Fuzzy Logic

3. Additional properties of propositional Łukasiewicz and Gödel–Dummett logics

Petr Cintula

Institute of Computer Science,
Czech Academy of Sciences, Prague, Czech Republic

www.cs.cas.cz/cintula/mfl-tuw

The classical case

Theorem 3.1 (Functional completeness)

Every Boolean function (i.e. any function $f\colon \{0,1\}^n \to \{0,1\}$ *for some* $n \geq 1$*) is representable by some formula of classical logic.*

The fuzzy case

Let L be either Ł of G.

Definition 3.2

A function $f\colon [0,1]^n \to [0,1]$ is *represented* by a formula $\varphi(v_1, \dots, v_n)$ in L if $e(\varphi) = f(e(v_1), e(v_2), \dots, e(v_n))$ for each $[0,1]_{\mathrm{L}}$-evaluation e.

Definition 3.3

The *functional representation* of L is the set $\mathcal{F}_{\mathrm{L}}$ of all functions from any power of $[0,1]$ into $[0,1]$ that are represented in L by some formula.

Relation with Lindenbaum–Tarski algebra

Let us fix $\mathrm{L} = \mathrm{Ł}$.
Let f_i be functions of n_i variables, $i \in \{1, 2\}$. We say that $f_1 = f_2$ iff $f_1(x_1, x_2, \ldots, x_{n_1}) = f_2(x_1, x_2, \ldots, x_{n_2})$ for every $x_j \in [0, 1]$. Let us for each $f \in \mathcal{F}_{Ł}$ define a class

$$[f] = \{g \in \mathcal{F}_{Ł} \mid f = g\} \qquad F = \{[f] \mid f \in \mathcal{F}_{Ł}\}$$

We define an MV-algebra $\boldsymbol{F}$ with domain F and operations:

$$\overline{0}^{\boldsymbol{F}} = [0] \quad \neg^{\boldsymbol{F}}[f] = [1 - f]_T \quad [f] \oplus^{\boldsymbol{F}} [g] = [\min\{1, f + g\}]$$

Theorem 3.4

The algebras $\boldsymbol{F}$ and $\mathbf{Lind}_{\emptyset}$ are isomorphic.

In the case of G, the definitions and the result are analogous.

A proof

Let the atoms be enumerated as $v_1, v_2, \ldots$. Any formula with variables with maximal index n is viewed as formula in variables $v_1, \ldots, v_n$.
We define the homomorphism:

$g\colon L_\emptyset \to F$ as $g([\varphi]) = [f_\varphi]$ where f_φ is the function represented by φ.

Then:

- the definition is sound and g is one-one: $[\varphi] = [\psi]$ iff $\vdash_Ł \varphi \leftrightarrow \psi$ iff (due to the standard completeness theorem) $e(\varphi) = e(\psi)$ for each $[0, 1]_Ł$-evaluation e iff $[f_\varphi] = [f_\psi]$.
- g is a homomorphism:
  $g([\varphi] \oplus [\psi]) = g([\varphi \oplus \psi]) = [f_{\varphi \oplus \psi}] = [f_\varphi \oplus f_\psi] = [f_\varphi] \oplus [f_\psi]$.
- g is onto (obvious).

How do the functions from $\mathcal{F}_{Ł}$ look like?

Observations

- they are all continuous
- they are piece-wise linear
- all pieces have integer coefficients
- if $x_1, \ldots, x_n \in \{0, 1\}^n$, then $f(x_1, \ldots, x_n) \in \{0, 1\}$
- if $x_1, \ldots, x_n \in ([0, 1] \cap \mathrm{Q})^n$, then $f(x_1, \ldots, x_n) \in [0, 1] \cap \mathrm{Q}$

Definition 3.5

A McNaughton function $f\colon [0, 1]^n \to [0, 1]$ is a continuous piece-wise linear function, where each of the pieces has integer coefficients.

Theorem 3.6 (McNaughton theorem)

$\mathcal{F}_{Ł}$ is the set of all McNaughton functions.

A lemma (for one variable only)

Lemma 3.7

Let $f\colon [0,1] \to$ R *be an* integer linear polynomial*, i.e. of the form*

$$f(x) = mx + n \qquad \textit{for some } m, n \in \mathrm{Z}$$

Then there is a formula φ_f *representing the function*

$$f^{\#} = \max\{0, \min\{1, f\}\}.$$

Proof.

By induction on $|m|$. If $|m| = 0$ then $f^{\#}$ is either constantly 0 or 1, then we can take as φ either the term $\overline{0}$ or $\overline{1}$, respectively. Assume now $|m| > 0$. WLOG we can assume $m > 0$: indeed otherwise we consider $f' = 1 - f$, here $m > 0$ and so we have φ_{1-f}. Note that clearly $\varphi_f = \neg\varphi_{1-f}$. . . .

A lemma: continuation of the proof

Let us consider the function $g = f - x$: by IH we have formulas φ_g and φ_{g+1}. If we show that

$$(g + x)^{\#} = (g^{\#} \oplus x) \,\&\, (g + 1)^{\#} \quad (1)$$

the proof is done as:

$$\varphi_f = \varphi_{g+x} = (\varphi_g \oplus x) \,\&\, \varphi_{g+1}.$$

We need to prove (1) for any $a \in [0, 1]$. Let L and R be its left/right side:

- if $|g(a)| > 1$ then $L = R = 1$ or $L = R = 0$
- $0 \leq g(a) \leq 1$ then $L = \min\{1, g(a) + a\}$, $g(a) = g^{\#}(a)$ and $(g + 1)^{\#}(a) = 1$. Hence $R = g(a) \oplus a = \min\{1, g(a) + a\} = L$.
- $-1 \leq g(a) \leq 0$ then $L = \max\{0, g(a) + a\}$, $g^{\#}(a) = 0$ and $(g + 1)^{\#}(a) = g(a) + 1$. Hence $g^{\#}(a) \oplus a = a$ and so $R = \max\{0, a + g(a) + 1 - 1\} = \max\{0, a + g(a)\} = L$.

The proof for one variable functions

Definition 3.8

Let $a, b \in [0, 1] \cap \mathbb{Q}$. Then any McNaughton function f s.t. $f(x) = 1$ iff $x \in [a, b]$ is called *pseudo characteristic function* of interval $[a, b]$.

Exercise 9

Prove that each interval has a pseudo characteristic function and find a formula representing it. Hint: use Lemma 3.7.

Lemma 3.9

Let $a, b \in [0, 1] \cap \mathbb{Q}$. Then for each $\epsilon > 0$ there is a pseudo characteristic function of the interval $[a, b]$, s.t. $f(x) = 0$ for $x \in [0, a - \epsilon] \cup [b + \epsilon, 1]$.

Proof.

If f is a pseudo char. function of some interval, so is f^n for each n. □

The proof for one variable functions

Let p be a McNaughton function of one variable given by n integer linear polynomials $p_1, \ldots, p_n$. For each $i \in \{1, 2, \ldots n\}$ let $P_i = [a_i, b_i]$ be the interval in which p coincides with p_i. Note that:

- $[0, 1] = \bigcup_i P_i$
- $a_i, b_i \in [0, 1] \cap \mathrm{Q}$
- there is a pseudo characteristic function f_i of $[a_i, b_i]$ such that $p(x) \geq (f_i \,\&\, p_i^{\#})(x)$ for each $x \notin P_i$.

Then

$$p(x) = \bigvee_i (f_i \,\&\, p_i^{\#})(x) \text{ and thus } \varphi_p = \bigvee_i \varphi_{f_i} \,\&\, \varphi_{p_i}.$$

The classical case, FMP and decidability

CL is complete with respect to a finite algebra, $\mathbf{2}$.

Definition 3.10

A logic has the finite model property (FMP) if it is complete with respect to a set of finite algebras.

From the FMP, we obtain decidability:

- Thanks to our finite notion of proof, the set of theorems is recursively enumerable.
- Thanks to FMP, the set of non-theorems is also recursively enumerable (we can check validity in bigger and bigger finite algebras until we find a countermodel).
- Therefore, theoremhood is a decidable problem.
- Note: provability from finitely-many premises is also decidable (using deduction theorem).

Finite chains

Lemma 3.11

Let $\boldsymbol{A_2}$ *be a subalgebra of an* MV*- or* G*-algebra* $\boldsymbol{A_1}$*. Then* $\models_{\boldsymbol{A}_1} \subseteq \models_{\boldsymbol{A}_2}$*.*

Exercise 10

(a) Prove that each n-valued G-chain is isomorphic to the subalgebra $\boldsymbol{G_n}$ of $[0,1]_{\mathrm{G}}$ with the domain $\{\frac{i}{n-1} \mid i \leq n-1\}$.

(b) Prove that each n-valued MV-chain is isomorphic to the subalgebra $\boldsymbol{Ł}_n$ of $[0,1]_{\mathrm{Ł}}$ with the domain $\{\frac{i}{n-1} \mid i \leq n-1\}$.

Lemma 3.12

$$\models_{\boldsymbol{G_m}} \subseteq \models_{\boldsymbol{G_n}} \quad \textit{iff} \quad n \leq m.$$
$$\models_{\boldsymbol{Ł}_m} \subseteq \models_{\boldsymbol{Ł}_n} \quad \textit{iff} \quad n-1 \textit{ divides } m-1.$$

Let us denote by $\mathbb{L}_{\mathrm{fin}}$ the class of finite L-chains.

The case of Gödel–Dummett logic

Theorem 3.13

Let φ *be a formula with* $n-2$ *variables. Then:* $\vdash_{\mathrm{G}} \varphi$ *iff* $\models_{\boldsymbol{G}_n} \varphi$.

Proof.

Contrapositively: assume that $\nvdash_{\mathrm{G}} \varphi$ and let e be a $[0,1]_{\mathrm{G}}$-evaluation s.t. $e(\varphi) \neq 1$. Let $X = \{0,1\} \cup \{e(v_i) \mid 1 \leq i \leq n-2\}$ and note that it is a subuniverse of $[0,1]_{\mathrm{G}}$, thus e can be seen as an $\boldsymbol{X}$-evaluation and so $\not\models_{\boldsymbol{X}} \varphi$. The previous exercise and lemma complete the proof. □

Theorem 3.14

For every *finite* *set of formulas* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*, TFAE:*

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{\mathrm{fin}}} \varphi$

The case of Łukasiewicz logic

Theorem 3.15

For every finite set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, *TFAE:*

1. $\Gamma \vdash_{Ł} \varphi$
2. $\Gamma \models_{[0,1]_Ł} \varphi$
3. $\Gamma \models_{\mathbb{MV}_{\text{fin}}} \varphi$

Proof: we show it for one variable v.

Let us define the set E of $[0,1]_Ł$-evaluations s.t. $e[\Gamma] \subseteq \{1\}$. Note that E can be seen as a union of real intervals. Assume that there is $e \in E$ s.t. $e(\varphi) \neq 1$. If we show that there is an evaluation $f \in E$, s.t. $f(v) = \frac{p}{n-1}$ and $f(\varphi) \neq 1$ we are done as f can be seen as $Ł_n$-evaluation.

- Either e lies on the border of some interval, then $f = e$ OR
- there has to be a neighborhood $X \subseteq E$ s.t. $g(\varphi) \neq \bar{1}$ for each $g \in X$. Therefore there has to be f we need. □

The classical case

- $\varphi \in \mathrm{SAT}(\mathrm{CL})$ if there is a $\mathbf{2}$-evaluation e such that $e(\varphi) = 1$.
- $\varphi \in \mathrm{TAUT}(\mathrm{CL})$ if for each $\mathbf{2}$-evaluation e holds $e(\varphi) = 1$.

Recall:

$$\begin{array}{lll} \varphi \in \mathrm{TAUT}(\mathrm{CL}) & \text{iff} & \neg\varphi \notin \mathrm{SAT}(\mathrm{CL}) \\ \varphi \in \mathrm{SAT}(\mathrm{CL}) & \text{iff} & \neg\varphi \notin \mathrm{TAUT}(\mathrm{CL}). \end{array}$$

Both problems, $\mathrm{SAT}(\mathrm{CL})$ and $\mathrm{TAUT}(\mathrm{CL})$, are decidable.

But how difficult are their computations?

Complexity classes

$f, g \colon \mathrm{N} \to \mathrm{N}$. $f \in O(g)$ iff there are $c, n_0 \in \mathrm{N}$ such that for each $n \geq n_0$ we have $f(n) \leq c\, g(n)$.

- **TIME**(f): the class of problems P such that there is a deterministic Turing machine M that accepts P and operates in time $O(f)$.
- **NTIME**(f): analogous class for nondeterministic Turing machines.
- **SPACE**(f): the class of problems P such that there is a deterministic Turing machine M that accepts P and operates in space $O(f)$.
- **NSPACE**(f): the analogous class for nondeterministic Turing machines.

Complexity classes

$$\mathbf{P} = \bigcup_{k \in \mathsf{N}} \mathbf{TIME}(n^k)$$

$$\mathbf{NP} = \bigcup_{k \in \mathsf{N}} \mathbf{NTIME}(n^k)$$

$$\mathbf{PSPACE} = \bigcup_{k \in \mathsf{N}} \mathbf{SPACE}(n^k)$$

If $\mathbf{C}$ is a complexity class, we denote $\mathbf{coC} = \{P \mid \overline{P} \in \mathbf{C}\}$, the class of complements of problems in $\mathbf{C}$.

Complexity classes

- Each deterministic complexity class $\mathbf{C}$ is closed under complementation: if $P \in \mathbf{C}$, then also $\overline{P} \in \mathbf{C}$.
- Is $\mathbf{NP}$ closed under complementation?
- $\mathbf{P} \subseteq \mathbf{NP}$, $\mathbf{P} \subseteq \mathbf{coNP}$, $\mathbf{NP} \subseteq \mathbf{PSPACE}$.
- Are the inclusions $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE}$ proper?
- Each of the classes $\mathbf{P}$, $\mathbf{NP}$, $\mathbf{coNP}$, and $\mathbf{PSPACE}$ is closed under finite unions and intersections.

Complexity classes

A problem P is said to be $\mathbf{C}$-hard iff any decision problem P' in $\mathbf{C}$ is polynomialy reducible to P, i.e.: there is a polynomially computable function f such that:

$$x \in P' \qquad \text{iff} \qquad f(x) \in P'$$

A problem P is $\mathbf{C}$-complete iff P is $\mathbf{C}$-hard and $P \in \mathbf{C}$.

The classical case

- $\mathrm{SAT}(\mathrm{CL}) \in \mathbf{NP}$: guess an evaluation and check whether it satisfies the formula (a polynomial matter).
- $\mathrm{TAUT}(\mathrm{CL}) \in \mathbf{coNP}$: $\varphi \in \mathrm{TAUT}(\mathrm{CL})$ iff $\neg\varphi \notin \mathrm{SAT}(\mathrm{CL})$.
- Cook Theorem: Let $\mathrm{SAT}^{\mathrm{CNF}}(\mathrm{CL})$ be the SAT problem for formulas in conjunctive normal form. Then: $\mathrm{SAT}^{\mathrm{CNF}}(\mathrm{CL})$ is $\mathbf{NP}$-complete.
- $\mathrm{SAT}^{\mathrm{CNF}}(\mathrm{CL})$ is a fragment of $\mathrm{SAT}(\mathrm{CL})$, therefore $\mathrm{SAT}(\mathrm{CL})$ is $\mathbf{NP}$-complete and $\mathrm{TAUT}(\mathrm{CL})$ is $\mathbf{coNP}$-complete.

The fuzzy case: basic definitions

Let L be either Łukasiewicz logic Ł or Gödel logic G. We define:

- $\varphi \in \mathrm{SAT}(\mathrm{L})$ if there is an evaluation e such that $e(\varphi) = 1$.
- $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{L})$ if there is an evaluation e such that $e(\varphi) > 0$.
- $\varphi \in \mathrm{TAUT}(\mathrm{L})$ if for each evaluation e holds $e(\varphi) = 1$.
- $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L})$ if for each evaluation e holds $e(\varphi) > 0$.

Note that $p \vee \neg p \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L})$ but $p \vee \neg p \notin \mathrm{TAUT}(\mathrm{L})$

Note that $p \wedge \neg p \in \mathrm{SAT}_{\mathrm{pos}}(\text{Ł})$ but $p \wedge \neg p \notin \mathrm{SAT}(\text{Ł})$

The fuzzy case: basic reductions

Lemma 3.16

Let L *be either Łukasiewicz logic* $Ł$ *or Gödel logic* G. *Then*

$$\varphi \in \mathrm{TAUT_{pos}}(\mathrm{L}) \quad \textit{iff} \quad \neg\varphi \notin \mathrm{SAT}(\mathrm{L})$$

$$\varphi \in \mathrm{SAT_{pos}}(\mathrm{L}) \quad \textit{iff} \quad \neg\varphi \notin \mathrm{TAUT}(\mathrm{L}).$$

Lemma 3.17

$$\varphi \in \mathrm{SAT}(Ł) \quad \textit{iff} \quad \neg\varphi \notin \mathrm{TAUT_{pos}}(Ł)$$

$$\varphi \in \mathrm{TAUT}(Ł) \quad \textit{iff} \quad \neg\varphi \notin \mathrm{SAT_{pos}}(Ł).$$

Exercise 11

Prove the above two lemmata, show that the last equivalence fails for G and the one but last holds there. (Hint: for the last part use properties of these sets proved in the next few slides).

The case of Łukasiewicz logic

Theorem 3.18

The sets $\mathrm{SAT}(Ł)$ *and* $\mathrm{SAT}_{\mathrm{pos}}(Ł)$ *are* $\mathbf{NP}$*-complete. Therefore the sets* $\mathrm{TAUT}(Ł)$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(Ł)$ *are* $\mathbf{coNP}$*-complete.*

We prove it in a series of lemmata. First we show that $\mathrm{SAT}(Ł)$ is $\mathbf{NP}$-hard:

Lemma 3.19

Let φ *be a formula with variables* $p_1, \ldots p_n$*.*

$$\varphi \in \mathrm{SAT}(\mathrm{CL}) \qquad \textit{iff} \qquad \varphi \wedge \bigwedge_{i=1}^{n} (p_i \vee \neg p_i) \in \mathrm{SAT}(Ł).$$

$\mathrm{SAT_{pos}}(\text{Ł})$ is **NP**-hard

Lemma 3.20

Let φ be a formula with variables $p_1, \ldots p_n$ built using: $\wedge, \vee, \neg$.

$$\varphi \in \mathrm{SAT(CL)} \qquad \textit{iff} \qquad \varphi^2 \wedge \bigwedge_{i=1}^{n} (p_i \vee \neg p_i)^2 \in \mathrm{SAT_{pos}}(\text{Ł}).$$

Proof.

Let e positively satisfy the right-hand formula. Then $e((p_i \vee \neg p_i)^2) > 0$ ergo $e(p_i) \neq 0.5$. We define the evaluation

$$e'(p_i) = \begin{cases} 1 & \text{if } e(p_i) > 0.5 \\ 0 & \text{if } e(p_i) < 0.5 \end{cases}$$

Clearly this can be extended to φ. And, since $e(\varphi^2) > 0$, we have $e(\varphi) > 0.5$ and so $e'(\varphi) = 1$. □

$\mathrm{SAT}(Ł)$ and $\mathrm{SAT}_{\mathrm{pos}}(Ł)$ are in $\mathbf{NP}$

Lemma 3.21

$$e(\varphi \to \psi) \geq r \quad \textit{IFF} \quad \exists x, y \in [0,1] \quad \begin{array}{rcl} e(\varphi) & \leq & x \\ e(\psi) & \geq & y \\ r & \leq & 1 - x + y \end{array}$$

$$e(\varphi \to \psi) \leq r \quad \textit{IFF} \quad \exists x, y \in [0,1], i \in \{0,1\} \quad \begin{array}{rcl} e(\varphi) & \geq & x \\ e(\psi) & \leq & y \\ i - r & \leq & 0 \\ i + x & \leq & 1 \\ i - y & \leq & 0 \\ i + r & \geq & 1 - x + y \end{array}$$

Using this lemma we can reduce the question of (positive) satisfiability to the question of Mixed Integer Programming (MIP) which is known to be in $\mathbf{NP}$:

For $\mathrm{SAT}(Ł)$ start with $e(\varphi) \geq 1$ for $\mathrm{SAT}_{\mathrm{pos}}(Ł)$ start with $\begin{array}{rcl} e(\varphi) & \geq & i_0 \\ i_0 & > & 0 \end{array}$

The case of Gödel–Dummett logic

Lemma 3.22

The mapping $f\colon [0,1] \to \{0,1\}$ *defined as* $f(0) = 0$ *and* $f(x) = 1$ *if* $x \neq 0$ *is a homomorphism from* $[0,1]_{\mathrm{G}}$ *to* $\boldsymbol{2}$.

Corollary 3.23

$$\mathrm{SAT}_{\mathrm{pos}}(\mathrm{G}) \subseteq \mathrm{SAT}(\mathrm{CL}) \qquad \mathrm{TAUT}(\mathrm{CL}) \subseteq \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G}).$$

The case of Gödel–Dummett logic

Corollary 3.24

$$\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{G}) \quad \textit{iff} \quad \varphi \in \mathrm{SAT}(\mathrm{G}) \quad \textit{iff} \quad \varphi \in \mathrm{SAT}(\mathrm{CL})$$

$$\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G}) \quad \textit{iff} \quad \neg\neg\varphi \in \mathrm{TAUT}(\mathrm{G}) \quad \textit{iff} \quad \varphi \in \mathrm{TAUT}(\mathrm{CL})$$

Proof.

Just observe that:

$$\mathrm{SAT}(\mathrm{G}) \subseteq \mathrm{SAT}_{\mathrm{pos}}(\mathrm{G}) \subseteq \mathrm{SAT}(\mathrm{CL}) \subseteq \mathrm{SAT}(\mathrm{G}).$$

And that

$$\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G}) \Rightarrow \neg\varphi \notin \mathrm{SAT}(\mathrm{G}) \Rightarrow \neg\varphi \notin \mathrm{SAT}_{\mathrm{pos}}(\mathrm{G})$$

$$\Rightarrow \neg\neg\varphi \in \mathrm{TAUT}(\mathrm{G}) \Rightarrow \varphi \in \mathrm{TAUT}(\mathrm{CL}) \Rightarrow \varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G}).$$

□

The case of Gödel–Dummett logic

Corollary 3.24

$\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{G})$ *iff* $\varphi \in \mathrm{SAT}(\mathrm{G})$ *iff* $\varphi \in \mathrm{SAT}(\mathrm{CL})$
$\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G})$ *iff* $\neg\neg\varphi \in \mathrm{TAUT}(\mathrm{G})$ *iff* $\varphi \in \mathrm{TAUT}(\mathrm{CL})$

Theorem 3.25

The sets $\mathrm{SAT}(\mathrm{G})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathrm{G})$ *are* $\mathbf{NP}$*-complete and the sets* $\mathrm{TAUT}(\mathrm{G})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G})$ *are* $\mathbf{coNP}$*-complete.*

Proof.

The only non clear case is $\mathrm{TAUT}(\mathrm{G})$: it is co$\mathbf{NP}$-hard due to the last reduction of the previous corollary. We present a non-deterministic polynomial 'algorithm' (sound due to Theorem 3.14) for $Fm_{\mathcal{L}} \setminus \mathrm{TAUT}(\mathrm{G})$:

`Step 1`: guess a $\boldsymbol{G}_n$-evaluation e (assuming that φ has $n-2$ variables)
`Step 2`: compute the value of $e(\varphi)$ (clearly in polynomial time)
`Output`: if $e(\varphi) \neq 1$ output $\varphi \notin \mathrm{TAUT}(\mathrm{G})$. □

Reasoning with imperfect information

To understand reasoning we need to handle imperfect information:

- Vagueness: Some facts are expressed using vague predicates (i.e. properties that admit borderline cases)
- Incompleteness: Some relevant facts are not known, there is uncertainty
- Inconsistency: The information contains (or entails) contradictions

These phenomena are mutually independent, but can often be found together

Uncertainty measures vs fuzzy logics

Uncertainty measures, in general,

- deal with uncertain classical events
- by assigning them numerical values with various interpretations (usually from $[0, 1]$)
- are not compositional

Fuzzy logics, in general,

- deal with vague events
- by assigning them truth values (usually from $[0, 1]$)
- but are compositional (truth-functional)

Uncertainty measures and fuzzy logic

Observation: uncertainty is itself a gradual notion:

$$P(\varphi) = \text{truth degree of "}\varphi\text{ is probable"}$$

belief degree of φ = truth degree of "φ is believed"

P. Hájek, D. Harmancová, F. Esteva, P. Garcia, and Lluís Godo.
On modal logics for qualitative possibility in a fuzzy setting.
In *Proceedings of Uncertainty in Artificial Intelligence*, 1994.

Petr Hájek, Lluís Godo, and Francesc Esteva.
Fuzzy logic and probability.
In *Proceedings of Uncertainty in Artificial Intelligence*, 1995.

Two-layer classical modal logics of probability

An older idea: interpret modality '$\Box\varphi$' as 'probably φ', and saying it is true if the probability of φ is bigger than a given threshold

Two-layer syntax consisting of:

- classical non-modal formulas describing the events
- atomic modal formulas of the form $\Box\varphi$, for each non-modal φ
- modal formulas built from atomic ones

Classical logic governs the behavior of modal and non-modal formulae

C. L. Hamblin. The modal 'probably'. *Mind*, 1959.

R. Fagin, J. Y. Halpern, and N. Megiddo. A logic for reasoning about probabilities. *Information and Computation*, 1990.

Fuzzy logic for reasoning about probability

Let us take:

- the classical logic CL in language $\rightarrow, \neg, \vee, \wedge, \overline{0}$
- Łukasiewicz logic Ł in language $\rightarrow_{Ł}, \neg_{Ł}, \oplus, \ominus$
- an extra symbol $\Box$

We define three kinds of formulas of a two-level language over a fixed set of variables Var:

- non-modal: built from Var using $\rightarrow, \neg, \vee, \wedge, \overline{0}$
- atomic modal: of the form $\Box\varphi$, for each non-modal φ
- modal: built from atomic ones using $\rightarrow_{Ł}, \neg_{Ł}, \oplus, \ominus$

Probability Kripke frames and Kripke models

Definition 3.26

A *probability Kripke frame* is a system $\mathbf{F} = \langle W, \mu \rangle$ where

- W is a set (of possible worlds)
- μ is a finitely additive probability measure defined on a sublattice of 2^W

Definition 3.27

A *Kripke model* $\mathbf{M}$ over a probability Kripke frame $\mathbf{F} = \langle W, \mu \rangle$ is a tuple $\mathbf{M} = \langle \mathbf{F}, \langle e_w \rangle_{w \in W} \rangle$ where:

- e_w is a classical evaluation of non-modal formulas
- for each non-modal formula φ, the domain of μ contains the set

$$\varphi^{\mathbf{M}} = \{w \mid e_w(\varphi) = 1\}$$

Truth definition

The truth values of non-modal formulas in world w are given by e_w

The truth values of atomic modal formulas are defined uniformly:

$$||\Box\varphi||_{\mathbf{M}} = \mu(\{w \mid e_w(\varphi) = 1\}) = \mu(\varphi^{\mathbf{M}})$$

The truth values of other modal formulas are computed as :

$$\begin{aligned}
||\neg_{Ł}\varphi||_{\mathbf{M}} &= 1 - ||\varphi||_{\mathbf{M}} \\
||\varphi \to_{Ł} \psi||_{\mathbf{M}} &= \min\{1, 1 - ||\varphi||_{\mathbf{M}} + ||\psi||_{\mathbf{M}}\} \\
||\varphi \oplus \psi||_{\mathbf{M}} &= \min\{1, ||\varphi||_{\mathbf{M}} + ||\psi||_{\mathbf{M}}\} \\
||\varphi \ominus \psi||_{\mathbf{M}} &= \max\{0, ||\varphi||_{\mathbf{M}} - ||\psi||_{\mathbf{M}}\}
\end{aligned}$$

Axiomatization

Definition 3.28

The logic FP of probability inside Łukasiewicz logic is given by the axiomatic system consisting of:

- the axioms and rules of CL for non-modal formulas,
- axioms and rules of Ł for modal formulas,
- modal axioms

  (FP0) $\neg_{Ł}\Box(\overline{0})$
  (FP1) $\Box(\varphi \to \psi) \to_{Ł} (\Box\varphi \to_{Ł} \Box\psi)$
  (FP2) $\neg_{Ł}\Box(\varphi) \to_{Ł} \Box(\neg\varphi)$
  (FP3) $\Box(\varphi \vee \psi) \to_{Ł} (\Box\psi \oplus (\Box\varphi \ominus \Box(\varphi \wedge \psi)))$

- a unary modal rule:

$$\varphi \vdash \Box\varphi$$

Completeness theorem

Theorem 3.29

Let $\Gamma \cup \{\Psi\}$ *be a set of modal formulas. TFAE:*

- $\Gamma \vdash_{\mathrm{FP}} \Psi$
- $||\Psi||_{\mathbf{M}} = 1$ *for each Kripke model* $\mathbf{M}$ *where* $||\Phi||_{\mathbf{M}} = 1$ *for each* $\Phi \in \Gamma$

Lluís Godo, Francesc Esteva, and Petr Hájek.
Reasoning about probability using fuzzy logic. *NNW*, 2000.

Variations: Changing the measure

Definition 3.30

A *necessity Kripke frame* is a system $\mathbf{F} = \langle W, \mu \rangle$ where

- W is a set (of possible worlds)
- μ is a necessity measure on a subset of 2^W

To keep the completeness we just replace the axioms:

(FN0) $\neg_{Ł}\Box(\bar{0})$

(FN1) $\Box(\varphi \to \psi) \to_{Ł} (\Box\varphi \to_{Ł} \Box\psi)$

(FN3) $(\Box\varphi \wedge_{Ł} \Box\psi) \to_{Ł} \Box(\varphi \wedge \psi)$

More variations

Variations considered in the literature:

- changing the measure
- changing the 'upper' logic: replacing the Łukasiewicz logic by any other t-norm-based logic
- changing the 'lower' logic: e.g. replacing CL by the Łukasiewicz logic to speak about probability of 'fuzzy' events
- adding more modalities, also binary ones
- any combination of the above four options

Lluís Godo, Tommaso Flaminio and Enrico Marchioni.
Reasoning about uncertainty of fuzzy events.
In *Understanding Vagueness*, 2011