A Gentle Introduction to Mathematical Fuzzy Logic

2. Basic properties of Łukasiewicz and Gödel–Dummett logic

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Outline

1. Completeness of Gödel–Dummett logic
2. Completeness of Łukasiewicz logic
3. Functional representation
4. Finite model property
5. Computational complexity
6. Algebraizability of Gödel–Dummett and Łukasiewicz logics
7. Axiomatic extensions of Gödel–Dummett and Łukasiewicz logics
8. Application: Fuzzy Logic and Probability
We consider primitive connectives $\mathcal{L} = \{\rightarrow, \land, \lor, \overline{0}\}$ and defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$:

\[
\neg \varphi = \varphi \rightarrow \overline{0} \quad \overline{1} = \neg \overline{0} \quad \varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)
\]

Formulas are built from a fixed countable set of atoms using the connectives.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulas.
A Hilbert-style proof system

Axioms:

(Tr) \((\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))\) \hspace{1cm} \text{transitivity}

(We) \(\varphi \to (\psi \to \varphi)\) \hspace{1cm} \text{weakening}

(Ex) \((\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))\) \hspace{1cm} \text{exchange}

(\wedge a) \(\varphi \land \psi \to \varphi\)

(\wedge b) \(\varphi \land \psi \to \psi\)

(\wedge c) \((\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \land \psi))\)

(\lor a) \(\varphi \to \varphi \lor \psi\)

(\lor b) \(\psi \to \varphi \lor \psi\)

(\lor c) \((\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \lor \psi \to \chi))\)

(Prl) \((\varphi \to \psi) \lor (\psi \to \varphi)\) \hspace{1cm} \text{prelinearity}

(EFQ) \(0 \to \varphi\) \hspace{1cm} \text{Ex falso quodlibet}

(Con) \((\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)\) \hspace{1cm} \text{contraction}

Inference rule: from \(\varphi\) and \(\varphi \to \psi\) infer \(\psi\) \hspace{1cm} \text{modus ponens}
The relation of provability

Proof: a proof of a formula $\varphi$ from a set of formulas (theory) $\Gamma$ is a finite sequence of formulas $\langle \psi_1, \ldots, \psi_n \rangle$ such that:

- $\psi_n = \varphi$
- for every $i \leq n$, either $\psi_i \in \Gamma$, or $\psi_i$ is an instance of an axiom, or there are $j, k < i$ such that $\psi_k = \psi_j \rightarrow \psi_i$.

We write $\Gamma \vdash_G \varphi$ if there is a proof of $\varphi$ from $\Gamma$.

A formula $\varphi$ is a theorem of Gödel–Dummett logic if $\vdash_G \varphi$.

Proposition 2.1

The provability relation of Gödel–Dummett logic is finitary: if $\Gamma \vdash_G \varphi$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_G \varphi$. 
Algebraic semantics

A Gödel algebra (or just G-algebra) is a structure\

\[ B = \langle B, \land^B, \lor^B, \rightarrow^B, 0^B, 1^B \rangle \] such that:

1. \( \langle B, \land^B, \lor^B, 0^B, 1^B \rangle \) is a bounded lattice
2. \( z \leq x \rightarrow^B y \) iff \( x \land^B z \leq y \) (residuation)
3. \( (x \rightarrow^B y) \lor^B (y \rightarrow^B x) = 1^B \) (prelinearity)

where \( x \leq^B y \) is defined as \( x \land^B y = x \) or (equivalently) as \( x \rightarrow^B y = 1^B \).

A G-algebra \( B \) is linearly ordered (or G-chain) if \( \leq^B \) is a total order.

By \( G \) (or \( G_{\text{lin}} \) resp.) we denote the class of all G-algebras (G-chains resp.)
Standard semantics

Consider algebra $[0, 1]_G = \langle [0, 1], \land [0, 1]^G, \lor [0, 1]^G, \rightarrow [0, 1]^G, 0, 1 \rangle$, where:

\[ a \land [0, 1]^G b = \min\{a, b\} \]

\[ a \lor [0, 1]^G b = \max\{a, b\} \]

\[ a \rightarrow [0, 1]^G b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise}. \end{cases} \]

Exercise 1

(a) Prove that $[0, 1]_G$ is the unique $G$-chain with the lattice reduct $\langle [0, 1], \min, \max, 0, 1 \rangle$. 
Semantical consequence

**Definition 2.2**

A \( B \)-evaluation is a mapping \( e \) from \( \text{Fm}_\mathcal{L} \) to \( B \) such that:

- \( e(0) = \overline{0}^B \)
- \( e(\varphi \land \psi) = e(\varphi) \land^B e(\psi) \)
- \( e(\varphi \lor \psi) = e(\varphi) \lor^B e(\psi) \)
- \( e(\varphi \rightarrow \psi) = e(\varphi) \rightarrow^B e(\psi) \)

**Definition 2.3**

A formula \( \varphi \) is a logical consequence of a set of formulas \( \Gamma \) w.r.t. a class \( \mathcal{K} \) of \( G \)-algebras, \( \Gamma \models_{\mathcal{K}} \varphi \), if for every \( B \in \mathcal{K} \) and every \( B \)-evaluation \( e \):

if \( e(\gamma) = \overline{1} \) for every \( \gamma \in \Gamma \), then \( e(\varphi) = \overline{1} \).
Completeness theorem

Theorem 2.4

The following are equivalent for every set of formulas $\Gamma \cup \{ \varphi \} \subseteq Fm_L$:

1. $\Gamma \vdash_G \varphi$
2. $\Gamma \models_G \varphi$
3. $\Gamma \models_{G_{\text{lin}}} \varphi$
4. $\Gamma \models_{[0,1]_G} \varphi$

Exercise 1
(a) Prove the implications from top to bottom.
Completeness theorem

Theorem 2.4

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_L$:

1. $\Gamma \vdash_G \varphi$
2. $\Gamma \models_G \varphi$
3. $\Gamma \models_{G_{\text{lin}}} \varphi$
4. $\Gamma \models_{[0,1]_G} \varphi$

Exercise 2

(a) Prove the implications from top to bottom.
Some theorems and derivations in $G$ 

**Proposition 2.5**

\begin{align*}
(T1) & \vdash_G \varphi \to \varphi \\
(T2) & \vdash_G \varphi \to (\psi \to \varphi \land \psi) \\
(D1) & \overline{1} \leftrightarrow \varphi \vdash_G \varphi \text{ and } \forall \varphi \vdash_G \overline{1} \leftrightarrow \varphi \\
(D2) & \varphi \to \psi \vdash_G \varphi \land \psi \leftrightarrow \varphi \text{ and } \forall \varphi \land \psi \leftrightarrow \varphi \vdash_G \varphi \to \psi \\
(D3) & \varphi \to (\psi \to \chi) \vdash_G \varphi \land \psi \to \chi \text{ and } \forall \varphi \land \psi \to \chi \vdash_G \varphi \to (\psi \to \chi)
\end{align*}

**Proposition 2.6**

\begin{align*}
\vdash_G \varphi \land \psi & \leftrightarrow \psi \land \varphi \\
\vdash_G \varphi \land (\psi \land \chi) & \leftrightarrow (\varphi \land \psi) \land \chi \\
\vdash_G \varphi \land (\varphi \lor \psi) & \leftrightarrow \varphi \\
\vdash_G \overline{1} \land \varphi & \leftrightarrow \varphi \\
\vdash_G (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi) & \leftrightarrow \overline{1} \\
\vdash_G \varphi \lor \psi & \leftrightarrow \psi \lor \varphi \\
\vdash_G \varphi \lor (\psi \lor \chi) & \leftrightarrow (\varphi \lor \psi) \lor \chi \\
\vdash_G \varphi \lor (\varphi \land \psi) & \leftrightarrow \varphi \\
\vdash_G \overline{0} \lor \varphi & \leftrightarrow \varphi \\
\end{align*}
The rule of substitution

**Proposition 2.7**

\[
\begin{align*}
\varphi \leftrightarrow \psi \vdash_G (\varphi \land \chi) & \iff (\psi \land \chi) \\
\varphi \leftrightarrow \psi \vdash_G (\chi \land \varphi) & \iff (\chi \land \psi) \\
\varphi \leftrightarrow \psi \vdash_G (\varphi \rightarrow \chi) & \iff (\psi \rightarrow \chi) \\
\end{align*}
\]

\[
\begin{align*}
\varphi \leftrightarrow \psi \vdash_G (\varphi \lor \chi) & \iff (\psi \lor \chi) \\
\varphi \leftrightarrow \psi \vdash_G (\chi \lor \varphi) & \iff (\chi \lor \psi) \\
\end{align*}
\]

\[
\vdash_G \varphi \leftrightarrow \varphi \quad \varphi \leftrightarrow \psi \vdash_G \psi \leftrightarrow \varphi \quad \varphi \leftrightarrow \psi, \psi \leftrightarrow \chi \vdash_G \varphi \leftrightarrow \chi
\]

**Corollary 2.8**

\[
\varphi \leftrightarrow \psi \vdash_G \chi \leftrightarrow \chi', \quad \text{where } \chi' \text{ results from } \chi \text{ by replacing its subformula } \varphi \text{ by } \psi.
\]

**Exercise 3**

(a) Prove this corollary and the two previous propositions.
Lindenbaum–Tarski algebra

**Definition 2.9**

Let $\Gamma$ be a theory. We define

$$[\varphi]_\Gamma = \{ \psi \mid \Gamma \vdash_G \varphi \leftrightarrow \psi \} \quad L_\Gamma = \{ [\varphi]_\Gamma \mid \varphi \in Fm_L \}$$

The **Lindenbaum–Tarski algebra** of a theory $\Gamma$ ($\text{Lind}_\Gamma$) as an algebra with the domain $L_\Gamma$ and operations:

$$\overline{0}_{\text{Lind}} = [\overline{0}]_\Gamma \quad \overline{1}_{\text{Lind}} = [\overline{1}]_\Gamma$$

$$[\varphi]_\Gamma \rightarrow_{\text{Lind}} [\psi]_\Gamma = [\varphi \rightarrow \psi]_\Gamma$$

$$[\varphi]_\Gamma \land_{\text{Lind}} [\psi]_\Gamma = [\varphi \land \psi]_\Gamma$$

$$[\varphi]_\Gamma \lor_{\text{Lind}} [\psi]_\Gamma = [\varphi \lor \psi]_\Gamma$$
**Proposition 2.10**

1. \([\varphi]_\Gamma = [\psi]_\Gamma \text{ iff } \Gamma \vdash_G \varphi \leftrightarrow \psi\)

2. \([\varphi]_\Gamma \leq^{\text{Lind}} [\psi]_\Gamma \text{ iff } \Gamma \vdash_G \varphi \rightarrow \psi\)

3. \(\overline{\text{Lind}}_\Gamma = [\varphi]_\Gamma \text{ iff } \Gamma \vdash_G \varphi\)

4. \(\text{Lind}_\Gamma \text{ is an } G\text{-algebra}\)

5. \(\text{Lind}_\Gamma \text{ is an } G\text{-chain iff } \Gamma \vdash_G \varphi \rightarrow \psi \text{ or } \Gamma \vdash_G \psi \rightarrow \varphi \text{ for each } \varphi, \psi\)

**Proof.**

1. Left-to-right is the just definition and ‘reflexivity’ of \(\leftrightarrow\). Conversely, we use ‘transitivity’ and ‘symmetry’ of \(\leftrightarrow\).

2. \([\varphi]_\Gamma \leq^{\text{Lind}} [\psi]_\Gamma \text{ iff } [\varphi]_\Gamma \wedge^{\text{Lind}} [\psi]_\Gamma = [\varphi]_\Gamma \text{ iff } [\varphi \wedge \psi]_\Gamma = [\varphi]_\Gamma \text{ iff (by 1.) } \Gamma \vdash_G \varphi \wedge \psi \leftrightarrow \varphi \text{ iff (by (D2)) } \Gamma \vdash_G \varphi \rightarrow \psi.\)

3. \(\overline{\text{Lind}}_\Gamma = [\varphi]_\Gamma \text{ iff (by 1.) } \Gamma \vdash_G \overline{I} \leftrightarrow \varphi \text{ iff (by (D1)) } \Gamma \vdash_G \varphi.\)

5. Trivial after we prove 4.
Lindenbaum–Tarski algebra: basic properties

Proposition 2.10

1. \([\varphi]\Gamma = [\psi]\Gamma \text{ iff } \Gamma \vdash_G \varphi \leftrightarrow \psi\)
2. \([\varphi]\Gamma \leq_{\text{Lind}} [\psi]\Gamma \text{ iff } \Gamma \vdash_G \varphi \rightarrow \psi\)
3. \(\overline{\text{Lind}} \Gamma = [\varphi]\Gamma \text{ iff } \Gamma \vdash_G \varphi\)
4. \(\text{Lind}_\Gamma \text{ is an } G\text{-algebra}\)
5. \(\text{Lind}_\Gamma \text{ is an } G\text{-chain iff } \Gamma \vdash_G \varphi \rightarrow \psi \text{ or } \Gamma \vdash_G \psi \rightarrow \varphi \text{ for each } \varphi, \psi\)

Proof.

4. First we note that the definition of \(\text{Lind}_\Gamma\) is sound due to 1. and Proposition 2.7.

The lattice identities hold due to 1. and Proposition 2.6, prelinearity due to 3. and axiom (Prl).

Finally, the residuation: \([\varphi]\Gamma \leq_{\text{Lind}} [\psi]\Gamma \rightarrow_{\text{Lind}} [\chi]\Gamma = [\psi \rightarrow \chi]\Gamma \text{ iff } \Gamma \vdash_G \varphi \rightarrow (\psi \rightarrow \chi) \text{ iff (by (D3)) } \Gamma \vdash_G \varphi \land \psi \rightarrow \chi\text{ iff } [\varphi]\Gamma \land_{\text{Lind}} [\psi]\Gamma \leq_{\text{Lind}} [\chi]\Gamma\).
Theorem 2.4

The following are equivalent for every set of formulas \( \Gamma \cup \{ \phi \} \subseteq Fm_\mathcal{L} \):

1. \( \Gamma \vdash_\mathcal{G} \phi \)
2. \( \Gamma \models_\mathcal{G} \phi \)
3. \( \Gamma \models_\mathcal{G}_{\text{lin}} \phi \)
4. \( \Gamma \models [0,1]_\mathcal{G} \phi \)

Proof.

2. implies 1.: contrapositively, assume that \( \Gamma \not\vdash_\mathcal{G} \phi \).

We know that \( \text{Lind}_\Gamma \in \mathcal{G} \) and the function \( e \) defined as \( e(\psi) = [\psi]_\Gamma \)

- is a \( \text{Lind}_\Gamma \)-evaluation and
- \( e(\psi) = \overline{1}^{\text{Lind}_\Gamma} \) iff \( \Gamma \vdash_\mathcal{G} \psi \).

Thus clearly \( e(\chi) = \overline{1}^{\text{Lind}_\Gamma} \) for each \( \chi \in \Gamma \) and \( e(\phi) \neq \overline{1}^{\text{Lind}_\Gamma} \).
Deduction Theorem

Theorem 2.11 (Deduction theorem)

For every set of formulas \( \Gamma \cup \{ \varphi, \psi \} \),

\[
\Gamma, \varphi \vdash_G \psi \iff \Gamma \vdash_G \varphi \rightarrow \psi
\]

Proof.

\( \Leftarrow \): follows from *modus ponens*

\( \Rightarrow \): let \( \alpha_1, \ldots, \alpha_n = \psi \) be the proof of \( \psi \) in \( \Gamma, \varphi \). We show by induction that \( \Gamma \vdash_G \varphi \rightarrow \alpha_i \) for each \( i \leq n \).
Theorem 2.11 (Deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{G} \psi \iff \Gamma \vdash_{G} \varphi \rightarrow \psi$$

Proof.

$\Leftarrow$: follows from *modus ponens*

$\Rightarrow$: let $\alpha_1, \ldots, \alpha_n = \psi$ be the proof of $\psi$ in $\Gamma, \varphi$. We show by induction that $\Gamma \vdash_{G} \varphi \rightarrow \alpha_i$ for each $i \leq n$.

If $\alpha_i = \varphi$ we use (T1); if $\alpha_i$ is an axiom or $\alpha_i \in \Gamma$ then $\Gamma \vdash_{G} \alpha_i$ and so we can use axiom (We) and MP.
Deduction Theorem

Theorem 2.11 (Deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{G} \psi \iff \Gamma \vdash_{G} \varphi \rightarrow \psi$$

Proof.

$\Leftarrow$: follows from modus ponens

$\Rightarrow$: let $\alpha_1, \ldots, \alpha_n = \psi$ be the proof of $\psi$ in $\Gamma, \varphi$. We show by induction that $\Gamma \vdash_{G} \varphi \rightarrow \alpha_i$ for each $i \leq n$.

Otherwise there has to be $k, j < i$ such that $\alpha_k = \alpha_j \rightarrow \alpha_i$.

Induction assumption gives: $\Gamma \vdash_{G} \varphi \rightarrow \alpha_j$ and $\Gamma \vdash \varphi \rightarrow (\alpha_j \rightarrow \alpha_i)$.

Using $\Gamma \vdash \varphi \rightarrow (\alpha_j \rightarrow \alpha_i)$, (Ex), and MP we get $\Gamma \vdash \alpha_j \rightarrow (\varphi \rightarrow \alpha_i)$, using this, $\Gamma \vdash_{G} \varphi \rightarrow \alpha_j$, (Tr), and MP twice we get $\Gamma \vdash \varphi \rightarrow (\varphi \rightarrow \alpha_i)$.

Finally we use (Con) and MP.
Lemma 2.12 (Semilinearity Property)

If $\Gamma, \varphi \rightarrow \psi \vdash G \chi$ and $\Gamma, \psi \rightarrow \varphi \vdash G \chi$, then $\Gamma \vdash G \chi$.

Proof.

By the deduction theorem: $\Gamma \vdash G (\varphi \rightarrow \psi) \rightarrow \chi$ and $\Gamma \vdash G (\psi \rightarrow \varphi) \rightarrow \chi$.

Thus by ($\lor \mathcal{c}$) we get $\Gamma \vdash G (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi) \rightarrow \chi$.

Axiom (Prl) completes the proof.
Linear Extension Property

Definition 2.13
A theory $\Gamma$ is linear if $\Gamma \vdash_G \varphi \rightarrow \psi$ or $\Gamma \vdash_G \psi \rightarrow \varphi$ for each $\varphi, \psi$.

Lemma 2.14 (Linear Extension Property)
If $\Gamma \not\vdash_G \varphi$, then there is linear theory $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \not\vdash_G \varphi$.

Proof.
Enumerate all pairs of formulas: $\langle \varphi_0, \psi_0 \rangle, \langle \psi_1, \varphi_1 \rangle, \ldots$
Construct theories $\Gamma_0, \Gamma_1, \ldots$ s.t. $\Gamma_0 = \Gamma$; $\Gamma_i \subseteq \Gamma_{i+1}$, and $\Gamma_i \not\vdash_G \varphi$:
- if $\Gamma_i, \varphi_i \rightarrow \psi_i \not\vdash_G \varphi$, then $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i \rightarrow \psi_i\}$
- if $\Gamma_i, \varphi_i \rightarrow \psi_i \vdash_G \varphi$, then $\Gamma_{i+1} = \Gamma_i \cup \{\psi_i \rightarrow \varphi_i\}$

Clearly $\Gamma_{i+1} \not\vdash_G \varphi$ (the 1st is obvious, in the 2nd would $\Gamma_{i+1} \vdash_G \varphi$ entail $\Gamma_i \vdash_G \varphi$ by the Semilinearity Property, a contradiction with the IH).
Define $\Gamma' = \bigcup \Gamma_i$. Clearly $\Gamma'$ is linear, $\Gamma' \supseteq \Gamma$, and $\Gamma' \not\vdash_G \varphi$. □
Theorem 2.4

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_L$:

1. $\Gamma \vdash G \varphi$
2. $\Gamma \models G \varphi$
3. $\Gamma \models_{G_{\text{lin}}} \varphi$
4. $\Gamma \models [0,1]_G \varphi$

Proof.

3. implies 1.: contrapositively, assume that $\Gamma \nvdash G \varphi$. Due to the Linear Extension Property there is a linear theory $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \nvdash G \varphi$.

We know that $\text{Lind}_{\Gamma'} \in G_{\text{lin}}$ and the function $e$ defined as $e(\psi) = [\psi]_{\Gamma'}$

- is a $\text{Lind}_{\Gamma'}$-evaluation and
- $e(\psi) = \overline{\text{Lind}}_{\Gamma'} \varphi$ iff $\Gamma' \vdash G \varphi$

Thus $e(\chi) = \overline{\text{Lind}}_{\Gamma'}$ for each $\chi \in \Gamma$ (as $\Gamma' \vdash_G \chi$) and $e(\varphi) \neq \overline{\text{Lind}}_{\Gamma'}$. $\square$
The proof of the standard completeness theorem

We continue the previous proof: note that the algebra $\text{Lind}_{\Gamma'}$ is countable.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f : L_{\Gamma'} \to [0, 1]$ such that $f(\overline{0}_{\text{Lind}_{\Gamma'}}) = 0$, $f(\overline{1}_{\text{Lind}_{\Gamma'}}) = 1$, and for each $a, b \in L_{\Gamma'}$ we have:

$$a \leq b \iff f(a) \leq f(b)$$

We define a mapping $\bar{e} : Fm_{\mathcal{L}} \to [0, 1]$ as

$$\bar{e}(\psi) = f(e(\psi))$$

and prove (by induction) that it is an $[0, 1]_G$-evaluation.

Then $\bar{e}(\psi) = 1$ iff $e(\psi) = \overline{1}_{\text{Lind}_{\Gamma'}}$ and so $\bar{e}[\Gamma] \subseteq \{1\}$ and $\bar{e}(\varphi) \neq 1$. 
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Syntax

We consider primitive connectives \( \mathcal{L} = \{ \rightarrow, \land, \lor, \overline{0} \} \) and defined connectives \( \neg, \overline{1}, \) and \( \leftrightarrow \):

\[
\neg \varphi = \varphi \rightarrow \overline{0} \\
\overline{1} = \neg \overline{0} \\
\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)
\]

Formulas are built from a fixed countable set of atoms using the connectives.

Let us by \( Fm_{\mathcal{L}} \) denote the set of all formulas.

We also use additional connectives \( \oplus \) and \( \& \) defined as:

\[
\varphi \oplus \psi = \neg \varphi \rightarrow \psi \\
\varphi \& \psi = \neg (\varphi \rightarrow \neg \psi)
\]
A Hilbert-style proof system

Axioms:

(Tr) \((\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))\)  
transitivity

(We) \(\varphi \rightarrow (\psi \rightarrow \varphi)\)  
weakening

(Ex) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))\)  
xchange

(\wedge a) \(\varphi \land \psi \rightarrow \varphi\)

(\wedge b) \(\varphi \land \psi \rightarrow \psi\)

(\wedge c) \((\chi \rightarrow \varphi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\chi \rightarrow \varphi \land \psi))\)

(\lor a) \(\varphi \rightarrow \varphi \lor \psi\)

(\lor b) \(\psi \rightarrow \varphi \lor \psi\)

(\lor c) \((\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \lor \psi \rightarrow \chi))\)

(Prl) \((\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)\)  
prelinearity

(EFQ) \(0 \rightarrow \varphi\)  
Ex falso quodlibet

(Waj) \(((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \varphi)\)  
Wajsberg axiom

Inference rule: from \(\varphi\) and \(\varphi \rightarrow \psi\) infer \(\psi\)  
modus ponens
The relation of provability

Proof: a proof of a formula $\varphi$ from a set of formulas (theory) $\Gamma$ is a finite sequence of formulas $\langle \psi_1, \ldots, \psi_n \rangle$ such that:

- $\psi_n = \varphi$
- for every $i \leq n$, either $\psi_i \in \Gamma$, or $\psi_i$ is an instance of an axiom, or there are $j, k < i$ such that $\psi_k = \psi_j \rightarrow \psi_i$.

We write $\Gamma \vdash_L \varphi$ if there is a proof of $\varphi$ from $\Gamma$.

A formula $\varphi$ is a theorem of Łukasiewicz logic if $\vdash_L \varphi$.

Proposition 2.15

The provability relation of Łukasiewicz logic is finitary: if $\Gamma \vdash_L \varphi$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_L \varphi$. 
Algebraic semantics

An **MV-algebra** is a structure \( B = \langle B, \oplus, \neg, \bar{0} \rangle \) such that:

1. \( \langle B, \oplus, \bar{0} \rangle \) is a commutative monoid,
2. \( \neg\neg x = x \),
3. \( x \oplus \neg\bar{0} = \neg\bar{0} \),
4. \( \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \).

In each MV-algebra we define additional operations:

- \( x \rightarrow y \) is \( \neg x \oplus y \) **implication**
- \( x \& y \) is \( \neg(\neg x \oplus \neg y) \) **strong conjunction**
- \( x \lor y \) is \( \neg(\neg x \oplus y) \oplus y \) **max-disjunction**
- \( x \land y \) is \( \neg(\neg x \lor \neg y) \) **min-conjunction**
- \( \bar{1} \) is \( \neg\bar{0} \) **top**

**Exercise 4**

Prove that \( \langle B, \land, \lor, \bar{0}, \bar{1} \rangle \) is a bounded lattice.
Algebraic semantics cont. and standard semantics

We say that an MV-algebra $B$ is linearly ordered (or MV-chain) if its lattice reduct is.

By $\text{MV}$ (or $\text{MV}_{\text{lin}}$ resp.) we denote the class of all MV-algebras (MV-chains resp.)

Take the algebra $[0, 1]_L = \langle [0, 1], \oplus, \neg, 0 \rangle$, with operations defined as:

$$\neg a = 1 - a \quad \quad a \oplus b = \min\{1, a + b\}.$$ 

**Proposition 2.16**

$[0, 1]_L$ is the unique (up to isomorphism) MV-chain with the lattice reduct $\langle [0, 1], \min, \max, 0, 1 \rangle$.

**Exercise 1**

(b) Check that $[0, 1]_L$ is an MV-chain and find another MV-chain isomorphic to $[0, 1]_L$ with the same lattice reduct.
Semantical consequence

**Definition 2.17**

A **$B$-evaluation** is a mapping $e$ from $Fm_L$ to $B$ such that:

- $e(\overline{0}) = \overline{0}^B$
- $e(\varphi \to \psi) = e(\varphi) \to^B e(\psi) = \neg^B e(\varphi) \oplus^B e(\psi)$
- $e(\varphi \land \psi) = e(\varphi) \land^B e(\psi) = \cdots$
- $e(\varphi \lor \psi) = e(\varphi) \lor^B e(\psi) = \cdots$

**Definition 2.18**

A formula $\varphi$ is a **logical consequence** of a set of formulas $\Gamma$ w.r.t. a class $\mathbb{K}$ of MV-algebras, $\Gamma \models^\mathbb{K} \varphi$, if for every $B \in \mathbb{K}$ and every $B$-evaluation $e$:

if $e(\gamma) = \overline{1}$ for every $\gamma \in \Gamma$, then $e(\varphi) = \overline{1}$. 
**Theorem 2.19**

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_L$:

1. $\Gamma \not\vdash L \varphi$
2. $\Gamma \not\models_{MV} \varphi$
3. $\Gamma \not\models_{MV_{lin}} \varphi$

If $\Gamma$ is finite we can add:

4. $\Gamma \not\models_{[0,1]L} \varphi$

**Exercise 2**

(b) Prove the implications from top to bottom.
Theorem 2.19

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_\mathcal{L}$:

1. $\Gamma \vdash_\mathcal{L} \varphi$
2. $\Gamma \models_{\text{MV}} \varphi$
3. $\Gamma \models_{\text{MV}_{\text{lin}}} \varphi$

If $\Gamma$ is finite we can add:

4. $\Gamma \models_{[0,1]_\mathcal{L}} \varphi$

Exercise 2

(b) Prove the implications from top to bottom.
Some theorems and derivations

Proposition 2.20

(T1) \( \vdash_L \varphi \rightarrow \varphi \)

(T2) \( \vdash_L \varphi \rightarrow (\psi \rightarrow \varphi \land \psi) \)

(T3) \( \vdash_L \varphi \lor \chi \rightarrow ((\varphi \rightarrow \psi) \lor \chi \rightarrow \psi \lor \chi) \)

(T4) \( \vdash_L \varphi \lor \varphi \rightarrow \varphi \)

(T5) \( \vdash_L \varphi \lor \psi \rightarrow \psi \lor \varphi \)

(D1) \( \overline{\mathbf{I}} \leftrightarrow \varphi \vdash_L \varphi \text{ and } \varphi \vdash_L \overline{\mathbf{I}} \leftrightarrow \varphi \)

(D2) \( \varphi \rightarrow \psi \vdash_L \varphi \land \psi \leftrightarrow \varphi \text{ and } \varphi \land \psi \leftrightarrow \varphi \vdash_L \varphi \rightarrow \psi \)

(D3') \( \varphi \rightarrow (\psi \rightarrow \chi) \vdash_G \varphi \land \psi \rightarrow \chi \text{ and } \varphi \land \psi \rightarrow \chi \vdash_G \varphi \rightarrow (\psi \rightarrow \chi) \)

Proposition 2.21

\( \vdash_L \varphi \oplus \psi \leftrightarrow \psi \oplus \varphi \)

\( \vdash_L \varphi \oplus (\psi \oplus \chi) \leftrightarrow (\varphi \oplus \psi) \oplus \chi \)

\( \vdash_L \overline{0} \oplus \varphi \leftrightarrow \varphi \)

\( \vdash_L \neg \neg \varphi \leftrightarrow \varphi \)

\( \vdash_L \varphi \oplus \overline{0} \leftrightarrow \overline{0} \)

\( \vdash_L \neg (\neg \varphi \oplus \psi) \oplus \psi \leftrightarrow \neg (\neg \psi \oplus \varphi) \oplus \varphi \)
The rule of substitution

### Proposition 2.22

\[ \varphi \leftrightarrow \psi \vdash L (\varphi \land \chi) \leftrightarrow (\psi \land \chi) \]
\[ \varphi \leftrightarrow \psi \vdash L (\chi \land \varphi) \leftrightarrow (\chi \land \psi) \]
\[ \varphi \leftrightarrow \psi \vdash L (\varphi \rightarrow \chi) \leftrightarrow (\psi \rightarrow \chi) \]

\[ \vdash L \varphi \leftrightarrow \varphi \quad \varphi \leftrightarrow \psi \vdash L \psi \leftrightarrow \varphi \quad \varphi \leftrightarrow \psi, \psi \vdash \chi \vdash L \varphi \leftrightarrow \chi \]

### Corollary 2.23

\[ \varphi \leftrightarrow \psi \vdash L \chi \leftrightarrow \chi', \quad \text{where } \chi' \text{ results from } \chi \text{ by replacing its subformula } \varphi \text{ by } \psi. \]

### Exercise 3

(b) Prove this corollary and the two previous propositions.
Definition 2.24
Let $\Gamma$ be a theory. We define

$$\left[\varphi\right]_\Gamma = \{\psi \mid \Gamma \vdash_L \varphi \leftrightarrow \psi\} \quad L_\Gamma = \{\left[\varphi\right]_\Gamma \mid \varphi \in Fm_L\}$$

The Lindenbaum–Tarski algebra of a theory $\Gamma$ ($\text{Lind}_\Gamma$) as an algebra with the domain $L_\Gamma$ and operations:

$$0^{\text{Lind}_\Gamma} = \left[0\right]_\Gamma$$

$$\neg^{\text{Lind}_\Gamma} [\varphi]_\Gamma = \left[\neg \varphi\right]_\Gamma$$

$$[\varphi]_\Gamma \oplus^{\text{Lind}_\Gamma} [\psi]_\Gamma = \left[\varphi \oplus \psi\right]_\Gamma$$
Proposition 2.25

1. \([\varphi]_\Gamma = [\psi]_\Gamma \iff \Gamma \vdash L \varphi \leftrightarrow \psi\)
2. \([\varphi]_\Gamma \leq^{\text{Lind}} [\psi]_\Gamma \iff \Gamma \vdash_L \varphi \rightarrow \psi\)
3. \(\overline{1}^{\text{Lind}}_\Gamma = [\varphi]_\Gamma \iff \Gamma \vdash_L \varphi\)
4. \(\text{Lind}_\Gamma \text{ is an MV-algebra}\)
5. \(\text{Lind}_\Gamma \text{ is an MV-chain iff } \Gamma \vdash_L \varphi \rightarrow \psi \text{ or } \Gamma \vdash_L \psi \rightarrow \varphi \text{ for each } \varphi, \psi\)

Proof.

1. Left-to-right is the just definition and ‘reflexivity’ of \(\leftrightarrow\). Conversely, we use ‘transitivity’ and ‘symmetry’ of \(\leftrightarrow\).
2. \([\varphi]_\Gamma \leq^{\text{Lind}} [\psi]_\Gamma \iff [\varphi]_\Gamma \wedge^{\text{Lind}} [\psi]_\Gamma = [\varphi]_\Gamma \iff [\varphi \wedge \psi]_\Gamma = [\varphi]_\Gamma \iff (by \ 1.) \Gamma \vdash_L \varphi \wedge \psi \leftrightarrow \varphi \iff (by (D2)) \Gamma \vdash_L \varphi \rightarrow \psi.\)
3. \(\overline{1}^{\text{Lind}}_\Gamma = [\varphi]_\Gamma \iff (by \ 2.) \Gamma \vdash_L \overline{1} \rightarrow \varphi \iff (by (D1)) \Gamma \vdash_L \varphi.\)
4. Trivial after we prove 4.
Proposition 2.25

1. $[\varphi]_{\Gamma} = [\psi]_{\Gamma}$ iff $\Gamma \vdash_{L} \varphi \leftrightarrow \psi$

2. $[\varphi]_{\Gamma} \leq_{Lind} [\psi]_{\Gamma}$ iff $\Gamma \vdash_{L} \varphi \rightarrow \psi$

3. $\overline{Lind}_{\Gamma} = [\varphi]_{\Gamma}$ iff $\Gamma \vdash_{L} \varphi$

4. $Lind_{\Gamma}$ is an MV-algebra

5. $Lind_{\Gamma}$ is an MV-chain iff $\Gamma \vdash_{L} \varphi \rightarrow \psi$ or $\Gamma \vdash_{L} \psi \rightarrow \varphi$ for each $\varphi, \psi$

Proof.

4. First we note that the definition of $Lind_{\Gamma}$ is sound due to 1. and Proposition 2.7.

The identities defining MV-algebras hold due to 1. and Proposition 2.21.
Łukasiewicz logic vs. Gödel–Dummett

Some things are the same, not only (T1), (T2), (D1), and (D2), but also:

\[
\varphi \land \psi \rightarrow \chi \vdash_{L} \varphi \rightarrow (\psi \rightarrow \chi) \\
\vdash_{L} \varphi \rightarrow \neg\neg\varphi \\
\vdash_{L} (\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi) \\
\]

\[
\varphi \land \psi \rightarrow \chi \vdash_{G} \varphi \rightarrow (\psi \rightarrow \chi) \\
\vdash_{G} \varphi \rightarrow \neg\neg\varphi \\
\vdash_{G} (\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi) \\
\]

Some are different:

\[
\varphi \rightarrow (\psi \rightarrow \chi) \nvdash_{L} \varphi \land \psi \rightarrow \chi \\
\vdash_{L} \neg\neg\varphi \rightarrow \varphi \\
\vdash_{L} (\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi) \\
\]

\[
\varphi \rightarrow (\psi \rightarrow \chi) \vdash_{G} \varphi \land \psi \rightarrow \chi \\
\nvdash_{G} \neg\neg\varphi \rightarrow \varphi \\
\nvdash_{G} (\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi) \\
\]

Contrast this with known derivation (D3’):

\[
\varphi \rightarrow (\psi \rightarrow \chi) \vdash_{L} \varphi \land \psi \rightarrow \chi \\
\varphi \land \psi \rightarrow \chi \vdash_{L} \varphi \rightarrow (\psi \rightarrow \chi) \\
\]

\[
\varphi \rightarrow (\psi \rightarrow \chi) \vdash_{G} \varphi \land \psi \rightarrow \chi \\
\varphi \land \psi \rightarrow \chi \vdash_{G} \varphi \rightarrow (\psi \rightarrow \chi) \\
\]
Failure of the Deduction Theorem

Assume that we would have that for every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$
\Gamma, \varphi \vdash_L \psi \text{ iff } \Gamma \vdash_L \varphi \rightarrow \psi
$$

Clearly (MP twice): $\varphi, \varphi \rightarrow (\varphi \rightarrow \psi) \vdash_L \psi$.

Thus by the deduction theorem we would get

$$
\vdash_L (\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi).
$$

This is the axiom of contraction known to fail in Łukasiewicz logic.
A possible solution

We can prove that:

\[
\vdash \phi \land \psi \leftrightarrow \psi \land \phi \quad \vdash \phi \land \top \leftrightarrow \phi \quad \vdash (\phi \land \psi) \land \chi \leftrightarrow \psi \land (\phi \land \chi)
\]

Thus it makes sense to define \( \phi^0 = \top \) and \( \phi^{n+1} = \phi^n \land \phi \)

Exercise 5

Let \( \chi \) be a \&-conjunction of \( n \) formulas \( \phi \) with arbitrary bracketing. Prove that \( \vdash_L \chi \leftrightarrow \phi^n \). Furthermore prove that \( \phi \vdash_L \phi^n \).
Theorem 2.26 (Local deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{\mathcal{L}} \psi \text{ iff there is } n \text{ such that } \Gamma \vdash_{\mathcal{L}} \varphi^n \rightarrow \psi$$

Proof.

$\Leftarrow$: follows from modus ponens and the previous exercise

$\Rightarrow$: let $\alpha_1, \ldots, \alpha_n = \psi$ be the proof of $\psi$ in $\Gamma, \varphi$. We show by induction that for each $i \leq n$ there is $n_i$ such that $\Gamma \vdash_{\mathcal{L}} \varphi^{n_i} \rightarrow \alpha_i$
Theorem 2.26 (Local deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_L \psi \text{ iff there is } n \text{ such that } \Gamma \vdash_L \varphi^n \rightarrow \psi$$

Proof.

$\Leftarrow$: follows from *modus ponens* and the previous exercise

$\Rightarrow$: let $\alpha_1, \ldots, \alpha_n = \psi$ be the proof of $\psi$ in $\Gamma, \varphi$. We show by induction that for each $i \leq n$ there is $n_i$ such that $\Gamma \vdash_L \varphi^{n_i} \rightarrow \alpha_i$

If $\alpha_i = \varphi$ we set $n_i = 1$ and use (T1); if $\alpha_i$ is an axiom or $\alpha_i \in \Gamma$, then $\Gamma \vdash_L \alpha_i$ and so we can set $n_i = 1$ and use axiom (We) and MP.
Local Deduction Theorem

Theorem 2.26 (Local deduction theorem)

For every set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{L} \psi \iff \text{there is } n \text{ such that } \Gamma \vdash_{L} \varphi^{n} \rightarrow \psi$$

Proof.

$\Leftarrow$: follows from modus ponens and the previous exercise

$\Rightarrow$: let $\alpha_1, \ldots, \alpha_n = \psi$ be the proof of $\psi$ in $\Gamma, \varphi$. We show by induction that for each $i \leq n$ there is $n_i$ such that $\Gamma \vdash_{L} \varphi^{n_i} \rightarrow \alpha_i$

Otherwise there has to be $k, j < i$ such that $\alpha_k = \alpha_j \rightarrow \alpha_i$.

Induction assumption gives: $\Gamma \vdash_{L} \varphi^{n_j} \rightarrow \alpha_j$ and $\Gamma \vdash \varphi^{n_k} \rightarrow (\alpha_j \rightarrow \alpha_i)$.

Using $\Gamma \vdash \varphi^{n_k} \rightarrow (\alpha_j \rightarrow \alpha_i)$, (Ex), and MP we get $\Gamma \vdash \alpha_j \rightarrow (\varphi^{n_k} \rightarrow \alpha_i)$, using this, $\Gamma \vdash_{L} \varphi^{n_j} \rightarrow \alpha_j$, (Tr), and MP we get $\Gamma \vdash \varphi^{n_j} \rightarrow (\varphi^{n_k} \rightarrow \alpha_i)$.

Finally we use (D3′) and the previous exercise to get $\Gamma \vdash \varphi^{n_j+n_k} \rightarrow \alpha_i$. 
Proof by Cases Property

**Theorem 2.27 (Proof by Cases Property)**

If $\Gamma, \varphi \vdash_L \chi$ and $\Gamma, \psi \vdash_L \chi$, then $\Gamma, \varphi \lor \psi \vdash_L \chi$.

**Proof.**

**Claim** If $\Gamma \vdash_L \varphi$, then $\Gamma \lor \chi \vdash_L \delta \lor \chi$ for each formula $\chi$ and each $\delta$ appearing in the proof of $\varphi$ from $\Gamma$.

**Proof of the claim:** trivial for $\delta \in \Gamma$ or $\delta$ an axiom; if we used MP, then by IH there has to be $\eta$ st.

$\Gamma \lor \chi \vdash_L \eta \lor \chi \quad \Gamma \lor \chi \vdash_L (\eta \rightarrow \delta) \lor \chi$ thus (T3) completes the proof.

Now using the claim: $\Gamma \lor \psi, \varphi \lor \psi \vdash_L \chi \lor \psi$ and $\Gamma \lor \chi, \psi \lor \chi \vdash_L \chi \lor \chi$. Using ($\lor$a), (T4), and (T5) we get $\Gamma, \varphi \lor \psi \vdash_L \psi \lor \chi$ and $\Gamma, \psi \lor \chi \vdash_L \chi$ and the rest is trivial.
Lemma 2.28 (Semilinearity Property)

If $\Gamma, \varphi \rightarrow \psi \vdash_L \chi$ and $\Gamma, \psi \rightarrow \varphi \vdash_L \chi$, then $\Gamma \vdash_L \chi$.

Proof.

By the Proof by Cases Property and axiom (Prl).
Linear Extensions Property

Definition 2.29
A theory $\Gamma$ is linear if $\Gamma \vdash \varphi \rightarrow \psi$ or $\Gamma \vdash \psi \rightarrow \varphi$ for each $\varphi, \psi$.

Lemma 2.30 (Linear Extension Property)

If $\Gamma \not\vdash \varphi$, then there is linear theory $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \not\vdash \varphi$.

Proof.
The same as in the case of Gödel–Dummett logic.

\[
\square
\]
Linear Extensions Property

**Definition 2.29**

A theory $\Gamma$ is **linear** if $\Gamma \vdash_L \varphi \rightarrow \psi$ or $\Gamma \vdash_L \psi \rightarrow \varphi$ for each $\varphi, \psi$.

**Lemma 2.30 (Linear Extension Property)**

If $\Gamma \nvdash_L \varphi$, then there is linear theory $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \nvdash_L \varphi$.

**Proof.**

Enumerate all pairs of formulas: $\langle \varphi_0, \psi_0 \rangle, \langle \psi_1, \varphi_1 \rangle, \ldots$

Construct theories $\Gamma_0, \Gamma_1, \ldots$ s.t. $\Gamma_0 = \Gamma; \Gamma_i \subseteq \Gamma_{i+1}$, and $\Gamma_i \nvdash_L \varphi$:

- **if** $\Gamma_i, \varphi_i \rightarrow \psi_i \nvdash_L \varphi$, then $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i \rightarrow \psi_i\}$
- **if** $\Gamma_i, \varphi_i \rightarrow \psi_i \vdash_L \varphi$, then $\Gamma_{i+1} = \Gamma_i \cup \{\psi_i \rightarrow \varphi_i\}$

Clearly $\Gamma_{i+1} \nvdash_L \varphi$ (the 1st is obvious, in the 2nd would $\Gamma_{i+1} \vdash_L \varphi$ entail $\Gamma_i \vdash_L \varphi$ by the Semilinearity Property, a contradiction with the IH.

Define $\Gamma' = \bigcup \Gamma_i$. Clearly $\Gamma'$ is linear, $\Gamma' \supseteq \Gamma$, and $\Gamma' \nvdash_L \varphi$. □
Theorem 2.19

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_L$:

1. $\Gamma \vdash_L \varphi$
2. $\Gamma \models_{\text{MV}} \varphi$
3. $\Gamma \models_{\text{MV}_{lin}} \varphi$

If $\Gamma$ is finite we can add:

4. $\Gamma \models_{[0,1]_L} \varphi$

The proof of the equivalence of the first three claims is the same as in the case of Gödel–Dummett logic.

We give a proof of 4. implies 1. but first . . .
MV-algebras and LOAGs

A lattice ordered Abelian group (LOAG for short) is a structure \( \langle G, +, 0, -, \leq \rangle \) s.t. \( \langle G, +, 0, - \rangle \) is an Abelian group and:

(i) \( \langle G, \leq \rangle \) is a lattice,

(ii) if \( x \leq y \), then \( x + z \leq y + z \) for all \( z \in G \).

A strong unit \( u \) is an element s.t.

\[
\forall x \in G \exists n \in N (x \leq nu)
\]

For LOAG \( G = \langle G, +, 0, -, \leq \rangle \) and strong unit \( u \) we define algebra

\[
\Gamma(G, u) = \langle [0, u], \oplus, \neg, \overline{0} \rangle,
\]

where \( x \oplus y = \min\{u, x + y\} \), \( \neg x = u - x \), \( \overline{0} = 0 \).

By \( R \) we denote the additive LOAG of reals.

**Proposition 2.31**

\( \Gamma(G, u) \) is an MV-algebra and for each \( u > 0 \), \( \Gamma(R, u) \) is isomorphic to the standard MV-algebra \( [0, 1]_L \).
The proof of the standard completeness theorem

If $\Gamma \not\models_{L} \varphi$ we know that there is a countable MV-chain $B$ s.t. $\Gamma \not\models_{B} \varphi$. Let $x_1, \ldots, x_n$ be variables occurring in $\Gamma \cup \{\varphi\}$. Then:

$$\not\models_{B} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma} (\psi \approx 1) \Rightarrow (\varphi \approx 1)$$

Let us define an algebra $B' = \langle Z \times B, +, -, 0 \rangle$ as:

$$\langle i, x \rangle + \langle j, y \rangle = \begin{cases} 
\langle i+j, x \oplus y \rangle & \text{if } x \& y = 0 \\
\langle i+j+1, x \& y \rangle & \text{otherwise}
\end{cases}$$

$$-\langle i, x \rangle = \langle -i - 1, \neg x \rangle \text{ and } 0 = \langle 0, \overline{0} \rangle$$

**Proposition 2.32**

$B'$ is a LOAG and $B = \Gamma(B', \langle 1, \overline{0} \rangle)$.
The proof of the standard completeness theorem

Let us fix an extra variable $u$, we define a translation of MV-terms into LOAG-terms:

$$x' = x, \quad 0' = 0, \quad (\neg t)' = u - t', \quad (t_1 \oplus t_2)' = (t'_1 + t'_2) \land u.$$

Recall that we have:

$$\not\models_B (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma} (\psi \approx \bar{1}) \Rightarrow (\varphi \approx \bar{1}),$$

Thus also:

$$\not\models_{B'} (\forall u)(\forall x_1, \ldots, x_n)[(0 < u) \land \bigwedge_{i \leq n} (x_i \leq u) \land (0 \leq x_i) \land \bigwedge_{\psi' \approx u} (\psi' \approx u) \Rightarrow (\varphi' \approx u)]$$
The proof of the standard completeness theorem

Gurevich–Kokorin theorem: each $\forall_1$-sentence of LOAGs is true in additive LOAG of reals iff it is true in all linearly ordered LOAGs. Thus

$$\not\models_R (\forall u)(\forall x_1, \ldots, x_n)[(0 < u) \land \bigwedge_{i \leq n}(x_i \leq u) \land (0 \leq x_i) \land \bigwedge_{\psi \in \Gamma} (\psi' \approx u) \Rightarrow (\varphi' \approx u)]$$

And so

$$\not\models_{\Gamma (R,u)} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma} (\psi \approx 1) \Rightarrow (\varphi \approx 1)$$

And so

$$\not\models_{[0,1]_L} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in \Gamma} (\psi \approx 1) \Rightarrow (\varphi \approx 1)$$

i.e., $\Gamma \not\models_{[0,1]_L} \varphi$
Non-theorem

For every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_\mathcal{L}$ we have:

$$\Gamma \vdash_\mathcal{L} \varphi \text{ if, and only if, } \Gamma \models_{[0,1]_\mathcal{L}} \varphi.$$
Failure of standard completeness for infinite theories

Non-theorem

For every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_L$ we have:

$$\Gamma \vdash_L \varphi \text{ if, and only if, } \Gamma \models_{[0,1]_L} \varphi.$$ 

Consider theory $\Gamma = \{(p \oplus .^n \oplus p) \to q \mid n \geq 1\} \cup \{\neg p \to q\}$. 
Non-theorem

For every set of formulas $\Gamma \cup \{ \varphi \} \subseteq Fm_\mathcal{L}$ we have:

$$\Gamma \vdash_\mathcal{L} \varphi \text{ if, and only if, } \Gamma \models_{[0,1]_\mathcal{L}} \varphi.$$ 

- Consider theory $\Gamma = \{(p \oplus ... \oplus p) \rightarrow q \mid n \geq 1\} \cup \{\lnot p \rightarrow q\}$.
- Note that for any $[0, 1]_\mathcal{L}$-evaluation $e$ s.t. $e[\Gamma] = \{1\}$ we have $e(q) = 1$ and so $\Gamma \models_{[0,1]_\mathcal{L}} q$. 

Failure of standard completeness for infinite theories
Failure of standard completeness for infinite theories

**Non-theorem**

For every set of formulas $\Gamma \cup \{ \varphi \} \subseteq Fm_{\mathcal{L}}$ we have:

$$\Gamma \vdash_{\mathcal{L}} \varphi \text{ if, and only if, } \Gamma \models_{[0,1]_{\mathcal{L}}} \varphi.$$ 

- Consider theory $\Gamma = \{(p \oplus .n \oplus p) \rightarrow q \mid n \geq 1\} \cup \{\neg p \rightarrow q\}$.
- Note that for any $[0,1]_{\mathcal{L}}$-evaluation $e$ s.t. $e[\Gamma] = \{1\}$ we have $e(q) = 1$ and so $\Gamma \models_{[0,1]_{\mathcal{L}}} q$.
- Thus by our *Non-theorem* $\Gamma \vdash_{\mathcal{L}} q$ and as proofs are finite, there must be a finite $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \vdash_{\mathcal{L}} q$.
- Thus by our *Non-theorem* $\Gamma_0 \models_{[0,1]_{\mathcal{L}}} q$.
Failure of standard completeness for infinite theories

Non-theorem

For every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_\mathcal{L}$ we have:

$$\Gamma \vdash_\mathcal{L} \varphi \text{ if, and only if, } \Gamma \models_{[0,1]_\mathcal{L}} \varphi.$$  

- Consider theory $\Gamma = \{(p \oplus . n \oplus p) \to q \mid n \geq 1\} \cup \{\neg p \to q\}$.
- Note that for any $[0, 1]_\mathcal{L}$-evaluation $e$ s.t. $e[\Gamma] = \{1\}$ we have $e(q) = 1$ and so $\Gamma \models_{[0,1]_\mathcal{L}} q$.
- Thus by our Non-theorem $\Gamma \vdash_\mathcal{L} q$ and as proofs are finite, there must be a finite $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \vdash_\mathcal{L} q$.
- Thus by our Non-theorem $\Gamma_0 \models_{[0,1]_\mathcal{L}} q$.
- Let $n$ be the maximal $n$ s.t. $(p \oplus . n \oplus p) \to q \in \Gamma_0$.
- $[0, 1]_\mathcal{L}$-evaluation $e(p) = \frac{1}{n+1}$ and $e(q) = \frac{n}{n+1}$ yields a contradiction.
Outline

1. Completeness of Gödel–Dummett logic
2. Completeness of Łukasiewicz logic
3. Functional representation
4. Finite model property
5. Computational complexity
6. Algebraizability of Gödel–Dummett and Łukasiewicz logics
7. Axiomatic extensions of Gödel–Dummett and Łukasiewicz logics
8. Application: Fuzzy Logic and Probability
The classical case

Theorem 2.33 (Functional completeness)

Every Boolean function (i.e. any function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ for some $n \geq 1$) is representable by some formula of classical logic.
The fuzzy case

Let $L$ be either $L$ of $G$.

**Definition 2.34**

A function $f : [0, 1]^n \rightarrow [0, 1]$ is *represented* by a formula $\varphi(v_1, \ldots, v_n)$ in $L$ if $e(\varphi) = f(e(v_1), e(v_2), \ldots, e(v_n))$ for each $[0, 1]_L$-evaluation $e$.

**Definition 2.35**

The *functional representation* of $L$ is the set $\mathcal{F}_L$ of all functions from any power of $[0, 1]$ into $[0, 1]$ that are represented in $L$ by some formula.
Relation with Lindenbaum–Tarski algebra

Let us fix $L = \mathbb{L}$.
Let $f_i$ be functions of $n_i$ variables, $i \in \{1, 2\}$. We say that $f_1 = f_2$ iff $f_1(x_1, x_2, \ldots, x_{n_1}) = f_2(x_1, x_2, \ldots, x_{n_2})$ for every $x_j \in [0, 1]$. Let us for each $f \in \mathcal{F}_L$ define a class

$$[f] = \{ g \in \mathcal{F}_L \mid f = g \} \quad F = \{ [f] \mid f \in \mathcal{F}_L \}$$

We define an MV-algebra $F$ with domain $F$ and operations:

$$0^F = [0] \quad \neg^F [f] = [1 - f]_T \quad [f] \oplus^F [g] = [\min\{1, f + g\}]$$
Relation with Lindenbaum–Tarski algebra

Let us fix $L = \mathcal{L}$. Let $f_i$ be functions of $n_i$ variables, $i \in \{1, 2\}$. We say that $f_1 = f_2$ iff $f_1(x_1, x_2, \ldots, x_{n_1}) = f_2(x_1, x_2, \ldots, x_{n_2})$ for every $x_j \in [0, 1]$. Let us for each $f \in \mathcal{F}_L$ define a class

$$[f] = \{ g \in \mathcal{F}_L \mid f = g \}$$

$$F = \{ [f] \mid f \in \mathcal{F}_L \}$$

We define an MV-algebra $F$ with domain $F$ and operations:

$$0^F = [0] \quad \neg^F [f] = [1 - f]_T \quad [f] \oplus^F [g] = [\min\{1, f + g\}]$$

**Theorem 2.36**

The algebras $F$ and $\text{Lind}_\emptyset$ are isomorphic.

In the case of $G$, the definitions and the result are analogous.
A proof

Let the atoms be enumerated as \( v_1, v_2, \ldots \). Any formula with variables with maximal index \( n \) is viewed as formula in variables \( v_1, \ldots, v_n \).

We define the homomorphism:

\[
g : L_\emptyset \to F \text{ as } g([\varphi]) = [f_\varphi] \text{ where } f_\varphi \text{ is the function represented by } \varphi.
\]

Then:

- the definition is sound and \( g \) is one-one: \( [\varphi] = [\psi] \iff \vdash_L \varphi \leftrightarrow \psi \) iff (due to the standard completeness theorem) \( e(\varphi) = e(\psi) \) for each \([0, 1]_L\)-evaluation \( e \) iff \([f_\varphi] = [f_\psi] \).
- \( g \) is a homomorphism:
  \[
g([\varphi] \oplus [\psi]) = g([\varphi \oplus \psi]) = [f_{\varphi \oplus \psi}] = [f_\varphi \oplus f_\psi] = [f_\varphi] \oplus [f_\psi].
\]
- \( g \) is onto (obvious).
How do the functions from $\mathcal{F}_L$ look like?

Observations

- they are all continuous
How do the functions from $\mathcal{F}_L$ look like?

**Observations**

- they are all continuous
- they are piece-wise linear

Definition 2.37

A McNaughton function $f : [0, 1]^n \rightarrow [0, 1]$ is a continuous piece-wise linear function, where each of the pieces has integer coefficients.

Theorem 2.38 (McNaughton theorem)

$\mathcal{F}_L$ is the set of all McNaughton functions.
How do the functions from $\mathcal{F}_L$ look like?

**Observations**
- they are all continuous
- they are piece-wise linear
- all pieces have integer coefficients
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A lemma

Lemma 2.39

Let $f : [0, 1]^n \to \mathbb{R}$ be an integer linear polynomial, i.e. of the form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i + b$$

for some $a_1, \ldots, a_n, b \in \mathbb{Z}$

Then there is a formula $\varphi_f$ representing the function

$$f^\# = \max\{0, \min\{1, f\}\}.$$

Proof.

By induction on $m = \sum_{i=1}^{n} |a_i|$. If $m = 0$ then $f^\#$ is either constantly 0 or 1, then we can take as $\varphi$ either the term $\bar{0}$ or $\bar{1}$, respectively. Assume now $m > 0$ and let $a_j$ be s.t. $|a_j| = \max_{i=1}^{n} |a_i|$. WLOG we can assume $a_j > 0$: indeed otherwise we consider $f' = 1 - f$, here $a_j > 0$ and so we have $\varphi_{1-f}$. Note that clearly $\varphi_f = \neg \varphi_{1-f}$. ...
A lemma: continuation of the proof

Let us consider the function \( g = f - x_j \): by IH we have formulas \( \varphi_g \) and \( \varphi_{g+1} \). If we show that

\[
(g + x_j)^\# = (g^\# \oplus x_j) \& (g + 1)^\#
\]

the proof is done as:

\[
\varphi_f = \varphi_{g+x_j} = (\varphi_g \oplus x_j) \& \varphi_{g+1}.
\]

So we need to prove (2.1). Let \( L \) and \( R \) be its left/right side:

- if \(|g(\bar{x})| > 1\) then \( L = R = 1 \) or \( L = R = 0 \)
- \( 0 \leq g(\bar{x}) \leq 1 \) then \( L = \min\{1, g(\bar{x}) + x_j\} \), \( g(\bar{x}) = g^\#(\bar{x}) \) and \( (g + 1)^\#(\bar{x}) = 1 \). Hence \( R = g(\bar{x}) \oplus x_j = \min\{1, g(\bar{x}) + x_j\} = L \).
- \( -1 \leq g(\bar{x}) \leq 0 \) then \( L = \max\{0, g(\bar{x}) + x_j\} \), \( g^\#(\bar{x}) = 0 \) and \( (g + 1)^\#(\bar{x}) = g(\bar{x}) + 1 \). Hence \( g^\#(\bar{x}) \oplus x_j = x_j \) and so \( R = \max\{0, x_j + g(\bar{x}) + 1 - 1\} = \max\{0, x_j + g(\bar{x})\} = L \).
# The proof for one variable functions

**Definition 2.40**

Let \( a, b \in [0, 1] \cap \mathbb{Q} \). Then any McNaughton function \( f \) s.t. \( f(x) = 1 \) iff \( x \in [a, b] \) is called *pseudo characteristic function* of interval \([a, b] \).

**Exercise 6**

Prove that each interval has a pseudo characteristic function and find a formula representing it. Hint: use Lemma 2.39.

**Lemma 2.41**

Let \( a, b \in [0, 1] \cap \mathbb{Q} \). Then for each \( \epsilon > 0 \) there is a pseudo characteristic function of the interval \([a, b]\), s.t. \( f(x) = 0 \) for \( x \in [0, a - \epsilon] \cup [b + \epsilon, 1] \).

**Proof.**

If \( f \) is a pseudo char. function of some interval, so is \( f^n \) for each \( n \).
The proof for one variable functions

Let $p$ be a McNaughton function of one variable given by $n$ integer linear polynomials $p_1, \ldots, p_n$. For each $i \in \{1, 2, \ldots, n\}$ let $P_i = [a_i, b_i]$ be the interval in which $p$ uses $p_i$. Note that:

- $[0, 1] = \bigcup_i P_i$
- $a_i, b_i \in [0, 1] \cap \mathbb{Q}$
- there is a pseudo characteristic function $f_i$ of $[a_i, b_i]$ such that $p(x) \geq (f_i \& p_i^\#)(x)$ for each $x \notin P_i$.

Then

$$p(x) = \bigvee_i (f_i \& p_i^\#)(x)$$

and thus $\varphi_p = \bigvee_i \varphi_{f_i} \& \varphi_{p_i}$.
Outline

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8. Application: Fuzzy Logic and Probability
The classical case, FMP and decidability

CL is complete with respect to a finite algebra, 2.

**Definition 2.42**

A logic has the **finite model property** (FMP) if it is complete with respect to a set of finite algebras.

From the FMP, we obtain **decidability**:

- Thanks to our finite notion of proof, the set of theorems is recursively enumerable.
- Thanks to FMP, the set of non-theorems is also recursively enumerable (we can check validity in bigger and bigger finite algebras until we find a countermodel).
- Therefore, theoremhood is a decidable problem.
- Note: provability from finitely-many premises is also decidable (using deduction theorem).
Finite chains

**Lemma 2.43**

Let $A_2$ be a subalgebra of an $MV$- or $G$-algebra $A_1$. Then $\models_{A_1} \subseteq \models_{A_2}$.

**Exercise 7**

(a) Prove that each $n$-valued $G$-chain is isomorphic to the subalgebra $G_n$ of $[0, 1]_G$ with the domain $\{\frac{i}{n-1} \mid i \leq n - 1\}$.

(b) Prove that each $n$-valued $MV$-chain is isomorphic to the subalgebra $\mathcal{L}_n$ of $[0, 1]_\mathcal{L}$ with the domain $\{\frac{i}{n-1} \mid i \leq n - 1\}$.

**Lemma 2.44**

\[ \models_{G_m} \subseteq \models_{G_n} \text{ iff } n \leq m. \]
\[ \models_{\mathcal{L}_m} \subseteq \models_{\mathcal{L}_n} \text{ iff } n - 1 \text{ divides } m - 1. \]

Let us denote by $\mathbb{L}_{\text{fin}}$ the class of finite $L$-chains.
The case of Gödel–Dummett logic

Theorem 2.45

Let $\varphi$ be a formula with $n - 2$ variables. Then: $\vdash_G \varphi$ iff $\models_{G_n} \varphi$.

Proof.

Contrapositively: assume that $\nvdash_G \varphi$ and let $e$ be a $[0, 1]_G$-evaluation s.t. $e(\varphi) \neq 1$. Let $X = \{0, 1\} \cup \{e(v_i) \mid 1 \leq i \leq n - 2\}$ and note that it is a subuniverse of $[0, 1]_G$, thus $e$ can be seen as an $X$-evaluation and so $\not\models_X \varphi$. The previous exercise and lemma complete the proof.

Theorem 2.46

For every finite set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_L$, TFAE:

1. $\Gamma \vdash_G \varphi$
2. $\Gamma \models_{[0,1]_G} \varphi$
3. $\Gamma \models_{G_{\text{fin}}} \varphi$
The case of Łukasiewicz logic

Theorem 2.47

For every finite set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_L$, TFAE:

1. $\Gamma \vdash_L \varphi$
2. $\Gamma \models_{[0,1]_L} \varphi$
3. $\Gamma \models_{MV_{\text{fin}}} \varphi$

Proof: we show it for one variable $\nu$.

Let us define the set $E$ of $[0,1]_L$-evaluations s.t. $e[\Gamma] \subseteq \{1\}$. Note that $E$ can be seen as a union of real intervals. Assume that there is $e \in E$ s.t. $e(\varphi) \neq 1$. If we show that there is an evaluation $f \in E$, s.t. $f(\nu) = \frac{p}{n-1}$ and $f(\varphi) \neq 1$ we are done as $f$ can be seen as $L_n$-evaluation.

- Either $e$ lies on the border of some interval, then $f = e$ OR
- there has to be a neighborhood $X \subseteq E$ s.t. $f(\varphi) \neq \overline{1}$ for each $f \in X$, then there has to be such $f$. 

\qed
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The classical case

- $\varphi \in \text{SAT}(\text{CL})$ if there is a 2-evaluation $e$ such that $e(\varphi) = 1$.
- $\varphi \in \text{TAUT}(\text{CL})$ if for each 2-evaluation $e$ holds $e(\varphi) = 1$.

Recall:

- $\varphi \in \text{TAUT}(\text{CL})$ iff $\neg \varphi \notin \text{SAT}(\text{CL})$
- $\varphi \in \text{SAT}(\text{CL})$ iff $\neg \varphi \notin \text{TAUT}(\text{CL})$.

Both problems, $\text{SAT}(\text{CL})$ and $\text{TAUT}(\text{CL})$, are decidable.

But how difficult are their computations?
Complexity classes

\( f, g : \mathbb{N} \to \mathbb{N}. f \in O(g) \) iff there are \( c, n_0 \in \mathbb{N} \) such that for each \( n \geq n_0 \) we have \( f(n) \leq c \cdot g(n) \).

- **TIME**\((f)\): the class of problems \( P \) such that there is a deterministic Turing machine \( M \) that accepts \( P \) and operates in time \( O(f) \).
- **NTIME**\((f)\): analogous class for nondeterministic Turing machines.
- **SPACE**\((f)\): the class of problems \( P \) such that there is a deterministic Turing machine \( M \) that accepts \( P \) and operates in space \( O(f) \).
- **NSPACE**\((f)\): the analogous class for nondeterministic Turing machines.
Complexity classes

\[
P = \bigcup_{k \in \mathbb{N}} \text{TIME}(n^k)
\]

\[
\text{NP} = \bigcup_{k \in \mathbb{N}} \text{NTIME}(n^k)
\]

\[
\text{PSPACE} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(n^k)
\]

If \( C \) is a complexity class, we denote \( \text{co}C = \{ P | \overline{P} \in C \} \), the class of complements of problems in \( C \).
Complexity classes

- Each deterministic complexity class $C$ is closed under complementation: if $P \in C$, then also $\overline{P} \in C$.
- Is $NP$ closed under complementation?
- $P \subseteq NP$, $P \subseteq coNP$, $NP \subseteq PSPACE$.
- Are the inclusions $P \subseteq NP \subseteq PSPACE$ proper?
- Each of the classes $P$, $NP$, $coNP$, and $PSPACE$ is closed under finite unions and intersections.
Complexity classes

A problem $P$ is said to be **C-hard** iff any decision problem $P'$ in $C$ is reducible to $P$.

A problem $P$ is **C-complete** iff $P$ is C-hard and $P \in C$. 
The classical case

- **SAT(\(CL\)) \in \textbf{NP}:** guess an evaluation and check whether it satisfies the formula (a polynomial matter).

- **TAUT(\(CL\)) \in \textbf{coNP}:** \(\varphi \in \text{TAUT}(\text{CL})\) iff \(\neg \varphi \notin \text{SAT}(\text{CL})\).

- **Cook Theorem:** Let \(\text{SAT}^{\text{CNF}}(\text{CL})\) be the SAT problem for formulas in conjunctive normal form. Then: \(\text{SAT}^{\text{CNF}}(\text{CL})\) is \textbf{NP}-complete.

- **\(\text{SAT}^{\text{CNF}}(\text{CL})\)** is a fragment of \(\text{SAT}(\text{CL})\), therefore **\(\text{SAT}(\text{CL})\)** is \textbf{NP}-complete and **\(\text{TAUT}(\text{CL})\)** is \textbf{coNP}-complete.
The fuzzy case: basic definitions

Let \( L \) be either Łukasiewicz logic \( \mathcal{L} \) or Gödel logic \( G \). We define:

- \( \varphi \in \text{SAT}(L) \) if there is an evaluation \( e \) such that \( e(\varphi) = 1 \).
- \( \varphi \in \text{SAT}_{\text{pos}}(L) \) if there is an evaluation \( e \) such that \( e(\varphi) > 0 \).
- \( \varphi \in \text{TAUT}(L) \) if for each evaluation \( e \) holds \( e(\varphi) = 1 \).
- \( \varphi \in \text{TAUT}_{\text{pos}}(L) \) if for each evaluation \( e \) holds \( e(\varphi) > 0 \).

Note that \( \varphi \lor \neg \varphi \in \text{TAUT}_{\text{pos}}(L) \) but \( \varphi \lor \neg \varphi \not\in \text{TAUT}(L) \).

Note that \( \varphi \land \neg \varphi \in \text{SAT}_{\text{pos}}(\mathcal{L}) \) but \( \varphi \land \neg \varphi \not\in \text{SAT}(\mathcal{L}) \).
The fuzzy case: basic reductions

Lemma 2.48

Let $L$ be either Łukasiewicz logic $Ł$ or Gödel logic $G$. Then

$\varphi \in \text{TAUT}_{\text{pos}}(L)$ iff $\neg \varphi \notin \text{SAT}(L)$

$\varphi \in \text{SAT}_{\text{pos}}(L)$ iff $\neg \varphi \notin \text{TAUT}(L)$.

Lemma 2.49

$\varphi \in \text{SAT}(Ł)$ iff $\neg \varphi \notin \text{TAUT}_{\text{pos}}(Ł)$

$\varphi \in \text{TAUT}(Ł)$ iff $\neg \varphi \notin \text{SAT}_{\text{pos}}(Ł)$.

Exercise 8

Prove the above two lemmata, show that the last equivalence fails for $G$ and the one but last holds there. (Hint: for the last part use properties of these sets proved in the next few slides).
The case of Łukasiewicz logic

**Theorem 2.50**

*The sets* $\text{SAT}(\mathcal{L})$ *and* $\text{SAT}^{\text{pos}}(\mathcal{L})$ *are NP-complete. Therefore the sets* $\text{TAUT}(\mathcal{L})$ *and* $\text{TAUT}^{\text{pos}}(\mathcal{L})$ *are coNP-complete.*

We prove it in a series of lemmata. First we show that $\text{SAT}(\mathcal{L})$ is NP-hard:

**Lemma 2.51**

*Let* $\varphi$ *be a formula with variables* $p_1, \ldots, p_n$.*

\[
\varphi \in \text{SAT}(\text{CL}) \iff \varphi \land \bigwedge_{i=1}^{n} (p_i \lor \lnot p_i) \in \text{SAT}(\mathcal{L}).
\]
Lemma 2.52

Let $\varphi$ be a formula with variables $p_1, \ldots, p_n$ built using: $\land, \lor, \neg$.

\[
\varphi \in \text{SAT(CL)} \iff \varphi^2 \land \bigwedge_{i=1}^{n} (p_i \lor \neg p_i)^2 \in \text{SAT}_{\text{pos}}(\mathcal{L}).
\]

Proof.

Let $e$ positively satisfy the right-hand formula. Then $e((p_i \lor \neg p_i)^2) > 0$ ergo $e(p_i) \neq 0.5$. We define the evaluation

\[
e'(p_i) = \begin{cases} 
1 & \text{if } e(p_i) > 0.5 \\
0 & \text{if } e(p_i) < 0.5
\end{cases}
\]

Clearly this can be extended to $\varphi$. And, since $e(\varphi^2) > 0$, we have $e(\varphi) > 0.5$ and so $e'(\varphi) = 1$. \qed
**Lemma 2.53**

\[ e(\varphi \rightarrow \psi) \geq r \quad \text{IFF} \quad \exists i, j \in [0, 1] \quad e(\varphi) \leq i \quad e(\psi) \geq j \quad r + i - j \leq 1 \]

\[ e(\varphi \rightarrow \psi) \leq r \quad \text{IFF} \quad \exists i, j \in [0, 1], y \in \{0, 1\} \quad e(\varphi) \geq i \quad e(\psi) \leq j \quad y - r \leq 0 \quad y + i \leq 1 \quad y - j \leq 0 \quad y + r + i - j \geq 1 \]

Using this lemma we can reduce the question of (positive) satisfiability to the question of Mixed Integer Programming (MIP) which is known to be in \( \text{NP} \):

For \( \text{SAT}(\mathcal{L}) \) start with \( e(\varphi) \geq 1 \) for \( \text{SAT}_{\text{pos}}(\mathcal{L}) \) start with \( e(\varphi) \geq i_0 \) where \( i_0 > 0 \).
Lemma 2.54

The mapping \( f : [0, 1] \rightarrow \{0, 1\} \) defined as \( f(0) = 0 \) and \( f(x) = 1 \) if \( x \neq 0 \) is a homomorphism from \([0, 1]_G\) to 2.

Corollary 2.55

\[ \text{SAT}_{\text{pos}}(G) \subseteq \text{SAT}(\text{CL}) \quad \text{TAUT}(\text{CL}) \subseteq \text{TAUT}_{\text{pos}}(G). \]
The case of Gödel–Dummett logic

**Corollary 2.56**

\[
\phi \in \text{SAT}_{\text{pos}}(G) \quad \text{iff} \quad \phi \in \text{SAT}(G) \quad \text{iff} \quad \phi \in \text{SAT}(\text{CL})
\]

\[
\phi \in \text{TAUT}_{\text{pos}}(G) \quad \text{iff} \quad \neg \neg \phi \in \text{TAUT}(G) \quad \text{iff} \quad \phi \in \text{TAUT}(\text{CL})
\]

**Proof.**

Just observe that:

\[
\text{SAT}(G) \subseteq \text{SAT}_{\text{pos}}(G) \subseteq \text{SAT}(\text{CL}) \subseteq \text{SAT}(G).
\]

And that

\[
\phi \in \text{TAUT}_{\text{pos}}(G) \Rightarrow \neg \phi \notin \text{SAT}(G) \Rightarrow \neg \phi \notin \text{SAT}_{\text{pos}}(G)
\]

\[
\Rightarrow \neg \neg \phi \in \text{TAUT}(G) \Rightarrow \phi \in \text{TAUT}(\text{CL}) \Rightarrow \phi \in \text{TAUT}_{\text{pos}}(G).
\]
The case of Gödel–Dummett logic

Corollary 2.56

\[ \varphi \in \text{SAT}_{\text{pos}}(G) \iff \varphi \in \text{SAT}(G) \iff \varphi \in \text{SAT}(\text{CL}) \]
\[ \varphi \in \text{TAUT}_{\text{pos}}(G) \iff \neg\neg\varphi \in \text{TAUT}(G) \iff \varphi \in \text{TAUT}(\text{CL}) \]

Theorem 2.57

The sets \( \text{SAT}(G) \) and \( \text{SAT}_{\text{pos}}(G) \) are \( \text{NP} \)-complete and the sets \( \text{TAUT}(G) \) and \( \text{TAUT}_{\text{pos}}(G) \) are \( \text{coNP} \)-complete.

Proof.

The only non clear case is \( \text{TAUT}(G) \): it is \( \text{coNP} \)-hard due to the last reduction of the previous corollary. We present a non-deterministic polynomial ‘algorithm’ (sound due to Theorem 2.46) for \( Fm_{\mathcal{L}} \setminus \text{TAUT}(G) \):

Step 1: guess a \( G_n \)-evaluation \( e \) (assuming that \( \varphi \) has \( n - 2 \) variables)

Step 2: compute the value of \( e(\varphi) \) (clearly in polynomial time)

Output: if \( e(\varphi) \neq 1 \) output \( \varphi \notin \text{TAUT}(G) \).
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Equational consequence

An equation in the language \( \mathcal{L} \) is a formal expression of the form \( \varphi \approx \psi \), where \( \varphi, \psi \in Fm_{\mathcal{L}} \).

We say that an equation \( \varphi \approx \psi \) is a consequence of a set of equations \( \Pi \) w.r.t. a class \( K \) of \( \mathcal{L} \)-algebras if for each \( A \in K \) and each \( A \)-evaluation \( e \) we have \( e(\varphi) = e(\psi) \) whenever \( e(\alpha) = e(\beta) \) for each \( \alpha \approx \beta \in \Pi \); we denote it by \( \Pi \models_K \varphi \approx \psi \).

A quasiequation in the language \( \mathcal{L} \) is a formal expression of the form \( (\bigwedge_{i=1}^{n} \varphi_i \approx \psi_i) \Rightarrow \varphi \approx \psi \), where \( \varphi_1, \ldots, \varphi_n, \varphi, \psi_1, \ldots, \psi_n, \psi \in Fm_{\mathcal{L}} \).
## Varieties and quasivarieties

<table>
<thead>
<tr>
<th>Type of class</th>
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<tr>
<td>variety</td>
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<td>quasivariety</td>
<td>quasiequations</td>
<td>I, S, P, and P_U</td>
</tr>
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</table>

**I** isomorphic images

**H** homomorphic images

**S** subalgebras

**P** direct products

**P_U** ultraproducts

**V** generated variety

**Q** generated quasivariety
Algebraization of Łukasiewicz logic

1. For every $\Gamma \cup \{ \varphi \} \subseteq Fm_L$,
   $$\Gamma \vdash_L \varphi \iff \{ \psi \approx 1 \mid \psi \in \Gamma \} \models_{MV} \varphi \approx 1$$

2. For every set of equations $\Pi \cup \{ \varphi \approx \psi \}$,
   $$\Pi \models_{MV} \varphi \approx \psi \iff \{ \alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi \} \vdash_L \varphi \leftrightarrow \psi$$

3. For every $\varphi \in Fm_L$,
   $$\varphi \vdash_L \varphi \leftrightarrow 1 \text{ and } \varphi \leftrightarrow 1 \vdash_L \varphi$$

4. For every $\varphi, \psi \in Fm_L$,
   $$\varphi \approx \psi \models_{MV} \varphi \leftrightarrow \psi \approx 1 \text{ and } \varphi \leftrightarrow \psi \approx 1 \models_{MV} \varphi \approx \psi$$

Translations:

- $\tau : \varphi \mapsto \varphi \approx 1$
- $\rho : \alpha \approx \beta \mapsto \alpha \leftrightarrow \beta$

MV-algebras are the equivalent algebraic semantics of $\mathbb{L}$. 
\textbf{MV} is a variety

\textbf{MV} is a variety of algebras, i.e. an equational class:

\begin{enumerate}
  \item \(x \oplus (y \oplus z) \approx (x \oplus y) \oplus z,\)
  \item \(x \oplus y \approx y \oplus x,\)
  \item \(x \oplus \overline{0} \approx x,\)
  \item \(\neg \neg x \approx x,\)
  \item \(x \oplus \neg \overline{0} \approx \neg \overline{0},\)
  \item \(\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x.\)
\end{enumerate}
Algebraization of Gödel–Dummett logic

1. For every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$,
   $$\Gamma \vdash_{G} \varphi \iff \{\psi \approx \overline{1} \mid \psi \in \Gamma\} \models_{G} \varphi \approx \overline{1}$$

2. For every set of equations $\Pi \cup \{\varphi \approx \psi\}$,
   $$\Pi \models_{G} \varphi \approx \psi \iff \{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{G} \varphi \leftrightarrow \psi$$

3. For every $\varphi \in Fm_{\mathcal{L}}$,
   $$\varphi \vdash_{G} \varphi \leftrightarrow \overline{1} \text{ and } \varphi \leftrightarrow \overline{1} \vdash_{G} \varphi$$

4. For every $\varphi, \psi \in Fm_{\mathcal{L}}$,
   $$\varphi \approx \psi \models_{G} \varphi \leftrightarrow \psi \approx \overline{1} \text{ and } \varphi \leftrightarrow \psi \approx \overline{1} \models_{G} \varphi \approx \psi$$

Translations:

- $\tau : \varphi \mapsto \varphi \approx \overline{1}$
- $\rho : \alpha \approx \beta \mapsto \alpha \leftrightarrow \beta$

$G$-algebras are the equivalent algebraic semantics of $G$. 
$\mathcal{G}$ is a variety

$\mathcal{G}$ is a variety of algebras, i.e. an equational class:

\begin{align*}
\text{E1} & \quad x \rightarrow x \approx \overline{1} \\
\text{E2} & \quad \overline{1} \rightarrow x \approx x \\
\text{E3} & \quad x \rightarrow (y \rightarrow z) \approx (x \rightarrow y) \rightarrow (x \rightarrow z) \\
\text{E4} & \quad (x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow y) \approx (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow x) \\
\text{E5} & \quad x \rightarrow x \lor y \approx \overline{1}, \quad y \rightarrow x \lor y \approx \overline{1} \\
\text{E6} & \quad (x \rightarrow y) \rightarrow (((y \rightarrow z) \rightarrow (x \lor y \rightarrow z)) \approx \overline{1} \\
\text{E7} & \quad x \land y \rightarrow x \approx \overline{1}, \quad x \land y \rightarrow y \approx \overline{1} \\
\text{E8} & \quad (x \rightarrow y) \rightarrow ((x \rightarrow z) \rightarrow (x \rightarrow y \land z)) \approx \overline{1} \\
\text{E9} & \quad \overline{0} \rightarrow x \approx \overline{1} \\
\text{E10} & \quad (x \rightarrow y) \lor (y \rightarrow x) \approx \overline{1}
\end{align*}
Algebraization of finitary extensions

Let $L$ be $L$ or $G$.

- $S = L + Ax + R$ ($Ax$ is a set of axioms and $R$ a set of finitary rules)
- $S = \{ A \in L \mid A \text{ satisfies } \tau(\varphi) \text{ for each } \varphi \in Ax \text{ and } \bigwedge_{i=1}^{n} \tau(\varphi_i) \Rightarrow \tau(\psi) \text{ for each } \langle \varphi_1, \ldots, \varphi_n, \psi \rangle \in R \}$.

We obtain the same relation between the logic and the algebraic semantics as before:

1. $\Gamma \vdash_S \varphi$ iff $\tau[\Gamma] \models_S \tau(\varphi)$
2. $\Pi \models_S \varphi \approx \psi$ iff $\rho[\Pi] \vdash_S \rho(\varphi \approx \psi)$
3. $\varphi \vdash_S \rho(\tau(\varphi))$ and $\rho(\tau(\varphi)) \vdash_S \varphi$
4. $\varphi \approx \psi \models_S \tau(\rho(\varphi \approx \psi))$ and $\tau(\rho(\varphi \approx \psi)) \models_S \varphi \approx \psi$

$S$ is the equivalent algebraic semantics of $S$. 
The translations $\tau$ and $\rho$ between formulas and equations give bijective correspondences (dual lattice isomorphisms):

1. between finitary extensions of $L$ and quasivarieties of $L$-algebras
2. between axiomatic extensions of $L$ and varieties of $L$-algebras.
**Theorem 2.58 (Proof by Cases Property)**

Assume that for each \( \langle \varphi_1, \ldots, \varphi_n, \psi \rangle \in R, \varphi_1 \lor \chi, \ldots \varphi_n \lor \chi \vdash_S \psi \lor \chi \). If \( \Gamma, \varphi \vdash_S \chi \) and \( \Gamma, \psi \vdash_S \chi \), then \( \Gamma, \varphi \lor \psi \vdash_S \chi \).

**Proof.**

*Claim* If \( \Gamma \vdash_S \varphi \), then \( \Gamma \lor \chi \vdash_S \delta \lor \chi \) for each formula \( \chi \) and each \( \delta \) appearing in the proof of \( \varphi \) from \( \Gamma \).

*Proof of the claim:* trivial for \( \delta \in \Gamma \) or \( \delta \) an axiom; if we used MP, then by IH there has to be \( \eta \) st.

\[
\Gamma \lor \chi \vdash \eta \lor \chi \quad \Gamma \lor \chi \vdash (\eta \rightarrow \delta) \lor \chi \text{ thus (T7) completes the proof.}
\]

Now using the claim: \( \Gamma \lor \psi, \varphi \lor \psi \vdash \chi \lor \psi \) and \( \Gamma \lor \chi, \psi \lor \chi \vdash \chi \lor \chi \).

Using (A6a), (T8), and (T9) we get \( \Gamma, \varphi \lor \psi \vdash \psi \lor \chi \) and \( \Gamma, \psi \lor \chi \vdash \chi \) and the rest is trivial.
Chain-completeness for extensions

Corollary 2.59

Assume that for each $\langle \varphi_1, \ldots, \varphi_n, \psi \rangle \in R$, $\varphi_1 \lor \chi, \ldots \varphi_n \lor \chi \vdash_S \psi \lor \chi$ (this is the case, in particular, if $S$ is an axiomatic extension). Then for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_L$: $\Gamma \vdash_S \varphi$ iff $\Gamma \models_{S_{\text{lin}}} \varphi$.

Exercise 9

Prove it.
Outline

1. Completeness of Gödel–Dummett logic
2. Completeness of Łukasiewicz logic
3. Functional representation
4. Finite model property
5. Computational complexity
6. Algebraizability of Gödel–Dummett and Łukasiewicz logics
7. Axiomatic extensions of Gödel–Dummett and Łukasiewicz logics
8. Application: Fuzzy Logic and Probability
The case of Gödel–Dummett logic

For each \( n \geq 1 \), recall the canonical \( n \)-valued \( G \)-chain:

\[
G_n = \langle \{ \frac{i}{n-1} \mid i \leq n - 1 \}, \min, \max, \rightarrow, 0, 1 \rangle.
\]

\[
G_n = G + \bigvee_{i=0}^{n-1} (p_i \rightarrow p_{i+1}).
\]

**Theorem 2.60**

- for each \( n \geq 1 \), \( G_n \)-algebras are the subvariety of \( G \)-algebras satisfying \( \bigvee_{i=0}^{n-1} (p_i \rightarrow p_{i+1}) \cong \top \) and it coincides with \( V(G_n) \).
- \( G \) is locally finite, i.e. each finite subset of a \( G \)-algebra generates a finite subalgebra.
- If \( C \) is an infinite \( G \)-chain, then \( V(C) = G \).
- the subvarieties of \( G \) are exactly:
  \[
  V(G_1) \subsetneq V(G_2) \subsetneq V(G_3) \subsetneq \ldots \subsetneq V(G_n) \subsetneq V(G_{n+1}) \subsetneq \ldots G.
  \]

**Exercise 10**

Prove it.
The case of Gödel–Dummett logic

**Theorem 2.61**

There are no other finitary extensions of $G$ than $G_n$s (i.e. $G$ has no proper subquasivarieties).

**Lemma 2.62**

Gödel–Dummett logic proves:

- $(\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$
- $(\varphi \rightarrow (\psi \land \chi)) \leftrightarrow ((\varphi \rightarrow \psi) \land (\varphi \rightarrow \chi))$
- $(\varphi \rightarrow (\psi \lor \chi)) \leftrightarrow ((\varphi \rightarrow \psi) \lor (\varphi \rightarrow \chi))$

Define a substitution $\sigma_\varphi(p) = \varphi \rightarrow p$. Then if $\bar{0}$ does not occur in $\varphi$ we have: $\vdash_G \sigma_\varphi(\psi) \leftrightarrow (\varphi \rightarrow \psi)$, $\psi \vdash_G \sigma_\varphi(\psi)$, and $\vdash_G \sigma_\varphi(\varphi)$. 
**Deduction theorems**

**Lemma 2.63**

*Any finitary extension $L$ of $G$ enjoys the deduction theorem.*

**Proof.**

Assume that $\phi \vdash_L \psi$. Let $\chi_f$ be the formula resulting from $\chi$ by replacing all occurrences of $\bar{0}$ by a fresh fixed variable $f$. Define a substitution $\sigma(q) = \bar{0}$ for $q = f$ and $q$ otherwise; observe $\sigma(\chi_f) = \chi$.

*Claim:* $\{f \rightarrow q \mid q \text{ in } \{\phi, \psi\}\}, \phi_f \vdash_L \psi_f$.

Thus $\sigma\sigma_{\phi_f} \{f \rightarrow q \mid q \text{ in } \{\phi, \psi\}\} \cup \{\phi_f\} \vdash_L \sigma\sigma_{\phi_f} (\psi_f)$. And so $\{(\phi \rightarrow \bar{0}) \rightarrow (\phi \rightarrow q) \mid q \text{ in } \{\phi, \psi\}\}, \sigma\sigma_{\phi_f} (\phi) \vdash_L \sigma\sigma_{\phi_f} (\psi)$. Since, clearly, $\vdash_L \sigma\sigma_{\phi_f} (\chi_f) \leftrightarrow (\phi \rightarrow \chi)$, we obtain $\vdash_L \phi \rightarrow \psi$.

**Exercise 11**

Complete the proof (including the claim!).
Structural completeness

The proof of Theorem 2.61.
Obvious as the previous lemma allows us to replace any additional rule of $L$ by an axiom.

Definition 2.64
A logic is structurally complete if each proper extension has some new theorems. A logic is hereditarily structurally complete if each of its extensions is structurally complete.

Corollary 2.65
$G$ is hereditarily structurally complete.

Exercise 12
$L$ is not structurally complete. (hint: use the rule $\varphi \leftrightarrow \neg \varphi \vdash \bot$)
Important MV-chains

Recall the functor $\Gamma$ which turns each Lattice ordered Abelian group with strong unit into and MV-algebra.

For each $n \geq 1$, recall the canonical $n$-valued MV-chain:

$$\mathcal{L}_n = \langle \{ \frac{i}{n-1} \mid i \leq n - 1 \}, \oplus, \neg, 0 \rangle.$$

- for each $u > 0$, $[0, 1]_L \cong \Gamma(R, u)$.
- $\mathcal{L}_n \cong \Gamma(Q_{n-1}, 1)$
- $K_n = \Gamma(Q_{n-1} \otimes \mathbb{Z}, \langle 1, 0 \rangle)$.

where on $Q_{n-1}$ is the additive group of rationals whose denominator is $n - 1$, and $Q_{n-1} \otimes \mathbb{Z}$ is the lexicographic product (direct product with the lexicographic order).
Varieties of MV-algebras

Proposition 2.66

- $\mathbf{V}([0, 1]_{\mathbb{L}}) = \mathbb{M}V$
- If $I \subseteq \mathbb{N}$ is infinite, then $\mathbf{V}(\{\mathbb{L}_i \mid i \in I\}) = \mathbb{M}V$
- $\mathbf{V}(\mathbb{L}_i) \subseteq \mathbf{V}(\mathbb{L}_j)$ iff $i - 1$ divides $j - 1$.

Theorem 2.67 (Komori)

Let $\mathbb{K} \subseteq \mathbb{M}V$ be a variety. $\mathbb{K} \neq \mathbb{M}V$ iff there are two finite disjoint sets $I, J \subseteq \mathbb{N}$ such that:

$$\mathbb{K} = \mathbf{V}(\{\mathbb{L}_i \mid i \in I\} \cup \{\mathbb{K}_j \mid j \in J\}).$$
Varieties of MV-algebras

**Definition 2.68**
If $i \in \mathbb{N}$, $\delta(i) = \{n \in \mathbb{N} \mid n \text{ is a divisor of } i\}$. If $J \subseteq \mathbb{N}$ is finite and nonempty, $\Delta(i, J) = \delta(i) \setminus \bigcup_{j \in J} \delta(j)$.

**Theorem 2.69 (Di Nola, Lettieri)**
Let $I, J \subseteq \mathbb{N}$ be finite disjoint sets. Then the variety $\mathbf{V} (\{L_i \mid i \in I\} \cup \{K_j \mid j \in J\})$ has the following equational base:

- $Eq(1) \quad ((n + 1)x^n)^2 \approx 2x^{n+1}$ \quad with \quad $n = \max(I \cup J)$,
- $Eq(2) \quad (px^{p-1})^{n+1} \approx (n + 1)x^p$,
- $Eq(3) \quad (n + 1)x^q \approx (n + 2)x^q$,

for every positive integer $1 < p < n$ such that $p$ is not a divisor of any $i \in I \cup J$ and for every $q \in \bigcup_{i \in I} \Delta(i, J)$. 
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Fuzzy logic for reasoning about probability

Fuzziness ≠ probability
Fuzzy logic for reasoning about probability

Fuzziness $\neq$ probability

Probability of $\varphi = \Box \varphi =$ truth degree of *it is probable that* $\varphi$
Fuzzy logic for reasoning about probability

Fuzziness ≠ probability

Probability of \( \varphi = \Box \varphi = \) truth degree of it is probable that \( \varphi \)

Let us take:

- the classical logic CL in language \( \to, \neg, \lor, \land, 0 \)
- Łukasiewicz logic \( \mathcal{L} \) in language \( \to_{\mathcal{L}}, \neg_{\mathcal{L}}, \oplus, \ominus \)
- an extra symbol \( \Box \)
Fuzzy logic for reasoning about probability

Fuzziness ≠ probability

Probability of $\varphi = \square \varphi = \text{truth degree of it is probable that } \varphi$

Let us take:

- the classical logic CL in language $\to, \neg, \lor, \land, \bar{0}$
- Łukasiewicz logic $\mathcal{L}$ in language $\to_{\mathcal{L}}, \neg_{\mathcal{L}}, \oplus, \ominus$
- an extra symbol $\square$

We define three kinds of formulas of a two-level language over a fixed set of variables $\text{Var}$:

- non-modal: built from $\text{Var}$ using $\to, \neg, \lor, \land, \bar{0}$
- atomic modal: of the form $\square \varphi$, for each non-modal $\varphi$
- modal: built from atomic ones using $\to_{\mathcal{L}}, \neg_{\mathcal{L}}, \oplus, \ominus$
Definition 2.70
A **probability Kripke frame** is a system $F = \langle W, \mu \rangle$ where
- $W$ is a set (of possible worlds)
- $\mu$ is a finitely additive probability measure defined on a sublattice of $2^W$

Definition 2.71
A **Kripke model** $M$ over a probability Kripke frame $F = \langle W, \mu \rangle$ is a tuple $M = \langle F, (e_w)_{w \in W} \rangle$ where:
- $e_w$ is a classical evaluation of non-modal formulas
- the domain of $\mu$ contains the set $\{ w \mid e_w(\varphi) = 1 \}$ for each non-modal formula $\varphi$
Truth definition

The truth values of modal formulas are defined uniformly:

\[ \|\Box \varphi\|_M = \mu(\{w \mid e_w(\varphi) = 1\}) \]
\[ \|\neg_L \Phi\|_M = 1 - \|\Phi\|_M \]
\[ \|\Phi \rightarrow_L \Psi\|_M = \min\{1, 1 - \|\Phi\|_M + \|\Psi\|_M\} \]
\[ \|\Phi \oplus \Psi\|_M = \min\{1, \|\Phi\|_M + \|\Psi\|_M\} \]
\[ \|\Phi \ominus \Psi\|_M = \max\{0, \|\Phi\|_M - \|\Psi\|_M\} \]
Axiomatization

**Definition 2.72**

The logic FP of probability inside Łukasiewicz logic is given by the axiomatic system consisting of:

- the axioms and rules of CL for non-modal formulas,
- axioms and rules of $\mathbb{L}$ for modal formulas,
- modal axioms
  
  (FP0) $\neg \mathbb{L} \Box (\overline{0})$
  (FP1) $\Box (\varphi \rightarrow \psi) \rightarrow \mathbb{L} (\Box \varphi \rightarrow \mathbb{L} \Box \psi)$
  (FP2) $\neg \mathbb{L} \Box (\varphi) \rightarrow \mathbb{L} \Box (\neg \varphi)$
  (FP3) $\Box (\varphi \vee \psi) \rightarrow \mathbb{L} (\Box \psi \oplus (\Box \varphi \ominus \Box (\varphi \land \psi)))$

- a unary modal rule:
  
  $\varphi \vdash \Box \varphi$

The notion of provability $\vdash_{FP}$ (from both modal and non-modal premises) is defined as usual.
Completeness theorem

Theorem 2.73 (Hájek)

Let $\Gamma \cup \{\Psi\}$ be a set of modal formulas. TFAE:

- $\Gamma \vdash_{FP} \Psi$
- $\|\Psi\|_M = 1$ for each Kripke model $M$ where $\|\Phi\|_M = 1$

for each $\Phi \in \Gamma$. 
Variations

- changing the measure of uncertainty (necessity, possibility, belief functions)
- changing the upper logic: replacing Łukasiewicz logic by any other fuzzy logic
- changing the lower logic: e.g. replacing CL by Łukasiewicz logic to speak about probability of vague events

Ex: Messi will score soon in the second half of the match

adding more modalities

any combination of the above four options

We can build also a general theory for these two-layer modal logics
Variations

- changing the measure of uncertainty (necessity, possibility, belief functions)
- changing the upper logic: replacing Łukasiewicz logic by any other fuzzy logic

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