftrightarrow (\chi \to \psi)$$

$$\vdash_{Ł} \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash_{Ł} \psi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi, \psi \leftrightarrow \chi \vdash_{Ł} \varphi \leftrightarrow \chi$$

Corollary 2.23

$\varphi \leftrightarrow \psi \vdash_{Ł} \chi \leftrightarrow \chi'$, *where χ' results from χ by replacing its subformula φ by ψ.*

Exercise 2

(b) Prove this corollary and the two previous propositions.

Lindenbaum–Tarski algebra

Definition 2.24

Let Γ be a theory. We define

$$[\varphi]_\Gamma = \{\psi \mid \Gamma \vdash_{Ł} \varphi \leftrightarrow \psi\} \qquad L_\Gamma = \{[\varphi]_\Gamma \mid \varphi \in Fm_{\mathcal{L}}\}$$

The Lindenbaum–Tarski algebra of a theory Γ ($\mathbf{Lind}_\Gamma$) as an algebra with the domain L_Γ and operations:

$$\begin{aligned} \bar{0}^{\mathbf{Lind}_\Gamma} &= [\bar{0}]_\Gamma \\ \neg^{\mathbf{Lind}_\Gamma}[\varphi]_\Gamma &= [\neg\varphi]_\Gamma \\ [\varphi]_\Gamma \oplus^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma &= [\varphi \oplus \psi]_\Gamma \end{aligned}$$

Lindenbaum–Tarski algebra: basic properties

Proposition 2.25

1. $[\varphi]_\Gamma = [\psi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi \leftrightarrow \psi$
2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi \to \psi$
3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi$
4. $\mathbf{Lind}_\Gamma$ *is an* MV*-algebra*
5. $\mathbf{Lind}_\Gamma$ *is an* MV*-chain iff* $\Gamma \vdash_{Ł} \varphi \to \psi$ *or* $\Gamma \vdash_{Ł} \psi \to \varphi$ *for each* φ, ψ

Proof.

1. Left-to-right is the just definition and 'reflexivity' of $\leftrightarrow$. Conversely, we use 'transitivity' and 'symmetry' of $\leftrightarrow$.

2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ iff $[\varphi]_\Gamma \wedge^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma = [\varphi]_\Gamma$ iff $[\varphi \wedge \psi]_\Gamma = [\varphi]_\Gamma$ iff (by 1.) $\Gamma \vdash_{Ł} \varphi \wedge \psi \leftrightarrow \varphi$ iff (by (D2)) $\Gamma \vdash_{Ł} \varphi \to \psi$.

3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ iff (by 2.) $\Gamma \vdash_{Ł} \overline{1} \to \varphi$ iff (by (D1)) $\Gamma \vdash_{Ł} \varphi$.

5. Trivial after we prove 4.

□

Lindenbaum–Tarski algebra: basic properties

Proposition 2.25

1. $[\varphi]_\Gamma = [\psi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi \leftrightarrow \psi$
2. $[\varphi]_\Gamma \leq^{\mathbf{Lind}_\Gamma} [\psi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi \to \psi$
3. $\overline{1}^{\mathbf{Lind}_\Gamma} = [\varphi]_\Gamma$ *iff* $\Gamma \vdash_{Ł} \varphi$
4. $\mathbf{Lind}_\Gamma$ *is an* MV*-algebra*
5. $\mathbf{Lind}_\Gamma$ *is an* MV*-chain iff* $\Gamma \vdash_{Ł} \varphi \to \psi$ *or* $\Gamma \vdash_{Ł} \psi \to \varphi$ *for each* φ, ψ

Proof.

4. First we note that the definition of $\mathbf{Lind}_\Gamma$ is sound due to 1. and Proposition 2.7.

The identities defining MV-algebras hold due to 1. and Proposition 2.21.

□

Łukasiewicz logic vs. Gödel–Dummett

Some things are the same, not only (T1), (T2), (D1), and (D2), but also:

$\varphi \wedge \psi \to \chi \vdash_{Ł} \varphi \to (\psi \to \chi)$ $\qquad$ $\varphi \wedge \psi \to \chi \vdash_{G} \varphi \to (\psi \to \chi)$

$\vdash_{Ł} \varphi \to \neg\neg\varphi$ $\qquad$ $\vdash_{G} \varphi \to \neg\neg\varphi$

$\vdash_{Ł} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ $\qquad$ $\vdash_{G} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

Some are different:

$\varphi \to (\psi \to \chi) \nvdash_{Ł} \varphi \wedge \psi \to \chi$ $\qquad$ $\varphi \to (\psi \to \chi) \vdash_{G} \varphi \wedge \psi \to \chi$

$\vdash_{Ł} \neg\neg\varphi \to \varphi$ $\qquad$ $\nvdash_{G} \neg\neg\varphi \to \varphi$

$\vdash_{Ł} (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$ $\qquad$ $\nvdash_{G} (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

Contrast this with known derivation (D3′):

$$\varphi \to (\psi \to \chi) \vdash_{Ł} \varphi \,\&\, \psi \to \chi \qquad \varphi \,\&\, \psi \to \chi \vdash_{Ł} \varphi \to (\psi \to \chi)$$

Failure of the Deduction Theorem

Assume that we would have that for every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{Ł} \psi \text{ iff } \Gamma \vdash_{Ł} \varphi \to \psi$$

Clearly (MP twice): $\varphi, \varphi \to (\varphi \to \psi) \vdash_{Ł} \psi$.

Thus by the deduction theorem we would get

$$\vdash_{Ł} (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi).$$

This is the axiom of contraction known to fail in Łukasiewicz logic

A possible solution

We can prove that:

$\vdash_{Ł} \varphi \mathbin{\&} \psi \leftrightarrow \psi \mathbin{\&} \varphi \qquad \vdash_{Ł} \varphi \mathbin{\&} \overline{1} \leftrightarrow \varphi \qquad \vdash_{Ł} (\varphi \mathbin{\&} \psi) \mathbin{\&} \chi \leftrightarrow \psi \mathbin{\&} (\varphi \mathbin{\&} \chi)$

Thus it makes sense to define $\varphi^0 = \overline{1}$ and $\varphi^{n+1} = \varphi^n \mathbin{\&} \varphi$

Exercise 4

Let χ be a $\&$-conjunction of n formulas φ with arbitrary bracketing. Prove that $\vdash_{Ł} \chi \leftrightarrow \varphi^n$. Furthermore prove that $\varphi \vdash_{Ł} \varphi^n$.

Local Deduction Theorem

Theorem 2.26 (Local deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$*,*

$$\Gamma, \varphi \vdash_{Ł} \psi \text{ iff there is } n \text{ such that } \Gamma \vdash_{Ł} \varphi^n \to \psi$$

Proof.

$\Leftarrow$: follows from *modus ponens* and the previous exercise
$\Rightarrow$: let $\alpha_1, \ldots, \alpha_n = \psi$ be the proof of ψ in Γ, φ. We show by induction that for each $i \leq n$ there is n_i such that $\Gamma \vdash_{Ł} \varphi^{n_i} \to \alpha_i$

If $\alpha_i = \varphi$ we set $n_i = 1$ and use (T1); if α_i is an axiom or $\alpha_i \in \Gamma$, then $\Gamma \vdash_{Ł} \alpha_i$ and so we can set $n_i = 1$ and use axiom (We) and MP.

□

Local Deduction Theorem

Theorem 2.26 (Local deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$*,*

$$\Gamma, \varphi \vdash_{Ł} \psi \textit{ iff there is } n \textit{ such that } \Gamma \vdash_{Ł} \varphi^n \to \psi$$

Proof.

$\Leftarrow$: follows from *modus ponens* and the previous exercise
$\Rightarrow$: let $\alpha_1, \dots, \alpha_n = \psi$ be the proof of ψ in Γ, φ. We show by induction that for each $i \leq n$ there is n_i such that $\Gamma \vdash_{Ł} \varphi^{n_i} \to \alpha_i$

Otherwise there has to be $k, j < i$ such that $\alpha_k = \alpha_j \to \alpha_i$.

Induction assumption gives: $\Gamma \vdash_{Ł} \varphi^{n_j} \to \alpha_j$ and $\Gamma \vdash \varphi^{n_k} \to (\alpha_j \to \alpha_i)$.

Using $\Gamma \vdash \varphi^{n_k} \to (\alpha_j \to \alpha_i)$, (Ex), and MP we get $\Gamma \vdash \alpha_j \to (\varphi^{n_k} \to \alpha_i)$,

using this, $\Gamma \vdash_{Ł} \varphi^{n_j} \to \alpha_j$, (Tr), and MP we get $\Gamma \vdash \varphi^{n_j} \to (\varphi^{n_k} \to \alpha_i)$.

Finally we use (D3′) and the previous exercise to get $\Gamma \vdash \varphi^{n_j + n_k} \to \alpha_i$. □

Proof by Cases Property

Theorem 2.27 (Proof by Cases Property)

If $\Gamma, \varphi \vdash_{Ł} \chi$ *and* $\Gamma, \psi \vdash_{Ł} \chi$, *then* $\Gamma, \varphi \vee \psi \vdash_{Ł} \chi$.

Proof.

Claim If $\Gamma \vdash_{Ł} \varphi$, then $\Gamma \vee \chi \vdash_{Ł} \delta \vee \chi$ for each formula χ and each δ appearing in the proof of φ from Γ.

Proof of the claim: trivial for $\delta \in \Gamma$ or δ an axiom; if we used MP, then by IH there has to be η st.

$\Gamma \vee \chi \vdash_{Ł} \eta \vee \chi \qquad \Gamma \vee \chi \vdash_{Ł} (\eta \to \delta) \vee \chi$ thus (T3) completes the proof.

Now using the claim: $\Gamma \vee \psi, \varphi \vee \psi \vdash_{Ł} \chi \vee \psi$ and $\Gamma \vee \chi, \psi \vee \chi \vdash_{Ł} \chi \vee \chi$. Using ($\vee$a), (T4), and (T5) we get $\Gamma, \varphi \vee \psi \vdash_{Ł} \psi \vee \chi$ and $\Gamma, \psi \vee \chi \vdash_{Ł} \chi$ and the rest is trivial. □

Semilinearity Property

Lemma 2.28 (Semilinearity Property)

If $\Gamma, \varphi \to \psi \vdash_{\text{Ł}} \chi$ *and* $\Gamma, \psi \to \varphi \vdash_{\text{Ł}} \chi$*, then* $\Gamma \vdash_{\text{Ł}} \chi$*.*

Proof.

By the Proof by Cases Property and axiom (Prl). □

Linear Extensions Property

Definition 2.29

A theory Γ is linear if $\Gamma \vdash_{\L} \varphi \to \psi$ or $\Gamma \vdash_{\L} \psi \to \varphi$ for each φ, ψ.

Lemma 2.30 (Linear Extension Property)

If $\Gamma \nvdash_{\L} \varphi$, then there is a linear theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash_{\L} \varphi$.

Proof.

The same as in the case of Gödel–Dummett logic. □

Linear Extensions Property

Definition 2.29

A theory Γ is linear if $\Gamma \vdash_{Ł} \varphi \to \psi$ or $\Gamma \vdash_{Ł} \psi \to \varphi$ for each φ, ψ.

Lemma 2.30 (Linear Extension Property)

If $\Gamma \nvdash_{Ł} \varphi$, then there is a linear theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash_{Ł} \varphi$.

Proof.

Enumerate all pairs of formulas: $\langle \varphi_0, \psi_0 \rangle, \langle \psi_1, \varphi_1 \rangle, \ldots$
Construct theories $\Gamma_0, \Gamma_1, \ldots$ such that $\Gamma_0 = \Gamma$; $\Gamma_i \subseteq \Gamma_{i+1}$, and $\Gamma_i \nvdash_{Ł} \varphi$:

- *if* $\Gamma_i, \varphi_i \to \psi_i \nvdash_{Ł} \varphi$, *then* $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i \to \psi_i\}$
- *if* $\Gamma_i, \varphi_i \to \psi_i \vdash_{Ł} \varphi$, *then* $\Gamma_{i+1} = \Gamma_i \cup \{\psi_i \to \varphi_i\}$

Clearly $\Gamma_{i+1} \nvdash_{Ł} \varphi$ (the 1st case is obvious; in the 2nd $\Gamma_{i+1} \vdash_{Ł} \varphi$ would entail $\Gamma_i \vdash_{Ł} \varphi$ by the Semilinearity Property, a contradiction with the IH.
Define $\Gamma' = \bigcup \Gamma_i$. Clearly Γ' is linear, $\Gamma' \supseteq \Gamma$, and $\Gamma' \nvdash_{Ł} \varphi$. □

General/linear/standard completeness theorem

Theorem 2.19

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{Ł} \varphi$
2. $\Gamma \models_{\mathbb{MV}} \varphi$
3. $\Gamma \models_{\mathbb{MV}_{lin}} \varphi$

If Γ *is finite we can add:*

4. $\Gamma \models_{[0,1]_{Ł}} \varphi$

The proof of the equivalence of the first three claims is the same as in the case of Gödel–Dummett logic.

We give a proof of 4. implies 1. but first . . .

MV-algebras and LOAGs

A lattice ordered Abelian group (*LOAG* for short) is a structure $\langle G, +, 0, -, \leq \rangle$ such that $\langle G, +, 0, - \rangle$ is an Abelian group and:

(i) $\langle G, \leq \rangle$ *is a lattice,*
(ii) *if* $x \leq y$*, then* $x + z \leq y + z$ *for all* $z \in G$*.*

A strong unit u is an element such that

$$(\forall x \in G)(\exists n \in N)(x \leq nu)$$

For LOAG $\boldsymbol{G} = \langle G, +, 0, -, \leq \rangle$ and strong unit u we define algebra $\Gamma(\boldsymbol{G}, u) = \langle [0, u], \oplus, \neg, \overline{0} \rangle$, where $x \oplus y = \min\{u, x + y\}$, $\neg x = u - x$, $\overline{0} = 0$.

We denote by $\boldsymbol{R}$ the additive LOAG of reals.

Proposition 2.31

$\Gamma(\boldsymbol{G}, u)$ *is an* MV*-algebra and for each* $u > 0$*,* $\Gamma(\boldsymbol{R}, u)$ *is isomorphic to the standard* MV*-algebra* $[0, 1]_{Ł}$*.*

The proof of the standard completeness theorem

If $\Gamma \nvdash_{Ł} \varphi$ we know that there is a countable MV-chain $\boldsymbol{B}$ s.t. $\Gamma \not\models_{\boldsymbol{B}} \varphi$. Let $x_1, \ldots, x_n$ be variables occurring in $\Gamma \cup \{\varphi\}$. Then:

$$\not\models_{\boldsymbol{B}} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma} (\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1})$$

Let us define an algebra $\boldsymbol{B}' = \langle Z \times B, +, -, 0 \rangle$ as:

$$\langle i, x \rangle + \langle j, y \rangle = \begin{cases} \langle i + j, x \oplus y \rangle & \text{if } x \,\&\, y = 0 \\ \langle i + j + 1, x \,\&\, y \rangle & \text{otherwise} \end{cases}$$

$$-\langle i, x \rangle = \langle -i - 1, \neg x \rangle \text{ and } 0 = \langle 0, \overline{0} \rangle$$

Proposition 2.32

$\boldsymbol{B}'$ is a LOAG and $\boldsymbol{B} = \boldsymbol{\Gamma}(\boldsymbol{B}', \langle 1, \overline{0} \rangle)$.

The proof of the standard completeness theorem

Let us fix an extra variable u, we define a translation of MV-terms into LOAG-terms:

$$x' = x \qquad \overline{0}' = 0 \qquad (\neg t)' = u - t' \qquad (t_1 \oplus t_2)' = (t_1' + t_2') \wedge u.$$

Recall that we have:

$$\not\models_{\boldsymbol{B}} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma} (\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1}),$$

Thus also:

$$\not\models_{\boldsymbol{B}'} (\forall u)(\forall x_1, \ldots, x_n)[(0 < u) \wedge \bigwedge_{i \leq n} (x_i \leq u) \wedge (0 \leq x_i) \wedge \bigwedge_{\psi \in \Gamma} (\psi' \approx u) \Rightarrow (\varphi' \approx u)]$$

The proof of the standard completeness theorem

Gurevich–Kokorin theorem: each $\forall_1$-sentence of LOAGs is true in additive LOAG of reals iff it is true in all linearly ordered LOAGs. Thus

$$\not\models_{\boldsymbol{R}} (\forall u)(\forall x_1, \ldots, x_n)[(0 < u) \wedge \bigwedge_{i \leq n}(x_i \leq u) \wedge (0 \leq x_i) \wedge \bigwedge_{\psi \in \Gamma}(\psi' \approx u) \Rightarrow (\varphi' \approx u)]$$

And so

$$\not\models_{\boldsymbol{\Gamma}(\boldsymbol{R},u)} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma}(\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1})$$

And so

$$\not\models_{[0,1]_{Ł}} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma}(\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1})$$

i.e., $\Gamma \not\models_{[0,1]_{Ł}} \varphi$

Failure of standard completeness for infinite theories

Non-theorem

For every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ we have:

$$\Gamma \vdash_{Ł} \varphi \text{ if, and only if, } \Gamma \models_{[0,1]_{Ł}} \varphi.$$

- Consider the theory $\Gamma = \{(p \oplus \overset{n}{\ldots} \oplus p) \to q \mid n \geq 1\} \cup \{\neg p \to q\}$.
- Note that for any $[0,1]_{Ł}$-evaluation e such that $e[\Gamma] = \{1\}$ we have $e(q) = 1$ and so $\Gamma \models_{[0,1]_{Ł}} q$.
- Thus by our *Non-theorem* $\Gamma \vdash_{Ł} q$ and, since proofs are finite, there must be a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{Ł} q$.
- Thus, $\Gamma_0 \models_{[0,1]_{Ł}} q$.
- Let n be the maximal n such that $(p \oplus \overset{n}{\ldots} \oplus p) \to q \in \Gamma_0$.
- The $[0,1]_{Ł}$-evaluation $e(p) = \frac{1}{n+1}$ and $e(q) = \frac{n}{n+1}$ yields a contradiction.

The classical case

Theorem 2.33 (Functional completeness)

Every Boolean function (i.e. any function $f\colon \{0,1\}^n \to \{0,1\}$ *for some* $n \geq 1$*) is representable by some formula of classical logic.*

The fuzzy case

Let L be either Ł of G.

Definition 2.34

A function $f\colon [0,1]^n \to [0,1]$ is *represented* by a formula $\varphi(v_1,\dots,v_n)$ in L if $e(\varphi) = f(e(v_1), e(v_2), \dots, e(v_n))$ for each $[0,1]_{\mathrm{L}}$-evaluation e.

Definition 2.35

The *functional representation* of L is the set $\mathcal{F}_{\mathrm{L}}$ of all functions from any power of $[0,1]$ into $[0,1]$ that are represented in L by some formula.

Relation with Lindenbaum–Tarski algebra

Let us fix $\mathrm{L} = \text{Ł}$.
Let f_i be functions of n_i variables, $i \in \{1, 2\}$. We say that $f_1 = f_2$ iff $f_1(x_1, x_2, \dots, x_{n_1}) = f_2(x_1, x_2, \dots, x_{n_2})$ for every $x_j \in [0, 1]$. Let us for each $f \in \mathcal{F}_{\text{Ł}}$ define a class

$$[f] = \{g \in \mathcal{F}_{\text{Ł}} \mid f = g\} \qquad F = \{[f] \mid f \in \mathcal{F}_{\text{Ł}}\}$$

We define an MV-algebra $\boldsymbol{F}$ with domain F and operations:

$$\overline{0}^{\boldsymbol{F}} = [0] \quad \neg^{\boldsymbol{F}}[f] = [1 - f]_T \quad [f] \oplus^{\boldsymbol{F}} [g] = [\min\{1, f + g\}]$$

Theorem 2.36

The algebras $\boldsymbol{F}$ and $\mathbf{Lind}_{\emptyset}$ are isomorphic.

In the case of G, the definitions and the result are analogous.

A proof

Let the atoms be enumerated as $v_1, v_2, \ldots$. Any formula with variables with maximal index n is viewed as formula in variables $v_1, \ldots, v_n$.
We define the homomorphism:

$$g \colon L_\emptyset \to F \text{ as } g([\varphi]) = [f_\varphi] \text{ where } f_\varphi \text{ is the function represented by } \varphi.$$

Then:

- the definition is sound and g is one-one: $[\varphi] = [\psi]$ iff $\vdash_{Ł} \varphi \leftrightarrow \psi$ iff (due to the standard completeness theorem) $e(\varphi) = e(\psi)$ for each $[0,1]_{Ł}$-evaluation e iff $[f_\varphi] = [f_\psi]$.
- g is a homomorphism:
  $g([\varphi] \oplus [\psi]) = g([\varphi \oplus \psi]) = [f_{\varphi \oplus \psi}] = [f_\varphi \oplus f_\psi] = [f_\varphi] \oplus [f_\psi]$.
- g is onto (obvious).

How do the functions from $\mathcal{F}_{Ł}$ look like?

Observations

- they are all continuous
- they are piece-wise linear
- all pieces have integer coefficients
- if $x_1, \dots, x_n \in \{0,1\}^n$, then $f(x_1, \dots, x_n) \in \{0,1\}$
- if $x_1, \dots, x_n \in ([0,1] \cap \mathrm{Q})^n$, then $f(x_1, \dots, x_n) \in [0,1] \cap \mathrm{Q}$

Definition 2.37

A McNaughton function $f\colon [0,1]^n \to [0,1]$ is a continuous piece-wise linear function, where each of the pieces has integer coefficients.

Theorem 2.38 (McNaughton theorem)

$\mathcal{F}_{Ł}$ is the set of all McNaughton functions.

A lemma

Lemma 2.39

Let $f\colon [0,1]^n \to \mathrm{R}$ *be an* integer linear polynomial*, i.e. of the form*

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i + b \qquad \textit{for some } a_1, \ldots, a_n, b \in \mathrm{Z}$$

Then there is a formula φ_f *representing the function*

$$f^{\#} = \max\{0, \min\{1, f\}\}.$$

Proof.

By induction on $m = \sum_{i=1}^{n} |a_i|$. If $m = 0$ then $f^{\#}$ is either constantly 0 or 1, then we can take as φ either the term $\overline{0}$ or $\overline{1}$, respectively. Assume now $m > 0$ and let a_j be such that $|a_j| = \max_{i=1}^{n} |a_i|$. WLOG we can assume $a_j > 0$: indeed otherwise we consider $f' = 1 - f$, here $a_j > 0$ and so we have φ_{1-f}. Note that clearly $\varphi_f = \neg\varphi_{1-f}$. ...

A lemma: continuation of the proof

Let us consider the function $g = f - x_j$: by IH we have formulas φ_g and φ_{g+1}. If we show that

$$(g + x_j)^\# = (g^\# \oplus x_j) \mathbin{\&} (g+1)^\# \tag{1}$$

the proof is done as:

$$\varphi_f = \varphi_{g+x_j} = (\varphi_g \oplus x_j) \mathbin{\&} \varphi_{g+1}.$$

So we need to prove (2.1). Let L and R be its left/right side :

- if $|g(\vec{x})| > 1$ then $L = R = 1$ or $L = R = 0$
- $0 \leq g(\vec{x}) \leq 1$ then $L = \min\{1, g(\vec{x}) + x_j\}$, $g(\vec{x}) = g^\#(\vec{x})$ and $(g+1)^\#(\vec{x}) = 1$. Hence $R = g(\vec{x}) \oplus x_j = \min\{1, g(\vec{x}) + x_j\} = L$.
- $-1 \leq g(\vec{x}) \leq 0$ then $L = \max\{0, g(\vec{x}) + x_j\}$, $g^\#(\vec{x}) = 0$ and $(g+1)^\#(\vec{x}) = g(\vec{x}) + 1$. Hence $g^\#(\vec{x}) \oplus x_j = x_j$ and so $R = \max\{0, x_j + g(\vec{x}) + 1 - 1\} = \max\{0, x_j + g(\vec{x})\} = L$.

The proof for one variable functions

Definition 2.40

Let $a, b \in [0, 1] \cap \mathbb{Q}$. Then any McNaughton function f such that $f(x) = 1$ iff $x \in [a, b]$ is called *pseudo characteristic function* of interval $[a, b]$.

Exercise 5

Prove that each interval has a pseudo characteristic function and find a formula representing it. Hint: use Lemma 2.39.

Lemma 2.41

Let $a, b \in [0, 1] \cap \mathbb{Q}$. Then for each $\epsilon > 0$ there is a pseudo characteristic function of the interval $[a, b]$, s.t. $f(x) = 0$ for $x \in [0, a - \epsilon] \cup [b + \epsilon, 1]$.

Proof.

If f is a pseudo char. function of some interval, so is f^n for each n. □

The proof for one variable functions

Let p be a McNaughton function of one variable given by n integer linear polynomials $p_1, \ldots, p_n$. For each $i \in \{1, 2, \ldots n\}$ let $P_i = [a_i, b_i]$ be the interval in which p uses p_i. Note that:

- $[0, 1] = \bigcup\limits_i P_i$
- $a_i, b_i \in [0, 1] \cap \mathrm{Q}$
- there is a pseudo characteristic function f_i of $[a_i, b_i]$ such that $p(x) \geq (f_i \,\&\, p_i^{\#})(x)$ for each $x \notin P_i$.

Then

$$p(x) = \bigvee_i (f_i \,\&\, p_i^{\#})(x) \text{ and thus } \varphi_p = \bigvee_i \varphi_{f_i} \,\&\, \varphi_{p_i}.$$

The classical case, FMP and decidability

CL is complete with respect to a finite algebra, $\mathbf{2}$.

Definition 2.42

A logic has the finite model property (FMP) if it is complete with respect to a set of finite algebras.

From the FMP, we obtain decidability:

- Thanks to our finite notion of proof, the set of theorems is recursively enumerable.
- Thanks to FMP, the set of non-theorems is also recursively enumerable (we can check validity in bigger and bigger finite algebras until we find a countermodel).
- Therefore, theoremhood is a decidable problem.
- Note: provability from finitely-many premises is also decidable (using deduction theorem).

Finite chains

Lemma 2.43

Let $\boldsymbol{B}$ be a subalgebra of an MV*- or* G*-algebra* $\boldsymbol{A}$*. Then* $\models_{\boldsymbol{A}} \subseteq \models_{\boldsymbol{B}}$*.*

Exercise 6

(a) Prove that each n-valued G-chain is isomorphic to the subalgebra $\boldsymbol{G}_n$ of $[0,1]_{\mathrm{G}}$ with the domain $\{\frac{i}{n-1} \mid i \leq n-1\}$.

(b) Prove that each n-valued MV-chain is isomorphic to the subalgebra $\boldsymbol{Ł}_n$ of $[0,1]_{\mathrm{Ł}}$ with the domain $\{\frac{i}{n-1} \mid i \leq n-1\}$.

Lemma 2.44

$$\models_{\boldsymbol{G}_m} \subseteq \models_{\boldsymbol{G}_n} \quad \textit{iff} \quad n \leq m.$$

$$\models_{\boldsymbol{Ł}_m} \subseteq \models_{\boldsymbol{Ł}_n} \quad \textit{iff} \quad n-1 \textit{ divides } m-1.$$

Let us denote by $\mathbb{L}_{\mathrm{fin}}$ the class of finite L-chains.

The case of Gödel–Dummett logic

Theorem 2.45

Let φ *be a formula with* $n-2$ *variables. Then:* $\vdash_{\mathrm{G}} \varphi$ *iff* $\models_{\boldsymbol{G}_n} \varphi$.

Proof.

Contrapositively: assume that $\nvdash_{\mathrm{G}} \varphi$ and let e be a $[0,1]_{\mathrm{G}}$-evaluation such that $e(\varphi) \neq 1$. Let $X = \{0,1\} \cup \{e(v_i) \mid 1 \leq i \leq n-2\}$ and note that it is a subuniverse of $[0,1]_{\mathrm{G}}$, thus e can be seen as an $\boldsymbol{X}$-evaluation and so $\not\models_{\boldsymbol{X}} \varphi$. The previous exercise and lemma complete the proof. □

Theorem 2.46

For every *finite* *set of formulas* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*. The following are equivalent:*

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{\mathrm{fin}}} \varphi$

The case of Łukasiewicz logic

Theorem 2.47

For every finite *set of formulas* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, *TFAE:*

1. $\Gamma \vdash_{Ł} \varphi$
2. $\Gamma \models_{[0,1]_{Ł}} \varphi$
3. $\Gamma \models_{\mathbb{MV}_{\text{fin}}} \varphi$

Proof: we show it for one variable v.

Let us define the set E of $[0,1]_{Ł}$-evaluations such that $e[\Gamma] \subseteq \{1\}$. Note that E can be seen as a union of real intervals. Assume that there is $e \in E$ such that $e(\varphi) \neq 1$. If we show that there is an evaluation $f \in E$, such that $f(v) = \frac{p}{n-1}$ and $f(\varphi) \neq 1$ we are done as f can be seen as $\boldsymbol{Ł}_n$-evaluation.

- Either e lies on the border of some interval, then $f = e$ OR
- there has to be a neighborhood $X \subseteq E$ such that $f(\varphi) \neq \overline{1}$ for each $f \in X$, then there has to be such f. □

The classical case

- $\varphi \in \mathrm{SAT}(\mathrm{CL})$ if there is a $\mathbf{2}$-evaluation e such that $e(\varphi) = 1$.
- $\varphi \in \mathrm{TAUT}(\mathrm{CL})$ if for each $\mathbf{2}$-evaluation e holds $e(\varphi) = 1$.

Recall:

$$\begin{array}{lll} \varphi \in \mathrm{TAUT}(\mathrm{CL}) & \text{iff} & \neg\varphi \notin \mathrm{SAT}(\mathrm{CL}) \\ \varphi \in \mathrm{SAT}(\mathrm{CL}) & \text{iff} & \neg\varphi \notin \mathrm{TAUT}(\mathrm{CL}). \end{array}$$

Both problems, $\mathrm{SAT}(\mathrm{CL})$ and $\mathrm{TAUT}(\mathrm{CL})$, are decidable.

But how difficult are their computations?

Complexity classes

$f, g \colon \mathrm{N} \to \mathrm{N}$. $f \in O(g)$ iff there are $c, n_0 \in \mathrm{N}$ such that for each $n \geq n_0$ we have $f(n) \leq c\, g(n)$.

- **TIME**(f): the class of problems P such that there is a deterministic Turing machine M that accepts P and operates in time $O(f)$.
- **NTIME**(f): analogous class for nondeterministic Turing machines.
- **SPACE**(f): the class of problems P such that there is a deterministic Turing machine M that accepts P and operates in space $O(f)$.
- **NSPACE**(f): the analogous class for nondeterministic Turing machines.

Complexity classes

$$\mathbf{P} = \bigcup_{k \in \mathrm{N}} \mathbf{TIME}(n^k)$$

$$\mathbf{NP} = \bigcup_{k \in \mathrm{N}} \mathbf{NTIME}(n^k)$$

$$\mathbf{PSPACE} = \bigcup_{k \in \mathrm{N}} \mathbf{SPACE}(n^k)$$

If $\mathbf{C}$ is a complexity class, we denote $\mathbf{coC} = \{P \mid \overline{P} \in \mathbf{C}\}$, the class of complements of problems in $\mathbf{C}$.

Complexity classes

- Each deterministic complexity class $\mathbf{C}$ is closed under complementation: if $P \in \mathbf{C}$, then also $\overline{P} \in \mathbf{C}$.
- Is $\mathbf{NP}$ closed under complementation?
- $\mathbf{P} \subseteq \mathbf{NP}$, $\mathbf{P} \subseteq \mathbf{coNP}$, $\mathbf{NP} \subseteq \mathbf{PSPACE}$.
- Are the inclusions $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE}$ proper?
- Each of the classes $\mathbf{P}$, $\mathbf{NP}$, $\mathbf{coNP}$, and $\mathbf{PSPACE}$ is closed under finite unions and intersections.

Complexity classes

A problem P is said to be $\mathbf{C}$-hard iff any decision problem P' in $\mathbf{C}$ is reducible to P.

A problem P is $\mathbf{C}$-complete iff P is $\mathbf{C}$-hard and $P \in \mathbf{C}$.

The classical case

- $\mathrm{SAT}(\mathrm{CL}) \in \mathbf{NP}$: guess an evaluation and check whether it satisfies the formula (a polynomial matter).
- $\mathrm{TAUT}(\mathrm{CL}) \in \mathbf{coNP}$: $\varphi \in \mathrm{TAUT}(\mathrm{CL})$ iff $\neg\varphi \notin \mathrm{SAT}(\mathrm{CL})$.
- Cook Theorem: Let $\mathrm{SAT}^{\mathrm{CNF}}(\mathrm{CL})$ be the SAT problem for formulas in conjunctive normal form. Then: $\mathrm{SAT}^{\mathrm{CNF}}(\mathrm{CL})$ is $\mathbf{NP}$-complete.
- $\mathrm{SAT}^{\mathrm{CNF}}(\mathrm{CL})$ is a fragment of $\mathrm{SAT}(\mathrm{CL})$, therefore $\mathrm{SAT}(\mathrm{CL})$ is $\mathbf{NP}$-complete and $\mathrm{TAUT}(\mathrm{CL})$ is $\mathbf{coNP}$-complete.

The fuzzy case: basic definitions

Let L be either Łukasiewicz logic Ł or Gödel logic G. We define:

- $\varphi \in \mathrm{SAT}(\mathrm{L})$ if there is an evaluation e such that $e(\varphi) = 1$.
- $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{L})$ if there is an evaluation e such that $e(\varphi) > 0$.
- $\varphi \in \mathrm{TAUT}(\mathrm{L})$ if for each evaluation e holds $e(\varphi) = 1$.
- $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L})$ if for each evaluation e holds $e(\varphi) > 0$.

Note that $\varphi \vee \neg\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L})$ but $\varphi \vee \neg\varphi \notin \mathrm{TAUT}(\mathrm{L})$

Note that $\varphi \wedge \neg\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\text{Ł})$ but $\varphi \wedge \neg\varphi \notin \mathrm{SAT}(\text{Ł})$

The fuzzy case: basic reductions

Lemma 2.48

Let L *be either Łukasiewicz logic* Ł *or Gödel logic* G. *Then*

$$\varphi \in \mathrm{TAUT_{pos}}(\mathrm{L}) \quad \textit{iff} \quad \neg\varphi \notin \mathrm{SAT}(\mathrm{L})$$

$$\varphi \in \mathrm{SAT_{pos}}(\mathrm{L}) \quad \textit{iff} \quad \neg\varphi \notin \mathrm{TAUT}(\mathrm{L}).$$

Lemma 2.49

$$\varphi \in \mathrm{SAT}(\text{Ł}) \quad \textit{iff} \quad \neg\varphi \notin \mathrm{TAUT_{pos}}(\text{Ł})$$

$$\varphi \in \mathrm{TAUT}(\text{Ł}) \quad \textit{iff} \quad \neg\varphi \notin \mathrm{SAT_{pos}}(\text{Ł}).$$

Exercise 7

Prove the above two lemmata, show that the last equivalence fails for G and the one but last holds there. (Hint: for the last part use properties of these sets proved in the next few slides).

The case of Łukasiewicz logic

Theorem 2.50

The sets $\mathrm{SAT}(Ł)$ *and* $\mathrm{SAT}_{\mathrm{pos}}(Ł)$ *are* **NP**-*complete. Therefore the sets* $\mathrm{TAUT}(Ł)$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(Ł)$ *are* **coNP**-*complete.*

We prove it in a series of lemmata. First we show that $\mathrm{SAT}(Ł)$ is **NP**-hard:

Lemma 2.51

Let φ *be a formula with variables* $p_1, \dots p_n$.

$$\varphi \in \mathrm{SAT}(\mathrm{CL}) \quad \textit{IFF} \quad \varphi \wedge \bigwedge_{i=1}^{n} (p_i \vee \neg p_i) \in \mathrm{SAT}(Ł).$$

$\mathrm{SAT_{pos}}(Ł)$ is **NP**-hard

Lemma 2.52

Let φ be a formula with variables $p_1, \dots p_n$ built using: $\wedge, \vee, \neg$.

$$\varphi \in \mathrm{SAT(CL)} \quad \textit{IFF} \quad \varphi^2 \wedge \bigwedge_{i=1}^{n} (p_i \vee \neg p_i)^2 \in \mathrm{SAT_{pos}}(Ł).$$

Proof.

Let e positively satisfy the right-hand formula. Then $e((p_i \vee \neg p_i)^2) > 0$ ergo $e(p_i) \neq 0.5$. We define the evaluation

$$e'(p_i) = \begin{cases} 1 & \text{if } e(p_i) > 0.5 \\ 0 & \text{if } e(p_i) < 0.5 \end{cases}$$

Clearly this can be extended to φ. And, since $e(\varphi^2) > 0$, we have $e(\varphi) > 0.5$ and so $e'(\varphi) = 1$. □

$\mathrm{SAT}(\text{Ł})$ and $\mathrm{SAT}_{\mathrm{pos}}(\text{Ł})$ are in $\mathbf{NP}$

Lemma 2.53

$$e(\varphi \to \psi) \geq r \quad \textit{IFF} \quad \exists i,j \in [0,1] \quad \begin{array}{rcl} e(\varphi) & \leq & i \\ e(\psi) & \geq & j \\ r+i-j & \leq & 1 \end{array}$$

$$e(\varphi \to \psi) \leq r \quad \textit{IFF} \quad \exists i,j \in [0,1], y \in \{0,1\} \quad \begin{array}{rcl} e(\varphi) & \geq & i \\ e(\psi) & \leq & j \\ y-r & \leq & 0 \\ y+i & \leq & 1 \\ y-j & \leq & 0 \\ y+r+i-j & \geq & 1 \end{array}$$

Using this lemma we can reduce the question of (positive) satisfiability to the question of Mixed Integer Programming (MIP) which is known to be in **NP**:

For $\mathrm{SAT}(\text{Ł})$ start with $e(\varphi) \geq 1$ for $\mathrm{SAT}_{\mathrm{pos}}(\text{Ł})$ start with $\begin{array}{rcl} e(\varphi) & \geq & i_0 \\ i_0 & > & 0 \end{array}$

The case of Gödel–Dummett logic

Lemma 2.54

The mapping $f\colon [0,1] \to \{0,1\}$ *defined as* $f(0) = 0$ *and* $f(x) = 1$ *if* $x \neq 0$ *is a homomorphism from* $[0,1]_{\mathrm{G}}$ *to* $\mathbf{2}$*.*

Corollary 2.55

$$\mathrm{SAT}_{\mathrm{pos}}(\mathrm{G}) \subseteq \mathrm{SAT}(\mathrm{CL}) \qquad \mathrm{TAUT}(\mathrm{CL}) \subseteq \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G}).$$

The case of Gödel–Dummett logic

Corollary 2.56

$$\begin{array}{lllll} \varphi \in \mathrm{SAT_{pos}}(\mathrm{G}) & \textit{iff} & \varphi \in \mathrm{SAT}(\mathrm{G}) & \textit{iff} & \varphi \in \mathrm{SAT}(\mathrm{CL}) \\ \varphi \in \mathrm{TAUT_{pos}}(\mathrm{G}) & \textit{iff} & \neg\neg\varphi \in \mathrm{TAUT}(\mathrm{G}) & \textit{iff} & \varphi \in \mathrm{TAUT}(\mathrm{CL}) \end{array}$$

Proof.

Just observe that:

$$\mathrm{SAT}(\mathrm{G}) \subseteq \mathrm{SAT_{pos}}(\mathrm{G}) \subseteq \mathrm{SAT}(\mathrm{CL}) \subseteq \mathrm{SAT}(\mathrm{G}).$$

And that

$$\varphi \in \mathrm{TAUT_{pos}}(\mathrm{G}) \Rightarrow \neg\varphi \notin \mathrm{SAT}(\mathrm{G}) \Rightarrow \neg\varphi \notin \mathrm{SAT_{pos}}(\mathrm{G})$$

$$\Rightarrow \neg\neg\varphi \in \mathrm{TAUT}(\mathrm{G}) \Rightarrow \varphi \in \mathrm{TAUT}(\mathrm{CL}) \Rightarrow \varphi \in \mathrm{TAUT_{pos}}(\mathrm{G}).$$

□

The case of Gödel–Dummett logic

Corollary 2.56

$\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{G})$ *iff* $\varphi \in \mathrm{SAT}(\mathrm{G})$ *iff* $\varphi \in \mathrm{SAT}(\mathrm{CL})$
$\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G})$ *iff* $\neg\neg\varphi \in \mathrm{TAUT}(\mathrm{G})$ *iff* $\varphi \in \mathrm{TAUT}(\mathrm{CL})$

Theorem 2.57

The sets $\mathrm{SAT}(\mathrm{G})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathrm{G})$ *are* $\mathbf{NP}$*-complete and the sets* $\mathrm{TAUT}(\mathrm{G})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G})$ *are* $\mathbf{coNP}$*-complete.*

Proof.

The only non clear case is $\mathrm{TAUT}(\mathrm{G})$: it is co$\mathbf{NP}$-hard due to the last reduction of the previous corollary. We present a non-deterministic polynomial 'algorithm' (sound due to Theorem 2.58) for $Fm_{\mathcal{L}} \setminus \mathrm{TAUT}(\mathrm{G})$:

`Step 1:` guess a $\boldsymbol{G}_n$-evaluation e (assuming that φ has $n-2$ variables)
`Step 2:` compute the value of $e(\varphi)$ (clearly in polynomial time)
`Output:` if $e(\varphi) \neq 1$ output $\varphi \notin \mathrm{TAUT}(\mathrm{G})$. □

Equational consequence

An equation in the language $\mathcal{L}$ is a formal expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in Fm_{\mathcal{L}}$.

We say that an equation $\varphi \approx \psi$ is a consequence of a set of equations Π w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-algebras if for each $\boldsymbol{A} \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e we have $e(\varphi) = e(\psi)$ whenever $e(\alpha) = e(\beta)$ for each $\alpha \approx \beta \in \Pi$; we denote it by $\Pi \models_{\mathbb{K}} \varphi \approx \psi$.

A quasiequation in the language $\mathcal{L}$ is a formal expression of the form $(\bigwedge_{i=1}^{n} \varphi_i \approx \psi_i) \Rightarrow \varphi \approx \psi$, where $\varphi_1, \ldots, \varphi_n, \varphi, \psi_1, \ldots, \psi_n, \psi \in Fm_{\mathcal{L}}$.

Varieties and quasivarieties

Type of class	Presented by	Closed under
variety	equations	$\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$
quasivariety	quasiequations	$\mathbf{I}$, $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{P}_{\mathrm{U}}$

$\mathbf{I}$	isomorphic images
$\mathbf{H}$	homomorphic images
$\mathbf{S}$	subalgebras
$\mathbf{P}$	direct products
$\mathbf{P}_{\mathrm{U}}$	ultraproducts
$\mathbf{V}$	generated variety
$\mathbf{Q}$	generated quasivariety

Algebraization of Łukasiewicz logic

1. For every $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$,
   $$\Gamma \vdash_{Ł} \varphi \text{ iff } \{\psi \approx \overline{1} \mid \psi \in \Gamma\} \models_{\mathbb{MV}} \varphi \approx \overline{1}$$
2. For every set of equations $\Pi \cup \{\varphi \approx \psi\}$,
   $$\Pi \models_{\mathbb{MV}} \varphi \approx \psi \text{ iff } \{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{Ł} \varphi \leftrightarrow \psi$$
3. For every $\varphi \in \boldsymbol{Fm}_{\mathcal{L}}$,
   $$\varphi \vdash_{Ł} \varphi \leftrightarrow \overline{1} \text{ and } \varphi \leftrightarrow \overline{1} \vdash_{Ł} \varphi$$
4. For every $\varphi, \psi \in \boldsymbol{Fm}_{\mathcal{L}}$,
   $$\varphi \approx \psi \models_{\mathbb{MV}} \varphi \leftrightarrow \psi \approx \overline{1} \text{ and } \varphi \leftrightarrow \psi \approx \overline{1} \models_{\mathbb{MV}} \varphi \approx \psi$$

Translations:

- $\tau : \varphi \mapsto \varphi \approx \overline{1}$
- $\rho : \alpha \approx \beta \mapsto \alpha \leftrightarrow \beta$

MV-algebras are the equivalent algebraic semantics of Ł.

$\mathbb{MV}$ is a variety

$\mathbb{MV}$ is a variety of algebras, i.e. an equational class:

$$
\begin{array}{ll}
(1) & x \oplus (y \oplus z) \approx (x \oplus y) \oplus z, \\
(2) & x \oplus y \approx y \oplus x, \\
(3) & x \oplus \overline{0} \approx x, \\
(4) & \neg\neg x \approx x, \\
(5) & x \oplus \neg\overline{0} \approx \neg\overline{0}, \\
(6) & \neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x.
\end{array}
$$

Algebraization of Gödel–Dummett logic

1. For every $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$,
$$\Gamma \vdash_{\mathrm{G}} \varphi \text{ iff } \{\psi \approx \overline{1} \mid \psi \in \Gamma\} \models_{\mathbb{G}} \varphi \approx \overline{1}$$
2. For every set of equations $\Pi \cup \{\varphi \approx \psi\}$,
$$\Pi \models_{\mathbb{G}} \varphi \approx \psi \text{ iff } \{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{\mathrm{G}} \varphi \leftrightarrow \psi$$
3. For every $\varphi \in \boldsymbol{Fm}_{\mathcal{L}}$,
$$\varphi \vdash_{\mathrm{G}} \varphi \leftrightarrow \overline{1} \text{ and } \varphi \leftrightarrow \overline{1} \vdash_{\mathrm{G}} \varphi$$
4. For every $\varphi, \psi \in \boldsymbol{Fm}_{\mathcal{L}}$,
$$\varphi \approx \psi \models_{\mathbb{G}} \varphi \leftrightarrow \psi \approx \overline{1} \text{ and } \varphi \leftrightarrow \psi \approx \overline{1} \models_{\mathbb{G}} \varphi \approx \psi$$

Translations:

- $\tau : \varphi \mapsto \varphi \approx \overline{1}$
- $\rho : \alpha \approx \beta \mapsto \alpha \leftrightarrow \beta$

G-algebras are the equivalent algebraic semantics of G.

$\mathbb{G}$ is a variety

$\mathbb{G}$ is a variety of algebras, i.e. an equational class:

- E1 $x \to x \approx \overline{1}$
- E2 $\overline{1} \to x \approx x$
- E3 $x \to (y \to z) \approx (x \to y) \to (x \to z)$
- E4 $(x \to y) \to ((y \to x) \to y) \approx (y \to x) \to ((x \to y) \to x)$
- E5 $x \to x \vee y \approx \overline{1}, \quad y \to x \vee y \approx \overline{1}$
- E6 $(x \to y) \to ((y \to z) \to (x \vee y \to z)) \approx \overline{1}$
- E7 $x \wedge y \to x \approx \overline{1}, \quad x \wedge y \to y \approx \overline{1}$
- E8 $(x \to y) \to ((x \to z) \to (x \to y \wedge z)) \approx \overline{1}$
- E9 $\overline{0} \to x \approx \overline{1}$
- E10 $(x \to y) \vee (y \to x) \approx \overline{1}$

Algebraization of finitary extensions

Let L be Ł or G.

- $\mathrm{S} = \mathrm{L} + Ax + R$ (Ax is a set of axioms and R a set of finitary rules)
- $\mathbb{S} = \{\boldsymbol{A} \in \mathbb{L} \mid \boldsymbol{A}$ satisfies $\tau(\varphi)$ for each $\varphi \in Ax$ and $\bigwedge_{i=1}^{n} \tau(\varphi_i) \Rightarrow \tau(\psi)$ for each $\langle \varphi_1, \ldots, \varphi_n, \psi \rangle \in R\}$.
- We obtain the same relation between the logic and the algebraic semantics as before:
  1. $\Gamma \vdash_{\mathrm{S}} \varphi$ iff $\tau[\Gamma] \models_{\mathbb{S}} \tau(\varphi)$
  2. $\Pi \models_{\mathbb{S}} \varphi \approx \psi$ iff $\rho[\Pi] \vdash_{\mathrm{S}} \rho(\varphi \approx \psi)$
  3. $\varphi \vdash_{\mathrm{S}} \rho(\tau(\varphi))$ and $\rho(\tau(\varphi)) \vdash_{\mathrm{S}} \varphi$
  4. $\varphi \approx \psi \models_{\mathbb{S}} \tau(\rho(\varphi \approx \psi))$ and $\tau(\rho(\varphi \approx \psi)) \models_{\mathbb{S}} \varphi \approx \psi$

$\mathbb{S}$ is the equivalent algebraic semantics of S.

Algebraization of finitary extensions

The translations τ and ρ between formulas and equations give bijective correspondences (dual lattice isomorphisms):

1. between finitary extensions of L and quasivarieties of L-algebras
2. between axiomatic extensions of L and varieties of L-algebras.

Proof by Cases Property for extensions

Theorem 2.58 (Proof by Cases Property)

Assume that for each $\langle\varphi_1, \ldots, \varphi_n, \psi\rangle \in R$*,* $\varphi_1 \vee \chi, \ldots \varphi_n \vee \chi \vdash_S \psi \vee \chi$*. If* $\Gamma, \varphi \vdash_S \chi$ *and* $\Gamma, \psi \vdash_S \chi$*, then* $\Gamma, \varphi \vee \psi \vdash_S \chi$*.*

Proof.

Claim If $\Gamma \vdash_S \varphi$, then $\Gamma \vee \chi \vdash_S \delta \vee \chi$ for each formula χ and each δ appearing in the proof of φ from Γ.

Proof of the claim: trivial for $\delta \in \Gamma$ or δ an axiom; if we used MP, then by IH there has to be η st.

$\Gamma \vee \chi \vdash_S \eta \vee \chi \qquad \Gamma \vee \chi \vdash_S (\eta \to \delta) \vee \chi$ thus (T7) completes the proof.

Now using the claim: $\Gamma \vee \psi, \varphi \vee \psi \vdash_S \chi \vee \psi$ and $\Gamma \vee \chi, \psi \vee \chi \vdash_S \chi \vee \chi$. Using (A6a), (T8), and (T9) we get $\Gamma, \varphi \vee \psi \vdash_S \psi \vee \chi$ and $\Gamma, \psi \vee \chi \vdash_S \chi$ and the rest is trivial. □

Chain-completeness for extensions

Corollary 2.59

Assume that for each $\langle\varphi_1, \ldots, \varphi_n, \psi\rangle \in \boldsymbol{R}$, $\varphi_1 \vee \chi, \ldots \varphi_n \vee \chi \vdash_{\mathrm{S}} \psi \vee \chi$ *(this is the case, in particular, if* S *is an axiomatic extension). Then for every set of formulas* $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$*:* $\Gamma \vdash_{\mathrm{S}} \varphi$ *iff* $\Gamma \models_{\mathbb{S}_{\mathrm{lin}}} \varphi$*.*

Exercise 8

Prove it.

The case of Gödel–Dummett logic

For each $n \geq 1$, recall the canonical n-valued G-chain:

$\boldsymbol{G}_n = \langle\{\frac{i}{n-1} \mid i \leq n-1\}, \min, \max, \rightarrow, 0, 1\rangle$.

$\mathrm{G}_n = \mathrm{G} + \bigvee_{i=0}^{n-1}(p_i \rightarrow p_{i+1})$.

Theorem 2.60

- *for each $n \geq 1$, G_n-algebras are the subvariety of G-algebras satisfying $\bigvee_{i=0}^{n-1}(p_i \rightarrow p_{i+1}) \approx \overline{1}$ and it coincides with $\mathbf{V}(\boldsymbol{G}_n)$.*
- *$\mathbb{G}$ is locally finite, i.e. each finite subset of a G-algebra generates a finite subalgebra.*
- *If $\boldsymbol{C}$ is an infinite G-chain, then $\mathbf{V}(\boldsymbol{C}) = \mathbb{G}$.*
- *the subvarieties of $\mathbb{G}$ are exactly:* $\mathbf{V}(\boldsymbol{G}_1) \subsetneq \mathbf{V}(\boldsymbol{G}_2) \subsetneq \mathbf{V}(\boldsymbol{G}_3) \subsetneq \ldots \subsetneq \mathbf{V}(\boldsymbol{G}_n) \subsetneq \mathbf{V}(\boldsymbol{G}_{n+1}) \subsetneq \ldots \mathbb{G}$.

Exercise 9

Prove it.

The case of Gödel–Dummett logic

Theorem 2.61

There are no other finitary extensions of G *than* G_n*s (i.e.* $\mathbb{G}$ *has no proper subquasivarieties).*

Lemma 2.62

Gödel–Dummett logic proves:

- $(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \to \psi) \to (\varphi \to \chi))$
- $(\varphi \to (\psi \wedge \chi)) \leftrightarrow ((\varphi \to \psi) \wedge (\varphi \to \chi))$
- $(\varphi \to (\psi \vee \chi)) \leftrightarrow ((\varphi \to \psi) \vee (\varphi \to \chi))$

Define a substitution $\sigma_\varphi(p) = \varphi \to p$*. Then if* $\overline{0}$ *does not occur in* φ *we have:* $\vdash_{\mathrm{G}} \sigma_\varphi(\psi) \leftrightarrow (\varphi \to \psi)$, $\psi \vdash_{\mathrm{G}} \sigma_\varphi(\psi)$, *and* $\vdash_{\mathrm{G}} \sigma_\varphi(\varphi)$.

Deduction theorems

Lemma 2.63

Any finitary extension L *of* G *enjoys the deduction theorem.*

Proof.

Assume that $\varphi \vdash_{\mathrm{L}} \psi$. Let χ_f be the formula resulting from χ by replacing all occurrences of $\overline{0}$ by a fresh fixed variable f. Define a substitution $\sigma(q) = \overline{0}$ for $q = f$ and q otherwise; observe $\sigma(\chi_f) = \chi$.

Claim: $\{f \to q \mid q \text{ in } \{\varphi, \psi\}\}, \varphi_f \vdash_{\mathrm{L}} \psi_f$.

Thus $\sigma\sigma_{\varphi_f}[\{f \to q \mid q \text{ in } \{\varphi, \psi\}\} \cup \{\varphi_f\}] \vdash_{\mathrm{L}} \sigma\sigma_{\varphi_f}(\psi_f)$. And so $\{(\varphi \to \overline{0}) \to (\varphi \to q) \mid q \text{ in } \{\varphi, \psi\}\}, \sigma\sigma_{\varphi_f}(\varphi) \vdash_{\mathrm{L}} \sigma\sigma_{\varphi_f}(\psi)$. Since, clearly, $\vdash_{\mathrm{L}} \sigma\sigma_{\varphi_f}(\chi_f) \leftrightarrow (\varphi \to \chi)$, we obtain $\vdash_{\mathrm{L}} \varphi \to \psi$. □

Exercise 10

Complete the proof (including the claim!).

Structural completeness

The proof of Theorem 2.88.

Obvious as the previous lemma allows us to replace any additional rule of L by an axiom. □

Definition 2.64

A logic is structurally complete if each proper extension has some new theorems. A logic is hereditarily structurally complete if each of its extensions is structurally complete.

Corollary 2.65

G *is hereditarily structurally complete.*

Exercise 11

Ł is not structurally complete. (hint: use the rule $\varphi \leftrightarrow \neg\varphi \vdash \overline{0}$)

Important MV-chains

Recall the functor Γ which turns each Lattice ordered Abelian group with strong unit into and MV-algebra

For each $n \geq 1$, recall the canonical n-valued MV-chain:
$\boldsymbol{Ł}_n = \langle\{\frac{i}{n-1} \mid i \leq n-1\}, \oplus, \neg, 0\rangle$.

- for each $u > 0$, $[0,1]_{Ł} \cong \Gamma(\boldsymbol{R}, u)$.
- $\boldsymbol{Ł}_n \cong \Gamma(\boldsymbol{Q}_{n-1}, 1)$
- $\boldsymbol{K}_n = \Gamma(\boldsymbol{Q}_{n-1} \otimes \boldsymbol{Z}, \langle 1, 0\rangle)$.

where on $\boldsymbol{Q}_{n-1}$ is the additive group of rationals whose denominator is $n-1$, and $\boldsymbol{Q}_{n-1} \otimes \boldsymbol{Z}$ is the lexicographic product (direct product with the lexicographic order).

Varieties of MV-algebras

Proposition 2.66

- $\mathbf{V}([0,1]_{Ł}) = \mathbb{MV}$
- *If* $I \subseteq \mathrm{N}$ *is infinite, then* $\mathbf{V}(\{\boldsymbol{Ł}_i \mid i \in I\}) = \mathbb{MV}$
- $\mathbf{V}(\boldsymbol{Ł}_i) \subseteq \mathbf{V}(\boldsymbol{Ł}_j)$ *iff* $i-1$ *divides* $j-1$*.*

Theorem 2.67 (Komori)

Let $\mathbb{K} \subseteq \mathbb{MV}$ *be a variety.* $\mathbb{K} \neq \mathbb{MV}$ *iff there are two finite disjoint sets* $I, J \subseteq \mathrm{N}$ *such that:*

$$\mathbb{K} = \mathbf{V}(\{\boldsymbol{Ł}_i \mid i \in I\} \cup \{\boldsymbol{K}_j \mid j \in J\}).$$

Varieties of MV-algebras

Definition 2.68

If $i \in \mathrm{N}$, $\delta(i) = \{n \in \mathrm{N} \mid n$ is a divisor of $i\}$. If $J \subseteq \mathrm{N}$ is finite and nonempty, $\Delta(i, J) = \delta(i) \setminus \bigcup_{j \in J} \delta(j)$.

Theorem 2.69 (Di Nola, Lettieri)

Let $I, J \subseteq \mathrm{N}$ be finite disjoint sets. Then the variety $\mathbf{V}(\{\text{Ł}_i \mid i \in I\} \cup \{\boldsymbol{K}_j \mid j \in J\})$ has the following equational base:

$$
\begin{aligned}
&Eq(1) \qquad ((n+1)x^n)^2 \approx 2x^{n+1} \qquad \text{with } n = \max(I \cup J),\\
&Eq(2) \qquad (px^{p-1})^{n+1} \approx (n+1)x^p,\\
&Eq(3) \qquad (n+1)x^q \approx (n+2)x^q,
\end{aligned}
$$

for every positive integer $1 < p < n$ such that p is not a divisor of any $i \in I \cup J$ and for every $q \in \bigcup_{i \in I} \Delta(i, J)$.

Fuzzy logic for reasoning about probability

Fuzziness $\neq$ probability

Probability of φ = $\Box\varphi =$ truth degree of *it is probable that* φ

Let us take:

- the classical logic CL in language $\to, \neg, \vee, \wedge, \overline{0}$
- Łukasiewicz logic Ł in language $\to_{Ł}, \neg_{Ł}, \oplus, \ominus$
- an extra symbol $\Box$

We define three kinds of formulas of a two-level language over a fixed set of variables *Var*:

- non-modal: built from *Var* using $\to, \neg, \vee, \wedge, \overline{0}$
- atomic modal: of the form $\Box\varphi$, for each non-modal φ
- modal: built from atomic ones using $\to_{Ł}, \neg_{Ł}, \oplus, \ominus$

Probability Kripke frames and Kripke models

Definition 2.70

A *probability Kripke frame* is a system $\mathbf{F} = \langle W, \mu \rangle$ where

- W is a set (of possible worlds)
- μ is a finitely additive probability measure defined on a sublattice of 2^W

Definition 2.71

A *Kripke model* $\mathbf{M}$ over a probability Kripke frame $\mathbf{F} = \langle W, \mu \rangle$ is a tuple $\mathbf{M} = \langle \mathbf{F}, (e_w)_{w \in W} \rangle$ where:

- e_w is a classical evaluation of non-modal formulas
- the domain of μ contains the set $\{w \mid e_w(\varphi) = 1\}$ for each non-modal formula φ

Truth definition

The truth values of modal formulas are defined uniformly:

$$
\begin{aligned}
||\Box\varphi||_{\mathbf{M}} =& \mu(\{w \mid e_w(\varphi) = 1\}) \\
||\neg_{Ł}\Phi||_{\mathbf{M}} =& 1 - ||\Phi||_{\mathbf{M}} \\
||\Phi \rightarrow_{Ł} \Psi||_{\mathbf{M}} =& \min\{1, 1 - ||\Phi||_{\mathbf{M}} + ||\Psi||_{\mathbf{M}}\} \\
||\Phi \oplus \Psi||_{\mathbf{M}} =& \min\{1, ||\Phi||_{\mathbf{M}} + ||\Psi||_{\mathbf{M}}\} \\
||\Phi \ominus \Psi||_{\mathbf{M}} =& \max\{0, ||\Phi||_{\mathbf{M}} - ||\Psi||_{\mathbf{M}}\}
\end{aligned}
$$

Axiomatization

Definition 2.72

The logic FP of probability inside Łukasiewicz logic is given by the axiomatic system consisting of:

- the axioms and rules of CL for non-modal formulas,
- axioms and rules of Ł for modal formulas,
- modal axioms

$$
\begin{array}{ll}
\text{(FP0)} & \neg_{Ł}\Box(\overline{0}) \\
\text{(FP1)} & \Box(\varphi \to \psi) \to_{Ł} (\Box\varphi \to_{Ł} \Box\psi) \\
\text{(FP2)} & \neg_{Ł}\Box(\varphi) \to_{Ł} \Box(\neg\varphi) \\
\text{(FP3)} & \Box(\varphi \vee \psi) \to_{Ł} (\Box\psi \oplus (\Box\varphi \ominus \Box(\varphi \wedge \psi)))
\end{array}
$$

- a unary modal rule:

$$\varphi \vdash \Box\varphi$$

The notion of provability $\vdash_{\mathsf{FP}}$ (from both modal and non-modal premises) is defined as usual.

Completeness theorem

Theorem 2.73 (Hájek)

Let $\Gamma \cup \{\Psi\}$ *be a set of modal formulas. TFAE:*

- $\Gamma \vdash_{\mathsf{FP}} \Psi$
- $||\Psi||_{\mathbf{M}} = 1$ *for each Kripke model* $\mathbf{M}$ *where* $||\Phi||_{\mathbf{M}} = 1$ *for each* $\Phi \in \Gamma$*.*

Variations

- changing the measure of uncertainty (necessity, possibility, belief functions)
- changing the upper logic: replacing Łukasiewicz logic by any other fuzzy logic
- changing the lower logic: e.g. replacing CL by Łukasiewicz logic to speak about probability of vague events
  Ex: *Messi will score soon in the second half of the match*
- adding more modalities
- any combination of the above four options

We can build also a general theory for these two-layer modal logics