Fuzzy Logic

4. Predicate Łukasiewicz and Gödel–Dummett logic

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Predicate language

Predicate language: $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$: predicate and function symbols with arity

Object variables: denumerable set OV

$\mathcal{P}$-terms:

- if $v \in \mathrm{OV}$, then v is a $\mathcal{P}$-term
- if $f \in \mathbf{F}$, $\mathbf{ar}(\mathbf{F}) = n$, and $t_1, \ldots, t_n$ are $\mathcal{P}$-terms, then so is $f(t_1, \ldots, t_n)$

Formulas

Atomic $\mathcal{P}$-formulas: propositional constant $\bar{0}$ and expressions of the form $R(t_1, \dots, t_n)$, where $R \in \mathbf{P}$, $\mathbf{ar}(\mathbf{R}) = n$, and $t_1, \dots, t_n$ are $\mathcal{P}$-terms.

$\mathcal{P}$-formulas:

- the atomic $\mathcal{P}$-formulas are $\mathcal{P}$-formulas
- if α and β are $\mathcal{P}$-formulas, then so are $\alpha \wedge \beta$, $\alpha \vee \beta$, and $\alpha \to \beta$
- if $x \in \mathrm{OV}$ and α is a $\mathcal{P}$-formula, then so are $\forall x\, \alpha$ and $\exists x\, \alpha$

Basic syntactical notions

$\mathcal{P}$-theory: a set of $\mathcal{P}$-formulas

A closed $\mathcal{P}$-term is a $\mathcal{P}$-term without variables.

An occurrence of a variable x in a formula φ is bound if it is in the scope of some quantifier over x; otherwise it is called a free occurrence.

A variable is free in a formula φ if it has a free occurrence in φ.
we write $\varphi(x_1, \dots, x_n)$ if free variables in φ are among $x_1, \dots, x_n$

A $\mathcal{P}$-sentence is a $\mathcal{P}$-formula with no free variables.

A term t is substitutable for the object variable x in a formula φ if no occurrence of any variable occurring in t is bound in $\varphi(x/t)$ unless it was already bound in φ.

Axiomatic system

A Hilbert-style proof system for $\mathrm{CL}\forall$ can be obtained as:

(P)	axioms of CL substituting propositional variables by $\mathcal{P}$-formulas	
$(\forall 1)$	$\forall x\, \varphi \to \varphi(x/t)$	t substitutable for x in φ
$(\forall 2)$	$\forall x\, (\chi \to \varphi) \to (\chi \to \forall x\, \varphi)$	x not free in χ
(MP)	*modus ponens* for $\mathcal{P}$-formulas	
(gen)	from φ infer $\forall x\, \varphi$.	

Let us denote as $\vdash_{\mathrm{CL}\forall}$ the provability relation.

Semantics

Classical $\mathcal{P}$-structure: a tuple $\mathbf{M} = \langle M, \langle P_\mathbf{M} \rangle_{P \in \mathbf{P}}, \langle f_\mathbf{M} \rangle_{f \in \mathbf{F}} \rangle$ where

- $M \neq \emptyset$
- $P_\mathbf{M} \subseteq M^n$, for each n-ary $P \in \mathbf{P}$
- $f_\mathbf{M} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$.

$\mathbf{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{OV} \to M$

For $x \in \mathrm{OV}, m \in M$, and $\mathbf{M}$-evaluation v, we define $\mathrm{v}_{x=m}$ as

$$\mathrm{v}_{x=m}(y) = \begin{cases} m & \text{if } y = x \\ \mathrm{v}(y) & \text{otherwise} \end{cases}$$

Tarski truth definition

Interpretation of $\mathcal{P}$-terms

$$\begin{array}{rcll} \|x\|^{\mathbf{M}}_{\mathrm{v}} & = & \mathrm{v}(x) & \text{for } x \in \mathrm{OV} \\ \|f(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & f_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}) & \text{for } n\text{-ary } f \in \mathbf{F} \end{array}$$

Truth-values of $\mathcal{P}$-formulas

$$\begin{array}{rcll} \|P(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \langle \|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}} \rangle \in P_{\mathbf{M}} & \text{for } P \in \mathbf{P} \\ \|\bar{0}\|^{\mathbf{M}}_{\mathrm{v}} = 0 & & & \\ \|\alpha \wedge \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 1 \text{ and } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \\ \|\alpha \vee \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 1 \text{ or } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \\ \|\alpha \to \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 0 \text{ or } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \\ \|\forall x\, \varphi\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \text{for each } m \in M \text{ we have } \|\varphi\|^{\mathbf{M}}_{\mathrm{v}_{x=m}} = 1 & \\ \|\exists x\, \varphi\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \text{there is } m \in M \text{ such that } \|\varphi\|^{\mathbf{M}}_{\mathrm{v}_{x=m}} = 1 & \end{array}$$

Model and semantical consequence

We write $\mathbf{M} \models \varphi$ if $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}} = 1$ for each $\mathbf{M}$-evaluation v.

Model: We say that a $\mathcal{P}$-structure $\mathbf{M}$ is a $\mathcal{P}$-model of a $\mathcal{P}$-theory T, $\mathbf{M} \models T$ in symbols, if $\mathbf{M} \models \varphi$ for each $\varphi \in T$.

Consequence: A $\mathcal{P}$-formula φ is a semantical consequence of a $\mathcal{P}$-theory T, $T \models_{\mathrm{CL}\forall} \varphi$, if each $\mathcal{P}$-model of T is also a model of φ.

The completeness theorem

Problem of completeness of $\mathrm{CL}\forall$: formulated by Hilbert and Ackermann (1928) and solved by Gödel (1929):

Theorem 4.1 (Gödel's completeness theorem)

For every predicate language $\mathcal{P}$ and for every set $T \cup \{\varphi\}$ of $\mathcal{P}$-formulas :

$$T \vdash_{\mathrm{CL}\forall} \varphi \qquad \textit{iff} \qquad T \models_{\mathrm{CL}\forall} \varphi$$

Some history

1947 Henkin: alternative proof of Gödel's completeness theorem

1961 Mostowski: interpretation of existential (resp. universal) quantifiers as suprema (resp. infima)

1963 Rasiowa, Sikorski: first-order intuitionistic logic

1963 Hay: infinitary standard Łukasiewicz first-order logic

1969 Horn: first-order Gödel–Dummett logic

1974 Rasiowa: first-order implicative logics

1990 Novák: first-order Pavelka logics

1992 Takeuti, Titani: first-order Gödel–Dummett logic with additional connectives

1998 Hájek: first-order axiomatic extensions of HL

2005 Cintula, Hájek: first-order core fuzzy logics

2011 Cintula, Noguera: first-order semilinear logics

Basic syntax is the again the same

Let L be G or $\mathrm{Ł}$ and $\mathbb{L}$ be $\mathbb{G}$ or $\mathbb{MV}$ correspondingly

Predicate language: $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$

Object variables: denumerable set OV

$\mathcal{P}$-terms, (atomic) $\mathcal{P}$-formulas, $\mathcal{P}$-theories: as in $\mathrm{CL}\forall$

free/bounded variables, substitutable terms, sentences: as in $\mathrm{CL}\forall$

Recall classical semantics

Classical $\mathcal{P}$-structure: a tuple $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ where

- $M \neq \emptyset$
- $P_{\mathbf{M}} \subseteq M^n$, for each n-ary $P \in \mathbf{P}$
- $f_{\mathbf{M}} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$.

$\mathbf{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{OV} \to M$

For $x \in \mathrm{OV}, m \in M$, and $\mathbf{M}$-evaluation v, we define $\mathrm{v}_{x=m}$ as

$$\mathrm{v}_{x=m}(y) = \begin{cases} m & \text{if } y = x \\ \mathrm{v}(y) & \text{otherwise} \end{cases}$$

Reformulating classical semantics

Classical $\mathcal{P}$-structure: a tuple $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ where

- $M \neq \emptyset$
- $P_{\mathbf{M}} \colon M^n \to \{0, 1\}$, for each n-ary $P \in \mathbf{P}$
- $f_{\mathbf{M}} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$.

$\mathbf{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{OV} \to M$

For $x \in \mathrm{OV}, m \in M$, and $\mathbf{M}$-evaluation v, we define $\mathrm{v}_{x=m}$ as

$$\mathrm{v}_{x=m}(y) = \begin{cases} m & \text{if } y = x \\ \mathrm{v}(y) & \text{otherwise} \end{cases}$$

And now the 'fuzzy' semantics for logic L ...

$\boldsymbol{A}$-$\mathcal{P}$-structure ($\boldsymbol{A} \in \mathbb{L}$): a tuple $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ where

- $M \neq \emptyset$
- $P_{\mathbf{M}} \colon M^n \to A$, for each n-ary $P \in \mathbf{P}$
- $f_{\mathbf{M}} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$.

$\mathbf{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{OV} \to M$

For $x \in \mathrm{OV}, m \in M$, and $\mathbf{M}$-evaluation v, we define $\mathrm{v}_{x=m}$ as

$$\mathrm{v}_{x=m}(y) = \begin{cases} m & \text{if } y = x \\ \mathrm{v}(y) & \text{otherwise} \end{cases}$$

Recall classical Tarski truth definition

Interpretation of $\mathcal{P}$-terms

$$\begin{aligned} \|x\|^{\mathbf{M}}_{\mathrm{v}} &= \mathrm{v}(x) && \text{for } x \in \mathrm{OV} \\ \|f(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} &= f_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}},\ \dots,\ \|t_n\|^{\mathbf{M}}_{\mathrm{v}}) && \text{for } n\text{-ary } f \in \mathbf{F} \end{aligned}$$

Truth-values of $\mathcal{P}$-formulas

$$\begin{aligned} \|P(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} = 1 &\quad \text{iff} \quad \langle \|t_1\|^{\mathbf{M}}_{\mathrm{v}},\ \dots,\ \|t_n\|^{\mathbf{M}}_{\mathrm{v}}\rangle \in P_{\mathbf{M}} \qquad \text{for } n\text{-ary } P \in \mathbf{P} \\ \|\overline{0}\|^{\mathbf{M}}_{\mathrm{v}} = 0 & \\ \|\alpha \wedge \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 &\quad \text{iff} \quad \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 1 \text{ and } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 \\ \|\alpha \vee \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 &\quad \text{iff} \quad \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 1 \text{ or } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 \\ \|\alpha \to \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 &\quad \text{iff} \quad \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 0 \text{ or } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 \\ \|\forall x\, \varphi\|^{\mathbf{M}}_{\mathrm{v}} = 1 &\quad \text{iff} \quad \text{for each } m \in M \text{ we have } \|\varphi\|^{\mathbf{M}}_{\mathrm{v}_{x=m}} = 1 \\ \|\exists x\, \varphi\|^{\mathbf{M}}_{\mathrm{v}} = 1 &\quad \text{iff} \quad \text{there is } m \in M \text{ such that } \|\varphi\|^{\mathbf{M}}_{\mathrm{v}_{x=m}} = 1 \end{aligned}$$

Reformulating classical Tarski truth definition

Interpretation of $\mathcal{P}$-terms

$$
\begin{aligned}
\|x\|^{\mathbf{M}}_{\mathrm{v}} &= \mathrm{v}(x) && \text{for } x \in \mathrm{OV} \\
\|f(t_1, \dots, t_n)\|^{\mathbf{M}}_{\mathrm{v}} &= f_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}) && \text{for } n\text{-ary } f \in \mathbf{F}
\end{aligned}
$$

Truth-values of $\mathcal{P}$-formulas

$$
\begin{aligned}
\|P(t_1, \dots, t_n)\|^{\mathbf{M}}_{\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}) && \text{for } n\text{-ary } P \in \mathbf{P} \\
\|\overline{0}\|^{\mathbf{M}}_{\mathrm{v}} &= \overline{0}^{\mathbf{2}} \\
\|\alpha \wedge \beta\|^{\mathbf{M}}_{\mathrm{v}} &= \min_{\leq_{\mathbf{2}}}\{\|\alpha\|^{\mathbf{M}}_{\mathrm{v}}, \|\beta\|^{\mathbf{M}}_{\mathrm{v}}\} \\
\|\alpha \vee \beta\|^{\mathbf{M}}_{\mathrm{v}} &= \max_{\leq_{\mathbf{2}}}\{\|\alpha\|^{\mathbf{M}}_{\mathrm{v}}, \|\beta\|^{\mathbf{M}}_{\mathrm{v}}\} \\
\|\alpha \to \beta\|^{\mathbf{M}}_{\mathrm{v}} &= \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} \to^{\mathbf{2}} \|\beta\|^{\mathbf{M}}_{\mathrm{v}} \\
\|\forall x\, \varphi\|^{\mathbf{M}}_{\mathrm{v}} &= \inf_{\leq_{\mathbf{2}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}_{x=m}} \mid m \in M\} \\
\|\exists x\, \varphi\|^{\mathbf{M}}_{\mathrm{v}} &= \sup_{\leq_{\mathbf{2}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}_{x=m}} \mid m \in M\}
\end{aligned}
$$

And now the Tarski truth definition for ‘fuzzy’ semantics

Interpretation of $\mathcal{P}$-terms

$$
\begin{array}{rcll}
\|x\|^{\mathbf{M}}_{\mathrm{v}} & = & \mathrm{v}(x) & \text{for } x \in \mathrm{OV} \\
\|f(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & f_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}},\dots,\|t_n\|^{\mathbf{M}}_{\mathrm{v}}) & \text{for } n\text{-ary } f \in \mathbf{F}
\end{array}
$$

Truth-values of $\mathcal{P}$-formulas

$$
\begin{array}{rcll}
\|P(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & P_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}},\dots,\|t_n\|^{\mathbf{M}}_{\mathrm{v}}) & \text{for } n\text{-ary } P \in \mathbf{P} \\
\|\overline{0}\|^{\mathbf{M}}_{\mathrm{v}} & = & \overline{0}^{\boldsymbol{A}} & \\
\|\alpha \wedge \beta\|^{\mathbf{M}}_{\mathrm{v}} & = & \min_{\leq_{\boldsymbol{A}}}\{\|\alpha\|^{\mathbf{M}}_{\mathrm{v}}, \|\beta\|^{\mathbf{M}}_{\mathrm{v}}\} & \\
\|\alpha \vee \beta\|^{\mathbf{M}}_{\mathrm{v}} & = & \max_{\leq_{\boldsymbol{A}}}\{\|\alpha\|^{\mathbf{M}}_{\mathrm{v}}, \|\beta\|^{\mathbf{M}}_{\mathrm{v}}\} & \\
\|\alpha \to \beta\|^{\mathbf{M}}_{\mathrm{v}} & = & \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} \to^{\boldsymbol{A}} \|\beta\|^{\mathbf{M}}_{\mathrm{v}} & \\
\|\forall x\, \varphi\|^{\mathbf{M}}_{\mathrm{v}} & = & \inf_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}_{x=m}} \mid m \in M\} & \\
\|\exists x\, \varphi\|^{\mathbf{M}}_{\mathrm{v}} & = & \sup_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}_{x=m}} \mid m \in M\} &
\end{array}
$$

Model and semantical consequence

Problem: the infimum/supremum need not exist! In such case we take its value (and values of all its superformulas) as undefined

Definition 4.2 (Model)

A tuple $\mathbf{M} = \langle \mathbf{A}, \mathbf{M} \rangle$ is a $\mathbb{K}$-$\mathcal{P}$-model of T, $\mathbf{M} \models T$ in symbols, if

- $\mathbf{M}$ is $\boldsymbol{A}$-$\mathcal{P}$-structure for some $\boldsymbol{A} \in \mathbb{K} \subseteq \mathbb{L}$
- $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}}$ is defined $\mathbf{M}$-evaluation v and each formula φ
- $\|\psi\|^{\mathbf{M}}_{\mathrm{v}} = \overline{1}^{\boldsymbol{A}}$ for each $\mathbf{M}$-evaluation v and each $\psi \in T$

Definition 4.3 (Semantical consequence)

A $\mathcal{P}$-formula φ is a semantical consequence of a $\mathcal{P}$-theory T w.r.t. the class $\mathbb{K}$ of L-algebras, $T \models_{\mathbb{K}} \varphi$ in symbols, if for each $\mathbb{K}$-$\mathcal{P}$-model $\mathbf{M}$ of T we have $\mathbf{M} \models \varphi$.

The semantics of chains

Proposition 4.4 (Assume that x is not free in ψ ...)

$$\varphi \models_{\mathbb{L}} \forall x\, \varphi \quad \textit{thus} \quad \varphi \models_{\mathbb{K}} \forall x\, \varphi$$

$$\varphi \vee \psi \models_{\mathbb{L}_{\mathrm{lin}}} (\forall x\, \varphi) \vee \psi \quad \textit{BUT} \quad \varphi \vee \psi \not\models_{\mathbb{G}} (\forall x\, \varphi) \vee \psi$$

Observation

Thus $\models_{\mathbb{L}} \subsetneq \models_{\mathbb{L}_{\mathrm{lin}}}$ even though in propositional logic $\models_{\mathbb{L}} = \models_{\mathbb{L}_{\mathrm{lin}}}$

Axiomatization: two first-order logics over L

Minimal predicate logic $\mathrm{L}\forall^{\mathrm{m}}$:

(P)	first-order substitutions of axioms and the rule of L	
$(\forall 1)$	$\forall x\, \varphi \to \varphi(x/t)$	t substitutable for x in φ
$(\exists 1)$	$\varphi(x/t) \to \exists x\, \varphi$	t substitutable for x in φ
$(\forall 2)$	$\forall x\, (\chi \to \varphi) \to (\chi \to \forall x\, \varphi)$	x not free in χ
$(\exists 2)$	$\forall x\, (\varphi \to \chi) \to (\exists x\, \varphi \to \chi)$	x not free in χ
(gen)	from φ infer $\forall x\, \varphi$	

Predicate logic $\mathrm{L}\forall$: an the extension of $\mathrm{L}\forall^{\mathrm{m}}$ by:

$(\forall 3)$	$\forall x\, (\varphi \vee \chi) \to (\forall x\, \varphi) \vee \chi$	x not free in χ

Theorems (for x not free in χ)

The logic $\mathrm{L}\forall^{\mathrm{m}}$ proves:

1. $\chi \leftrightarrow \forall x\, \chi$
2. $\exists x\, \chi \leftrightarrow \chi$
3. $\forall x\, (\varphi \to \psi) \to (\forall x\, \varphi \to \forall x\, \psi)$
4. $\forall x\, \forall y\, \varphi \leftrightarrow \forall y\, \forall x\, \varphi$
5. $\forall x\, (\varphi \to \psi) \to (\exists x\, \varphi \to \exists x\, \psi)$
6. $\exists x\, \exists y\, \varphi \leftrightarrow \exists y\, \exists x\, \varphi$
7. $\forall x\, (\chi \to \varphi) \leftrightarrow (\chi \to \forall x\, \varphi)$
8. $\forall x\, (\varphi \to \chi) \leftrightarrow (\exists x\, \varphi \to \chi)$
9. $\exists x\, (\chi \to \varphi) \to (\chi \to \exists x\, \varphi)$
10. $\exists x\, (\varphi \to \chi) \to (\forall x\, \varphi \to \chi)$
11. $\exists x\, (\varphi \vee \psi) \leftrightarrow \exists x\, \varphi \vee \exists x\, \psi$
12. $\exists x\, (\varphi \,\&\, \chi) \leftrightarrow \exists x\, \varphi \,\&\, \chi$
13. $\exists x\, (\varphi^n) \leftrightarrow (\exists x\, \varphi)^n$

The logic $\mathrm{L}\forall$ furthermore proves:

14. $(\forall x\, \varphi) \vee \chi \leftrightarrow \forall x\, (\varphi \vee \chi)$
15. $\exists x\, (\varphi \wedge \chi) \leftrightarrow (\exists x\, \varphi) \wedge \chi$

Exercise 12

Prove at least 7 of these theorems.

$Ł\forall = Ł\forall^m$

Proposition 4.5

$Ł\forall = Ł\forall^m$.

Proof.

It is enough to show that $Ł\forall^m$ proves $(\forall 3)$. From $(\alpha \vee \beta) \leftrightarrow ((\alpha \to \beta) \to \beta)$ and (3) we obtain $\forall x\, (\varphi \vee \psi) \to \forall x\, ((\psi \to \varphi) \to \varphi)$. Now, again by (3), we have $\forall x\, ((\psi \to \varphi) \to \varphi) \to (\forall x\, (\psi \to \varphi) \to \forall x\, \varphi)$. By (7) and suffixing, $(\forall x\, (\psi \to \varphi) \to \forall x\, \varphi) \to ((\psi \to \forall x\, \varphi) \to \forall x\, \varphi)$, and finally we have $((\psi \to \forall x\, \varphi) \to \forall x\, \varphi) \to \forall x\, \varphi \vee \psi$. Transitivity ends the proof. $\square$

Syntactical properties of $\vdash_{\mathrm{L}\forall^m}$ and $\vdash_{\mathrm{L}\forall}$

Let $\vdash$ be either $\vdash_{\mathrm{L}\forall^m}$ or $\vdash_{\mathrm{L}\forall}$.

Theorem 4.6 (Congruence Property)

Let φ, ψ be sentences, χ a formula, and $\hat{\chi}$ a formula resulting from χ by replacing some occurrences of φ by ψ. Then

$$\vdash \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash \psi \leftrightarrow \varphi$$

$$\varphi \leftrightarrow \psi \vdash \chi \leftrightarrow \hat{\chi} \qquad \varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash \varphi \leftrightarrow \psi.$$

Theorem 4.7 (Constants Theorem)

Let $\Sigma \cup \{\varphi\}$ be a theory and c a constant not occurring there. Then $\Sigma \vdash \varphi(x/c)$ iff $\Sigma \vdash \varphi$.

Exercise 13

Prove the Constants Theorem for $\vdash_{\mathrm{G}\forall^m}$.

Deduction theorems

Theorem 4.8

For each $\mathcal{P}$-theory $T \cup \{\varphi, \psi\}$:

- $T, \varphi \vdash_{\mathrm{G}\forall^{\mathrm{m}}} \psi$ *iff* $T \vdash_{\mathrm{G}\forall^{\mathrm{m}}} \varphi \to \psi$.
- $T, \varphi \vdash_{\mathrm{G}\forall} \psi$ *iff* $T \vdash_{\mathrm{G}\forall} \varphi \to \psi$.
- $T, \varphi \vdash_{\text{Ł}\forall} \psi$ *iff* $T \vdash_{\text{Ł}\forall} \varphi^n \to \psi$ *for some* $n \in \mathrm{N}$.

Exercise 14

Prove the Deduction theorems.

Syntactical properties of $\vdash_{\text{L}\forall}$

Theorem 4.9 (Proof by Cases Property)

For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:

$$\frac{T, \varphi \vdash_{\text{L}\forall} \chi \qquad T, \psi \vdash_{\text{L}\forall} \chi}{T, \varphi \vee \psi \vdash_{\text{L}\forall} \chi} \qquad \text{(PCP)}$$

Proof.

We show by induction $T \vee \chi \vdash \varphi \vee \chi$ whenever $T \vdash \varphi$ and χ is a sentence; the rest is the same as in the propositional case.

Let δ be an element of the proof of φ from T: the claim is

- trivial if $\delta \in T$ or δ is an axiom;
- proved as in the propositional case if δ is obtained using (MP)
- easy if $\delta = \forall x\, \psi$ is obtained using (gen): from the IH we get $T \vee \chi \vdash \psi \vee \chi$ and using (gen), $(\forall 3)$, and (MP) we obtain $T \vee \chi \vdash (\forall x\, \psi) \vee \chi$.

□

Syntactical properties of $\vdash_{L\forall}$

Theorem 4.9 (Proof by Cases Property)

For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:

$$\frac{T, \varphi \vdash_{L\forall} \chi \qquad T, \psi \vdash_{L\forall} \chi}{T, \varphi \vee \psi \vdash_{L\forall} \chi} \tag{PCP}$$

Theorem 4.10 (Semilinearity Property)

For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:

$$\frac{T, \varphi \to \psi \vdash_{L\forall} \chi \qquad T, \psi \to \varphi \vdash_{L\forall} \chi}{T \vdash_{L\forall} \chi} \tag{SLP}$$

Proof.

Easy using PCP and $\vdash_{L\forall} (\varphi \to \psi) \vee (\psi \to \varphi)$. □

Soundness

Exercise 15

Prove for L be either $\mathrm{Ł}$ of G that

- $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \subseteq \models_{\mathbb{L}}$
- $\vdash_{\mathrm{L}\forall} \subseteq \models_{\mathbb{L}_{\mathrm{lin}}}$
- $\vdash_{\mathrm{Ł}\forall} \subseteq \models_{\mathbb{MV}}$

Recall that $\vdash_{\mathrm{G}\forall} \not\subseteq \models_{\mathbb{G}}$

Failure of certain classical theorems (for x not free in χ)

Recall:

$$\vdash_{\mathrm{L}\forall} (\forall x\, \varphi) \vee \chi \leftrightarrow \forall x\, (\varphi \vee \chi) \qquad \vdash_{\mathrm{L}\forall} \exists x\, (\varphi \wedge \chi) \leftrightarrow (\exists x\, \varphi) \wedge \chi$$

$$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \forall x\, (\chi \to \varphi) \leftrightarrow (\chi \to \forall x\, \varphi) \qquad \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \forall x\, (\varphi \to \chi) \leftrightarrow (\exists x\, \varphi \to \chi)$$

$$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \exists x\, (\chi \to \varphi) \to (\chi \to \exists x\, \varphi) \qquad \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \exists x\, (\varphi \to \chi) \to (\forall x\, \varphi \to \chi)$$

Proposition 4.11

The formulas in the first row are not provable in $\mathrm{G}\forall^{\mathrm{m}}$ *and the converse directions of formulas in the last row are provable* $\text{Ł}\forall^{\mathrm{m}}$ *but not in* $\mathrm{G}\forall$.

Exercise 16

Prove the second part of the previous proposition.

Towards completeness: Lindenbaum–Tarski algebra

Let L be G or $\mathrm{Ł}$ and $\vdash$ be either $\vdash_{\mathrm{L}\forall^{\mathrm{m}}}$ or $\vdash_{\mathrm{L}\forall}$. Let T be a $\mathcal{P}$-theory.

Lindenbaum–Tarski algebra of T ($\mathbf{LindT}_T$):

- domain $L_T = \{[\varphi]_T \mid \varphi$ a $\mathcal{P}$-sentence$\}$ where

$$[\varphi]_T = \{\psi \mid \psi \text{ a } \mathcal{P}\text{-sentence and } T \vdash \varphi \leftrightarrow \psi\}.$$

- operations:

$$\circ^{\mathbf{LindT}_T}([\varphi_1]_T, \dots, [\varphi_n]_T) = [\circ(\varphi_1, \dots, \varphi_n)]_T$$

Exercise 17

- $\mathbf{LindT}_T \in \mathbb{L}$
- $[\varphi]_T \leq_{\mathbf{LindT}_T} [\psi]_T$ iff $T \vdash \varphi \to \psi$
- $\mathbf{LindT}_T \in \mathbb{L}_{\text{lin}}$ if, and only if, T is linear.

Towards completeness: Canonical model

Canonical model ($\mathfrak{CM}_T$) of a $\mathcal{P}$-theory T (in $\vdash$): $\mathcal{P}$-structure $\langle \mathbf{LindT}_T, \mathbf{M} \rangle$ such that

- domain of $\mathbf{M}$: the set CT of closed $\mathcal{P}$-terms
- $f_{\mathbf{M}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ for each n-ary $f \in \mathbf{F}$, and
- $P_{\mathbf{M}}(t_1, \ldots, t_n) = [P(t_1, \ldots, t_n)]_T$ for each n-ary $P \in \mathbf{P}$.

A $\mathcal{P}$-theory T is $\forall$-Henkin if for each $\mathcal{P}$-formula $\psi(x)$ such that $T \nvdash \forall x\, \psi$ there is a constant c in $\mathcal{P}$ such that $T \nvdash \psi(x/c)$.

Towards completeness: Canonical model

$\forall$-Henkin: $T \nvdash \forall x\, \psi$ implies $T \nvdash \psi(x/c)$ for some constant c

Proposition 4.12

Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-sentence φ we have $\|\varphi\|^{\mathfrak{CM}_T} = [\varphi]_T$ and so $\mathfrak{CM}_T \models \varphi$ iff $T \vdash \varphi$.

Proof.

Let v be evaluation s.t. $\mathrm{v}(x_i) = t_i$ for some $t_i \in CT$. We show by induction that $\|\varphi\|^{\mathfrak{CM}_T}_{\mathrm{v}} = [\varphi(x_1/t_1, \ldots, x_n/t_n)]_T$ for each $\varphi(x_1, \ldots, x_n)$

Towards completeness: Canonical model

$\forall$-Henkin: $T \nvdash \forall x\, \psi$ implies $T \nvdash \psi(x/c)$ for some constant c

Proposition 4.12

Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-sentence φ we have $\|\varphi\|^{\mathfrak{CM}_T} = [\varphi]_T$ and so $\mathfrak{CM}_T \models \varphi$ iff $T \vdash \varphi$.

Proof.

Let v be evaluation s.t. $\mathrm{v}(x_i) = t_i$ for some $t_i \in CT$. We show by induction that $\|\varphi\|^{\mathfrak{CM}_T}_{\mathrm{v}} = [\varphi(x_1/t_1, \dots, x_n/t_n)]_T$ for each $\varphi(x_1, \dots, x_n)$

The base case and the induction step for connectives is just the definition.

Towards completeness: Canonical model

$\forall$-Henkin: $T \nvdash \forall x\, \psi$ implies $T \nvdash \psi(x/c)$ for some constant c

Proposition 4.12

Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-sentence φ we have $\|\varphi\|^{\mathfrak{CM}_T} = [\varphi]_T$ and so $\mathfrak{CM}_T \models \varphi$ iff $T \vdash \varphi$.

Proof.

Let v be evaluation s.t. $\mathrm{v}(x_i) = t_i$ for some $t_i \in CT$. We show by induction that $\|\varphi\|^{\mathfrak{CM}_T}_{\mathrm{v}} = [\varphi(x_1/t_1, \dots, x_n/t_n)]_T$ for each $\varphi(x_1, \dots, x_n)$

Quantifiers: $[\forall x\, \varphi]_T \stackrel{?}{=} \|\forall x\, \varphi\|^{\mathfrak{CM}_T} = \inf_{\leq_{\mathbf{LindT}_T}} \{[\varphi(x/t)]_T \mid t \in CT\}$

From $T \vdash \forall x\, \varphi \to \varphi(x/t)$ we get that $[\forall x\, \varphi]_T$ is a lower bound.

We show it is the largest one: take any χ s.t. $[\chi]_T \not\leq_{\mathbf{LindT}_T} [\forall x\, \varphi]_T$; thus $T \nvdash \chi \to \forall x\, \varphi$, and so $T \nvdash \forall x\, (\chi \to \varphi)$. As T is $\forall$-Henkin there is $c \in CT$ s.t. $T \nvdash (\chi \to \varphi)(x/c)$, thus $T \nvdash \chi \to \varphi(x/c)$, i.e., $[\chi]_T \not\leq_{\mathbf{LindT}_T} [\varphi(x/c)]_T$. $\square$

Completeness theorem for $\mathrm{L}\forall^{\mathrm{m}}$

Theorem 4.13 (Completeness theorem for $\mathrm{L}\forall^{\mathrm{m}}$)

Let L *be either* Ł *or* G *and* $T \cup \{\varphi\}$ *a* $\mathcal{P}$*-theory. Then:*
$T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$ *iff* $T \models_{\mathbb{L}} \varphi$.

All we need to prove this theorem is to show that:

Lemma 4.14 (Extension lemma for $\mathrm{L}\forall^{\mathrm{m}}$)

Let $T \cup \{\varphi\}$ *be a* $\mathcal{P}$*-theory such that* $T \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$*. Then there is* $\mathcal{P}' \supseteq \mathcal{P}$ *and a* $\forall$*-Henkin* $\mathcal{P}'$*-theory* $T' \supseteq T$ *such that* $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$.

Proof.

$\mathcal{P}' = \mathcal{P}$ + countably many new object constants. Let T' be T as $\mathcal{P}'$-theory. Take any $\mathcal{P}'$-formula $\psi(x)$, such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \forall x\, \psi$. Thus $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi$ and so $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(x/c)$ for some $c \in \mathcal{P}'$ not occurring in $T' \cup \{\psi\}$ (by Constants Theorem). $\square$

Completeness theorem for $\mathrm{L}\forall$

Theorem 4.15 (Completeness theorem for $\mathrm{L}\forall$)

Let L *be either* Ł *or* G *and* $T \cup \{\varphi\}$ *a* $\mathcal{P}$*-theory. Then*

$$T \vdash_{\mathrm{L}\forall} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}_{\mathrm{lin}}} \varphi.$$

All we need to prove this theorem is to show that:

Lemma 4.16 (Extension lemma for $\mathrm{L}\forall$)

Let $T \cup \{\varphi\}$ *be a* $\mathcal{P}$*-theory such that* $T \nvdash_{\mathrm{L}\forall} \varphi$*. Then there is a predicate language* $\mathcal{P}' \supseteq \mathcal{P}$ *and a linear* $\forall$*-Henkin* $\mathcal{P}'$*-theory* $T' \supseteq T$ *such that* $T' \nvdash_{\mathrm{L}\forall} \varphi$*.*

Initializing the construction

Let $\mathcal{P}'$ be the expansion of $\mathcal{P}$ by countably many new constants.

We enumerate all $\mathcal{P}'$-formulas with one free variable: $\{\chi_i(x) \mid i \in \mathrm{N}\}$.

We construct a sequence of $\mathcal{P}'$-sentences φ_i and an increasing chain of $\mathcal{P}'$-theories T_i such that $T_i \nvdash \varphi_j$ for each $j \leq i$.

Take $T_0 = T$ and $\varphi_0 = \varphi$, which fulfils our conditions.

In the induction step we distinguish two possibilities and show that the required conditions are met:

The induction step

(H1) If $T_i \vdash \varphi_i \vee \forall x\, \chi_{i+1}$: then we define $\varphi_{i+1} = \varphi_i$ and

$$T_{i+1} = T_i \cup \{\forall x\, \chi_{i+1}\}.$$

(H2) If $T_i \nvdash \varphi_i \vee \forall x\, \chi_{i+1}$, then we define $T_{i+1} = T_i$ and

$\varphi_{i+1} = \varphi_i \vee \chi_{i+1}(x/c)$ for some c not occurring in $T_i \cup \{\varphi_j \mid j \leq i\}$.

Assume, for a contradiction, that $T_{i+1} \vdash \varphi_j$ for some $j \leq i+1$. Then also $T_{i+1} \vdash \varphi_{i+1}$.

Thus in case (H1) we have $T_i \cup \{\forall x\, \chi_{i+1}\} \vdash \varphi_i$. Since, trivially, $T_i \cup \{\varphi_i\} \vdash \varphi_i$ we obtain by Proof by Cases Property that
$T_i \cup \{\varphi_i \vee \forall x\, \chi_{i+1}\} \vdash \varphi_i$ and so $T_i \vdash \varphi_i$; a contradiction!

The induction step

(H1) If $T_i \vdash \varphi_i \vee \forall x\, \chi_{i+1}$: then we define $\varphi_{i+1} = \varphi_i$ and

$$T_{i+1} = T_i \cup \{\forall x\, \chi_{i+1}\}.$$

(H2) If $T_i \nvdash \varphi_i \vee \forall x\, \chi_{i+1}$, then we define $T_{i+1} = T_i$ and

$\varphi_{i+1} = \varphi_i \vee \chi_{i+1}(x/c)$ for some c not occurring in $T_i \cup \{\varphi_j \mid j \leq i\}$.

Assume, for a contradiction, that $T_{i+1} \vdash \varphi_j$ for some $j \leq i+1$. Then also $T_{i+1} \vdash \varphi_{i+1}$.

Thus in case (H2) we have $T_i \vdash \varphi_i \vee \chi_{i+1}(x/c)$. Using Constants Theorem we obtain $T_i \vdash \varphi_i \vee \chi_{i+1}$ and thus by (gen), $(\forall 3)$, and (MP) we obtain $T_i \vdash \varphi_i \vee \forall x\, \chi_{i+1}$; a contradiction!

Final touches . . .

Let T' be a maximal theory extending $\bigcup T_i$ s.t. $T' \nvdash \varphi_i$ for each i. Such T' exists thanks to Zorn's Lemma: let $\mathcal{T}$ be a chain of such theories then clearly so is $\bigcup \mathcal{T}$.

T' is *linear:* assume that $T' \nvdash \psi \to \chi$ and $T' \nvdash \chi \to \psi$. Then there are i, j such that $T', \psi \to \chi \vdash \varphi_i$ and $T', \chi \to \psi \vdash \varphi_j$. Thus also

$$T', \psi \to \chi \vdash \varphi_{\max\{i,j\}} \text{ and } T', \chi \to \psi \vdash \varphi_{\max\{i,j\}}.$$

Thus by Semilinearity Property also $T' \vdash \varphi_{\max\{i,j\}}$; a contradiction!

T' is *$\forall$-Henkin:* if $T' \nvdash \forall x\, \chi_{i+1}$, then we must have used case (H2); since $T' \nvdash \varphi_{i+1}$ and $\varphi_{i+1} = \varphi_i \vee \chi_{i+1}(x/c))$ we also have $T' \nvdash \chi_{i+1}(x/c)$.

It works in Gödel–Dummett logic

Theorem 4.17

The following are equivalent for every set of $\mathcal{P}$-formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$:

1. $\Gamma \vdash_{\mathrm{G}\forall} \varphi$
2. $\Gamma \models_{\mathbb{G}_{\mathrm{lin}}} \varphi$
3. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Recall the proof in the propositional case

Contrapositively: assume that $T \not\vdash_{\mathrm{G}} \varphi$. Let $\boldsymbol{B}$ be a countable G-chain and e a $\boldsymbol{B}$-evaluation such that $e[T] \subseteq \{\overline{1}^{\boldsymbol{B}}\}$ and $e(\varphi) \neq \overline{1}^{\boldsymbol{B}}$.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f\colon B \to [0,1]$ such that $f(\overline{0}) = 0, f(\overline{1}) = 1$, and for each $a, b \in B$ we have:

$$a \leq b \quad \text{iff} \quad f(a) \leq f(a)$$

We define a mapping $\overline{e}\colon Fm_{\mathcal{L}} \to [0,1]$ as

$$\overline{e}(\psi) = f(e(\psi))$$

and prove (by induction) that it is $[0,1]_{\mathrm{G}}$-evaluation.

Then $\overline{e}(\psi) = 1$ iff $e(\psi) = \overline{1}^{\boldsymbol{B}}$ and so $\overline{e}[T] \subseteq \{1\}$ and $\overline{e}(\varphi) \neq 1$.

Would it work in the first-order level?

Contrapositively: assume that $T \not\vdash_{\mathrm{G}\forall} \varphi$. Let $\boldsymbol{B}$ be a countable G-chain and $\mathbf{M} = \langle \boldsymbol{B}, \mathbf{M} \rangle$ a model of T such that $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}} \neq \overline{1}^{\boldsymbol{B}}$.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f\colon B \to [0,1]$ such that $f(\overline{0}) = 0, f(\overline{1}) = 1$, and for each $a, b \in B$ we have:

$$a \leq b \quad \text{iff} \quad f(a) \leq f(a)$$

Would it work in the first-order level?

Contrapositively: assume that $T \nvdash_{\mathrm{G}\forall} \varphi$. Let $\boldsymbol{B}$ be a countable G-chain and $\mathbf{M} = \langle \boldsymbol{B}, \mathbf{M} \rangle$ a model of T such that $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}} \neq \overline{1}^{\boldsymbol{B}}$.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f \colon B \to [0, 1]$ such that $f(\overline{0}) = 0, f(\overline{1}) = 1$, and for each $a, b \in B$ we have:

$$f(a \wedge b) = f(a) \wedge f(b) \text{ and } f^{-1}(a \wedge b) = f^{-1}(a) \wedge f^{-1}(b)$$

Would it work in the first-order level?

Contrapositively: assume that $T \not\vdash_{\mathrm{G}\forall} \varphi$. Let $\boldsymbol{B}$ be a countable G-chain and $\mathbf{M} = \langle \boldsymbol{B}, \mathbf{M} \rangle$ a model of T such that $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}} \neq \bar{1}^{\boldsymbol{B}}$.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f\colon B \to [0,1]$ such that $f(\bar{0}) = 0, f(\bar{1}) = 1$, and for each $a, b \in B$ we have:

$$f(\bigwedge_{a \in X} a) = \bigwedge_{a \in X} f(a) \text{ and } f^{-1}(\bigwedge_{a \in X} a) = \bigwedge_{a \in X} f^{-1}(a)$$

We define a $[0,1]_{\mathrm{G}}$-structure $\bar{\mathbf{M}}$ with the same domain, functions and

$$P_{\bar{\mathbf{M}}}(x_1, \ldots, x_n) = f(P_{\mathbf{M}}(x_1, \ldots, x_n))$$

and prove (by induction) that $\|\psi\|^{\bar{\mathbf{M}}}_{\mathrm{v}} = f(\|\psi\|^{\mathbf{M}}_{\mathrm{v}})$. Then $\|\psi\|^{\bar{\mathbf{M}}}_{\mathrm{v}} = 1$ iff $\|\psi\|^{\mathbf{M}}_{\mathrm{v}} = \bar{1}^{\boldsymbol{B}}$ and so $\langle [0,1]_{\mathrm{G}}, \bar{\mathbf{M}} \rangle$ is model of T and $\|\varphi\|^{\bar{\mathbf{M}}}_{\mathrm{v}} \neq 1$.

What about the case of Łukasiewicz logic?

Theorem 4.18

There is a formula φ such that $\models_{[0,1]_{Ł}} \varphi$ and $\not\vdash_{Ł\forall} \varphi$.

Neither the set of theorems nor the set of satisfiable formulas w.r.t. the models of standard MV-algebra $[0,1]_{Ł}$ are recursively enumerable. In fact we have:

Theorem 4.19 (Ragaz, Goldstern, Hájek)

The set $\mathrm{stTAUT}(Ł\forall)$ is Π_2-complete and $\mathrm{stSAT}(Ł\forall)$ is Π_1-complete.

Finite model property: The classical case

- Valid sentences of $\mathrm{CL}\forall$ (in any predicate language) are recursively enumerable thanks to the completeness theorem.
- Löwenheim (1915): Monadic classical logic (the fragment of $\mathrm{CL}\forall$ only with unary predicates and no functional symbols) has the finite model property, and hence it is decidable.
- Church (1936) and Turing (1937): if the predicate language contains at least a binary predicate, then $\mathrm{CL}\forall$ is undecidable.
- Surány (1959): The fragment of $\mathrm{CL}\forall$ with three variables is undecidable.
- Mortimer (1975): The fragment of $\mathrm{CL}\forall$ with two variables has the finite model property, and hence it is decidable.

Finite model property: the fuzzy case

In Gödel–Dummett logic the FMP does not even hold for formulas with one variable (a model is finite if it has a finite domain).

Example in $\mathrm{G}\forall = \models_{[0,1]_\mathrm{G}}$

$$\varphi = \neg\forall x\, P(x) \;\wedge\; \neg\exists x\, \neg P(x).$$

Evidently φ has no finite model and so $\varphi \models^{\mathrm{fin}}_{[0,1]_\mathrm{G}} \overline{0}$. But consider $[0,1]_\mathrm{G}$-model $\mathbf{M}$ with domain N, where $P_\mathbf{M}(n) = \frac{1}{n+1}$. Then clearly for each $n \in \mathrm{N}$: $\|P(n)\| > 0$ and $\inf_{n\in\mathrm{N}} \|P(n)\| = 0$, i.e., $\mathbf{M} \models \varphi$, and so $\varphi \not\models_{[0,1]_\mathrm{G}} \overline{0}$.

The infimum is not the minimum, it is not *witnessed*.

Exercise 18

Show that $\models_{[0,1]_{\text{Ł}}}$ does not have the FMP (hint: use the formula $\exists x\,(P(x) \leftrightarrow \neg P(x))\ \&\ \forall x\,\exists y\,(P(x) \leftrightarrow P(y)\ \&\ P(y))$).

Witnessed models

Definition 4.20

A $\mathcal{P}$-model $\mathbf{M}$ is witnessed if for each $\mathcal{P}$-formula $\varphi(x, \vec{y})$ and for each $\mathbf{M}$-evaluation v, there are $s, i \in M$ such that:

$$\|\forall x\, \varphi\|^{\mathbf{M}}_{\mathrm{v}} = \|\varphi\|^{\mathbf{M}}_{\mathrm{v}_{x=i}} \qquad \|\exists x\, \varphi\|^{\mathbf{M}}_{\mathrm{v}} = \|\varphi\|^{\mathbf{M}}_{\mathrm{v}_{x=s}}.$$

Exercise 19

Consider formulas

$$(W\exists)\ \ \exists x\, (\exists y\, \psi(x/y) \to \psi) \qquad (W\forall)\ \ \exists x\, (\psi \to \forall y\, \psi(x/y))$$

Show that not all models of these formulas are witnessed and these formulas are

- true in all witnessed models of $\mathrm{G}\forall$
- not provable in $\mathrm{G}\forall$
- provable in (true in all models of) $\mathrm{Ł}\forall$

Witnessed logic and witnessed completeness

Theorem 4.21 (Witnessed completeness theorem for $Ł\forall$)

Let $T \cup \{\varphi\}$ *a theory. Then* $T \vdash_{Ł\forall} \varphi$ *iff for each witnessed* $\mathbb{MV}_{\mathrm{lin}}$*-model* $\mathbf{M}$ *of* T *we have* $\mathbf{M} \models \varphi$*.*

Definition 4.22

The logic $\mathrm{G}\forall^{\mathrm{w}}$ is the extension of $\mathrm{G}\forall$ by the axioms $(W\exists)$ and $(W\forall)$.

(note that the analogous definition for L would yield $Ł\forall^{\mathrm{w}} = Ł\forall$)

Theorem 4.23 (Witnessed completeness theorem for $\mathrm{G}\forall^{\mathrm{w}}$)

Let $T \cup \{\varphi\}$ *be a theory. Then* $T \vdash_{\mathrm{G}\forall^{\mathrm{w}}} \varphi$ *iff for each witnessed* $\mathbb{G}_{\mathrm{lin}}$*-model* $\mathbf{M}$ *of* T *we have* $\mathbf{M} \models \varphi$*.*

A proof

A theory T is Henkin if it is $\forall$-Henkin and for each $\varphi(x)$ such that $T \vdash (\exists x)\varphi$ there is a constant such that $T \vdash \varphi(x/c)$.

Assume that we can prove:

Lemma 4.24 (Full Extension lemma for $\mathrm{L}\forall$)

Let $T \cup \{\varphi\}$ be a $\mathcal{P}$-theory such that $T \nvdash_{\mathrm{L}\forall^{\mathrm{w}}} \varphi$. Then there is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$ and a linear Henkin $\mathcal{P}'$-theory $T' \supseteq T$ such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{w}}} \varphi$.

Then the proof of the witnessed completeness is an easy corollary of the following straightforward proposition

Proposition 4.25

Let T be a Henkin $\mathcal{P}$-theory. Then $\mathfrak{CM}_T$ is a witnessed model.

Before we prove the full extension lemma . . .

Definition 4.26

Let $\mathcal{P}_1 \subseteq \mathcal{P}_2$. A $\mathcal{P}_2$-theory T_2 is a conservative expansion of a $\mathcal{P}_1$-theory T_1 if for each $\mathcal{P}_1$-formula φ, $T_2 \vdash \varphi$ iff $T_1 \vdash \varphi$.

Proposition 4.27

For each predicate language $\mathcal{P}$, each $\mathcal{P}$-theory T, each $\mathcal{P}$-formula $\varphi(x)$, and any constant $c \notin \mathcal{P}$ holds that $T \cup \{\varphi(x/c)\}$ is a conservative expansion (in the logic $\mathrm{L}\forall$) of $T \cup \{\exists x\, \varphi\}$.

Proof.

Assume that $T \cup \{\varphi(x/c)\} \vdash_{\mathrm{L}\forall} \psi$. Then, by Deduction Theorem, there is n such that $T \vdash_{\mathrm{L}\forall} \varphi(x/c)^n \to \psi$. Thus by the Constants Theorem and $(\exists 2)$ we obtain $T \vdash_{\mathrm{L}\forall} \exists x\, (\varphi^n) \to \psi$. Using (13) we obtain $T \vdash_{\mathrm{L}\forall} (\exists x\, \varphi)^n \to \psi$. Deduction Theorem completes the proof. $\square$

A proof of full extension lemma

Modify the proof of the extension lemma, s.t. after going through options (H1) and (H2) on the i-th step we construct theories T'_{i+1}. Then we distinguish two new options:

(W1) If $T'_{i+1}, (\exists x)\chi_{i+1} \nvdash \varphi_{i+1}$: then we define $T_{i+1} = T'_{i+1} \cup \{\chi_{i+1}(x/c)\}$.
for some c not occurring in $T'_i \cup \{\varphi_j \mid j \leq i\}$.

(W2) If $T'_{i+1}, (\exists x)\chi_{i+1} \vdash \varphi_{i+1}$: then we define $T_{i+1} = T'_{i+1}$

The induction assumption $T_{i+1} \nvdash \varphi_{i+1}$ holds: in (W2) trivially, in case of (W1) we use the fact that $T'_{i+1} \cup \{\chi_{i+1}(x/c)\}$ is a conservative expansion of $T'_{i+1} \cup \{(\exists x)\chi_{i+1}\}$.

The rest is the same as the proof of the extension lemma, we only show that T' is Henkin: it $T' \vdash (\exists x)\chi_{i_1}$ then we used case (W1) (from $T', (\exists x)\chi_{i+1} \vdash \varphi_{i+1}$, a contradiction). Thus $T' \vdash \chi_{i+1}(x/c)$.

Skolemization

Theorem 4.28

For Gödel–Dummett logic we have: $T \cup \{\forall \vec{y}\, \varphi(x/f_\varphi(\vec{y}))\}$ *is a conservative expansion of* $T \cup \{\forall \vec{y}\, \exists x\, \varphi\}$ *for each* $\mathcal{P}$*-theory* $T \cup \{\varphi(x, \vec{y})\}$*, and a functional symbol* $f_\varphi \notin \mathcal{P}$ *of the proper arity.*

A hint of the proof.

Take $\mathcal{P}$-formula χ s.t. $T \cup \{\forall y\, \exists x\, \varphi\} \nvdash \chi$. Let T' be a Henkin $\mathcal{P}'$-theory $T' \supseteq T \cup \{\forall y\, \exists x\, \varphi\}$ s.t. $T' \nvdash \chi$, and hence $\mathfrak{CM}_{T'} \not\models \chi$.
For each closed $\mathcal{P}'$-term t we have $T' \vdash \exists x\, \varphi(x, y/t)$ (by $(\forall 1)$) and hence there is a $\mathcal{P}'$-constant c_t such that $T' \vdash \varphi(x/c_t, y/t)$.
We define a model $\mathbf{M}$ by expanding $\mathfrak{CM}_{T'}$ with one functional symbol defined as: $(f_\varphi)_{\mathbf{M}}(t) = c_t$
Observe that for each $\mathcal{P}'$-formula: $\mathbf{M} \models \psi$ iff $\mathfrak{CM}'_T \models \psi$
Thus $\mathbf{M} \models T$ and $\mathbf{M} \not\models \chi$ and so clearly $\mathbf{M} \models \forall y\, \varphi(x/f_\varphi(y))$
And so we have established $T \cup \{\forall y\, \varphi(x/f_\varphi(y))\} \nvdash \chi$. □

Traditional fuzzy mathematics/logic

Fuzzy set: a mapping $D \to [0, 1]$

Fuzzy mathematics: a collection of classical mathematical results with sets replaced by fuzzy sets — there are exceptions

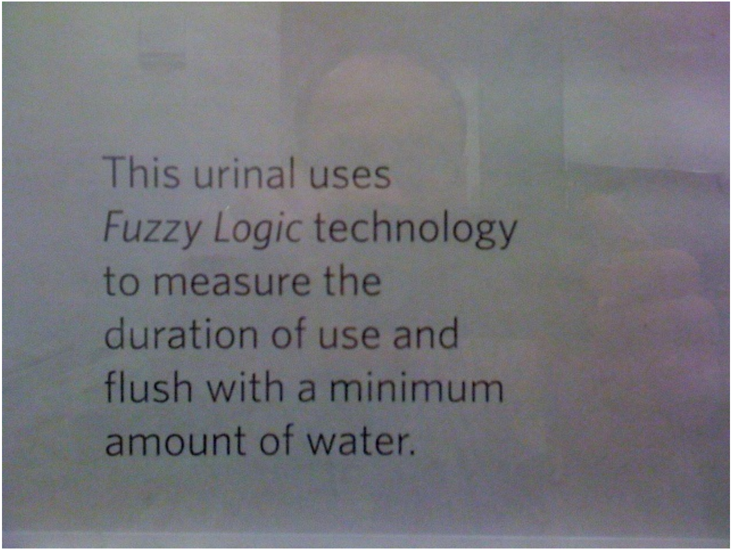
Fuzzy logic (1st meaning): a classical mathematical logic with

- two-valued evaluation is replaced by [0,1]-evaluation
- ‘truth functions’ of $\wedge, \vee, \neg$ computed as: $\min$, $\max$, $1 - x$

Fuzzy logic (2nd meaning): a label to anything involving fuzzy sets

A success of Fuzzy Logic: a useful way to describe (some aspects of) ‘graded’ behavior of real-word predicates

Traditional fuzzy mathematics/logic



Mathematical fuzzy logic

MFL is a bunch of non-classical logics with syntax, semantics, ...

MFL aims at describing *the laws of truth preservation* in reasoning under (a certain form of) vagueness/gradedness

MFL has an interpretation in terms of truth degrees, which is to be seen as just a *model* (a classical rendering of vague phenomena)

Such models have originally been employed for 'suggesting' the laws of reasoning under gradedness, BUT they must be regarded as secondary (essentially classical, not genuinely 'fuzzy')

A logic-based approach towards fuzzy mathematics?

> *"It is the opinion of the author that from a mathematical viewpoint the important feature of fuzzy set theory is the replacement of the two-valued logic by a multiple-valued logic. [. . . I]t is now clear how we can find for every mathematical notion its 'fuzzy counterpart'. Since every mathematical notion can be written as a formula in a formal language, we have only to internalize, i.e. to interpret these expressions by the given multiple-valued logic."*
>
> *Höhle, Fuzzy real numbers as Dedekind cuts w.r.t. a multiple-valued logic, FSS* **1987**

He suggests a use of multiple-valued logic *to obtain* 'fuzzy counterparts' of classical mathematical notions

We go further: we also would like to *reason* about these concepts

Formal fuzzy mathematics

Architecture of classical mathematics:

- Logic: (first-order) Boolean logic

  governs reasoning in mathematical theories

- Foundations: set theory (type theory, ...)

  a formal theory giving a general framework

- Particular disciplines: graph theory, topology, ...

  formalized within the foundational theory

Proposed architecture of fuzzy mathematics:

- Logic: (first-order) mathematical fuzzy logic

  developed enough for building formal theories

- Foundations: a kind of formal fuzzy set theory

  that's what this talk is all about

- Particular disciplines: fuzzy graph theory, fuzzy topology, ...

  formalized within the foundational theory

Formal fuzzy mathematics

First-order fuzzy logic is strong enough to support non-trivial formal mathematical foundational theories

Mathematical concepts in such theories show gradual rather than bivalent structure

Examples:

- Skolem, Hájek (1960, 2005): naïve set theory over Ł
- Takeuti–Titani (1994): ZF-style fuzzy set theory
  in a system close to Gödel logic ($\Rightarrow$ contractive)
- Hájek–Haniková (2003): ZF-style set theory over $HL_\triangle$
- Novák (2004): Church-style fuzzy type theory over $IMTL_\triangle$
- Běhounek–Cintula (2005): Fuzzy Class Theory

Hájek–Haniková fuzzy set theory

Logic: First-order $\mathrm{HL}_{\triangle}$ with identity

Language: $\in$

Axioms (z not free in φ):

- $\triangle\forall u\,(u \in x \leftrightarrow u \in y) \to x = y$ (extensionality)
- $\exists z\,\triangle\forall y\,\neg(y \in z)$ (empty set $\emptyset$)
- $\exists z\,\triangle\forall u\,(u \in z \leftrightarrow (u = x \vee u = y))$ (pair $\{x, y\}$)
- $\exists z\,\triangle\forall u\,(u \in z \leftrightarrow \exists y\,(u \in y \,\&\, y \in x))$ (union $\bigcup$)
- $\exists z\,\triangle\forall u\,(u \in z \leftrightarrow \triangle(\forall x \in u)(x \in y))$ (weak power)
- $\exists z\,\triangle(\emptyset \in z \,\&\, (\forall x \in z)(x \cup \{x\} \in z))$ (infinity)
- $\exists z\,\triangle\forall u\,(u \in z \leftrightarrow (u \in x \,\&\, \varphi(u, x))$ (separation)
- $\exists z\,\triangle[(\forall u \in x)\exists v\,\varphi(u, v) \to (\forall u \in x)(\exists v \in z)\varphi(u, v)]$ (collection)
- $\triangle\forall x\,((\forall y \in x)\varphi(y) \to \varphi(x)) \to \triangle\forall x\,\varphi(x)$ ($\in$-induction)
- $\exists z\,\triangle(\forall u\,(u \in z \vee \neg(u \in z))\,\&\,(\forall u \in x)(u \in z))$ (support)

Properties

Semantics: A cumulative hierarchy of HL-valued fuzzy sets

Features:

- Contains an inner model of classical ZF:
  (as the subuniverse of hereditarily crisp sets)
- Conservatively extends classical ZF with fuzzy sets
- Generalizes Takeuti–Titani's construction
  in a non-contractive fuzzy logic

Cantor–Łukasiewicz set theory

Logic: First-order Łukasiewicz logic Ł∀

Language: $\in$, set comprehension terms $\{x \mid \varphi\}$

Axioms:

- $y \in \{x \mid \varphi\} \leftrightarrow \varphi(y)$ (unrestricted comprehension)

Features:

- Non-contractivity of Ł blocks Russell's paradox
- Consistency conjectured by Skolem (1960—still open: in 2010 a gap found by Terui in White's 1979 consistency proof)
- Adding extensionality is inconsistent with CŁ
- Open problem: define a reasonable arithmetic in CŁ (some negative results by Hájek, 2005)

Proposed foundations: Fuzzy Class Theory

Great flexibility and generality	$\Rightarrow$	"arbitrary" fuzzy logic L
Analogy with classical foundations	$\Rightarrow$	higher-order
Axiomatizability	$\Rightarrow$	Henkin-style

Henkin-style higher-order fuzzy logic L = FCT over L

Intended models = Zadeh's fuzzy sets of any order over a fixed domain

Soundness $\Rightarrow$ results are valid of real fuzzy sets

Běhounek–C: *Fuzzy class theory.* FSS 2005

Fuzzy Class Theory—FCT

Fuzzy Class Theory is:

- Henkin-style higher-order fuzzy *logic*, which is proposed for the foundational theory of:
- an open *project* with the aim to develop formal fuzzy mathematics within a unified axiomatic framework provided by a suitable *mathematical fuzzy logic* by the 'deductive means' provided by that logic.

Formal fuzzy mathematics is NOT:

- 'incompatible' with classical mathematics $\Rightarrow$ 1) it contains CM
  $\Rightarrow$ 2) CM can be viewed as its 'limit' case
- an *alternative* mathematics
  $\Rightarrow$ we keep classical metamathematics

Setting the stage (for a simplified account)—algebra

Recall the standard MV-algebra $[0,1]_{Ł} = \langle [0,1], \&, \to, \wedge, \vee, 0, 1\rangle$ where

$$x \mathbin{\&} y = \max\{x+y-1, 0\} \qquad x \to y = \min\{1-x+y, 1\}$$
$$x \wedge y = \min\{x, y\} \qquad x \vee y = \max\{x, y\}$$

Add a unary connective $\triangle$ interpreted as $\triangle 1 = 1$ and $\triangle x = 0$ for $x < 1$

Setting the stage—language

Consider three-sorted predicate language with sorts for

- O objects
- C classes of objects
- $\mathcal{C}$ classes of classes of objects

binary predicates

- $\in \subseteq O \times C$ and $\in \subseteq C \times \mathcal{C}$
- $= \subseteq O \times O$ and $= \subseteq C \times C$ and $= \subseteq \mathcal{C} \times \mathcal{C}$

and terms:

- $\langle \cdot, \cdot \rangle : O^2 \to O$, we write Rxy for '$\langle x, y \rangle \in X$'
- $\{x \mid \varphi\}$ gives a class and $\{X \mid \varphi\} \in \mathcal{C}$ a class of classes

We shall also use defined binary predicates:

- $A \subseteq B = \forall x\, (x \in A \to x \in B)$
- $A \approx B = (A \subseteq B) \wedge (B \subseteq A)$

Setting the stage—models

Intended models:

- Object variables range over a set (universe) U
- Class variables range over $[0,1]^U$
- Class-class variables range over $[0,1]^{[0,1]^U}$

'General' models for an MV-chain $\boldsymbol{A}$:

- Object variables range over a set (universe) U
- Class variables range over a subset of A^U
- Class-class variables range over a subset of A^{A^U}

Setting the stage—axiomatization?

W.r.t. intended models: not a nice one (it contains second-order logic)

W.r.t. general models: yes (due to the completeness theorem)

But even soundness w.r.t. intended models is very usefull

Setting the stage—axiomatization

First-order axioms: those of first order Łukasiewicz logic with $\triangle$

Equality axioms: as usual plus $\exists x, y\, (\triangle(x = y) \leftrightarrow x = y)$

Additional axioms:

- *Comprehension axioms:*

$$\forall y\, (y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)) \text{ and } \forall Y\, (Y \in \{X \mid \varphi(X)\} \leftrightarrow \varphi(Y))$$

- *Extensionality:*

$$\exists x\, \triangle(x \in A \leftrightarrow x \in B) \rightarrow A = B$$

$$\exists X\, \triangle(X \in \mathcal{A} \leftrightarrow X \in \mathcal{B}) \rightarrow \mathcal{A} = \mathcal{B}$$

- *Axioms for tuples:* tuples equal iff all components equal, etc.

Properties of fuzzy relations

$$\begin{aligned} \mathrm{Refl}(R) &= \forall x\, Rxx \\ \mathrm{Sym}(R) &= \forall xy\, (Rxy \to Ryx) \\ \mathrm{Trans}(R) &= \forall xyz\, (Rxy \,\&\, Ryz \to Rxz) \end{aligned}$$

Consider the domain $U = \{1, \ldots, 6\}$:

$$P_1 = \begin{pmatrix} 1.0 & 1.0 & 0.5 & 0.4 & 0.3 & 0.0 \\ 0.8 & 1.0 & 0.4 & 0.4 & 0.3 & 0.0 \\ 0.7 & 0.9 & 1.0 & 0.8 & 0.7 & 0.4 \\ 0.9 & 1.0 & 0.7 & 1.0 & 0.9 & 0.6 \\ 0.6 & 0.8 & 0.8 & 0.7 & 1.0 & 0.7 \\ 0.3 & 0.5 & 0.6 & 0.4 & 0.7 & 1.0 \end{pmatrix}$$

We compute: $\mathrm{Refl}(P_1) = 1$, $\mathrm{Trans}(P_1) = 1$, $\mathrm{Sym}(P_1) = 0.4$

Properties of fuzzy relations

$$\begin{array}{rcl} \mathrm{Refl}(R) & = & \forall x\, Rxx \\ \mathrm{Sym}(R) & = & \forall xy\,(Rxy \to Ryx) \\ \mathrm{Trans}(R) & = & \forall xyz\,(Rxy \,\&\, Ryz \to Rxz) \end{array}$$

Add some disturbances to P_1 and get

$$P_2 = \begin{pmatrix} 1.00 & 1.00 & 0.56 & 0.40 & 0.30 & 0.00 \\ 0.87 & 1.00 & 0.33 & 0.44 & 0.26 & 0.02 \\ 0.67 & 0.92 & 0.93 & 0.87 & 0.70 & 0.39 \\ 0.93 & 1.00 & 0.64 & 1.00 & 0.97 & 0.67 \\ 0.52 & 0.79 & 0.82 & 0.71 & 1.00 & 0.59 \\ 0.27 & 0.50 & 0.61 & 0.41 & 0.72 & 1.00 \end{pmatrix}$$

We compute: $\mathrm{Refl}(P_2) = 0.93$, $\mathrm{Sym}(P_2) = 0.41$, $\mathrm{Trans}(P_2) = 0.85$

Graded theories

Consequence of the methodology and apparatus:

Not only $\in$, but all notions are naturally graded

In FCT we have:

- Graded inclusion $\subseteq$ … inclusion to a degree
- Graded reflexivity Refl … reflexivity to a degree
- Graded property “being a fuzzy topological space”
- …

Graded properties of fuzzy relations pursued to a certain extent already by Gottwald (1993, 2001), Bělohlávek (2002), …

Reading of graded theorems

Consider theorem:

$$\mathrm{Trans}\, R \,\&\, \mathrm{Trans}\, S \to \mathrm{Trans}(R \cap S)$$

It says: "*The more* both R and S are transitive,
the more their intersection is transitive"

Compare it to the theorem of traditional fuzzy mathematics:

Assume $\mathrm{Trans}\, R$ and $\mathrm{Trans}\, S$ *(to degree 1)*. Then $\mathrm{Trans}(R \cap S)$.

$\Rightarrow$ Theorems should be formulated as implications,
proved by chains of provable implications in fuzzy logic

The fully graded approach

- Graded notions generalize the traditional (non-graded) ones
  traditional = $\triangle$(graded)
- Graded notions allow to infer relevant information when the traditional conditions are *almost* fulfilled cf. 0.999-reflexivity
- Graded notions are easily be handled by FCT
  inferring by the rules of fuzzy logic
- Graded notions are more fuzzy properties of fuzzy sets crisp?

Exponents

Classical theorems: $\varphi_1 \mathbin{\&} \ldots \mathbin{\&} \varphi_n \to \psi$

FFL theorems: $\varphi_1^{k_1} \mathbin{\&} \ldots \mathbin{\&} \varphi_n^{k_n} \to \psi$

If we can prove

$$\varphi_1^2 \mathbin{\&} \varphi_2^1 \to \psi_1 \quad \text{and} \quad \varphi_1^1 \mathbin{\&} \varphi_2^2 \to \psi_2$$

then the best we can get

$$\begin{aligned} \varphi_1^3 \mathbin{\&} \varphi_2^3 &\quad\to\quad \psi_1 \mathbin{\&} \psi_2 \\ \varphi_1^2 \mathbin{\&} \varphi_2^2 &\quad\to\quad \psi_1 \wedge \psi_2 \end{aligned}$$

Preconditions and Compound Notions

Traditional fuzzy mathematics

$$\begin{aligned} \mathrm{Preord}\, R &= \mathrm{Refl}\, R \,\&\, \mathrm{Trans}\, R \\ \mathrm{Sim}\, R &= \mathrm{Refl}\, R \,\&\, \mathrm{Sym}\, R \,\&\, \mathrm{Trans}\, R \end{aligned}$$

We should define

$$\begin{aligned} \mathrm{Preord}^{r,t}\, R &= \mathrm{Refl}^{r}\, R \,\&\, \mathrm{Trans}^{t}\, R \\ \mathrm{Sim}^{r,s,t}\, R &= \mathrm{Refl}^{r}\, R \,\&\, \mathrm{Sym}^{s}\, R \,\&\, \mathrm{Trans}^{t}\, R \end{aligned}$$

In fact we define (r, t)-preorders and (r, s, t)-similarities.

$(2, 5)$-preorders are more sensitive to imperfections in transitivity than in reflexivity

$(10, 1)$-preorders are much more sensitive to flaws in reflexivity than in transitivity.

Fuzzy relations and fuzzy partitions

Similarity

$$
\begin{aligned}
\mathrm{Refl}(R) &= \forall x\, Rxx \\
\mathrm{Sym}(R) &= \forall xy\, (Rxy \to Ryx) \\
\mathrm{Trans}(R) &= \forall xyz\, (Rxy \,\&\, Ryz \to Rxz) \\
\mathrm{Sim}(R) &= \mathrm{Refl}(R) \,\&\, \mathrm{Sym}(R) \,\&\, \mathrm{Trans}(R)
\end{aligned}
$$

Partitions

$$
\begin{aligned}
\mathrm{Crisp}(\mathcal{A}) &= \forall A \,\triangle(A \in \mathcal{A} \vee \neg(A \in \mathcal{A})) \\
\mathrm{NormM}(\mathcal{A}) &= \forall A \in \mathcal{A}\, \exists x \,\triangle(x \in A) \\
\mathrm{Cover}(\mathcal{A}) &= \forall x\, \exists A \in \mathcal{A} \,\triangle(x \in A) \\
\mathrm{Disj}(\mathcal{A}) &= \forall A, B \in \mathcal{A}\, \exists x\, ((x \in A \,\&\, x \in B) \to A \approx B) \\
\mathrm{Part}(\mathcal{A}) &= \mathrm{Crisp}(\mathcal{A}) \,\&\, \mathrm{NormM}(\mathcal{A}) \,\&\, \mathrm{Cover}(\mathcal{A}) \,\&\, \mathrm{Disj}(\mathcal{A})
\end{aligned}
$$

This part of the talk is based on: Běhounek–Bodenhofer–C: *Relations in Fuzzy Class Theory: Initial steps.* FSS 2008.

Note: $\triangle\,\mathrm{Part}(\mathcal{A})$ iff $\mathcal{A}$ is T-partition   De Baets, Mesiar: *T-partitions.* FSS 1998

From similarities to partitions

Definitions

$$[x]_R = \{y \mid Ryx\}.$$

$$\mathrm{V}/R \;=\; \{A \mid \exists x\,(A = [x]_R)\}$$

Results

- $\mathrm{Crisp}(\mathrm{V}/R)$
- $\triangle\,\mathrm{Refl}(R) \to \mathrm{Cover}(\mathrm{V}/R)\;\&\;\mathrm{NormM}(\mathrm{V}/R)$ $\qquad \mathrm{Refl}(R) \to \exists x\,(x \in [x]_R)$
- $\mathrm{Trans}^2(R)\;\&\;\mathrm{Sym}(R) \to \mathrm{Disj}(\mathrm{V}/R)$ $\qquad \mathrm{Trans}(R) \to \exists xy\,(Rxy \to [x]_R \subseteq [y]_R)$

1. $Rzx\;\&\;Rxy \to Rzy$ — $\mathrm{Trans}(R)$
2. $(Rzx\;\&\;Rxy \to Rzy) \to (Rxy \to (Rzx \to Rzy))$ — propositional axiom
3. $Rxy \to (Rzx \to Rzy)$ — 1., 2., MP
4. $Rxy \to (z \in [x]_R \to z \in [y]_R)$ — 3., comprehension axioms
5. $\exists xy\,(Rxy \to [x]_R \subseteq [y]_R)$ — generalization
5. $\mathrm{Trans}(R) \to \exists xy\,(Rxy \to [x]_R \subseteq [y]_R)$ — deduction theorem

From similarities to partitions

Definitions

$$[x]_R = \{y \mid Ryx\}.$$

$$\mathrm{V}/R \;=\; \{A \mid \exists x\,(A = [x]_R)\}$$

Results

- $\mathrm{Crisp}(\mathrm{V}/R)$
- $\triangle\,\mathrm{Refl}(R) \to \mathrm{Cover}(\mathrm{V}/R) \,\&\, \mathrm{NormM}(\mathrm{V}/R)$ $\qquad \mathrm{Refl}(R) \to \exists x\,(x \in [x]_R)$
- $\mathrm{Trans}^2(R) \,\&\, \mathrm{Sym}(R) \to \mathrm{Disj}(\mathrm{V}/R)$ $\qquad \mathrm{Trans}(R) \to \exists xy\,(Rxy \to [x]_R \subseteq [y]_R)$

1. $Rxy \,\&\, Rxz \to Ryz$ $\qquad \mathrm{Trans}(R), \mathrm{Sym}(R)$
2. $x \in [y]_R \,\&\, x \in [z]_R \to Ryz$ $\qquad$ 1., comprehension axioms
3. $x \in [y]_R \,\&\, x \in [z]_R \to [y]_R \subseteq [z]_R$ $\qquad$ 2., $\mathrm{Trans}(R)$
4. $x \in [y]_R \,\&\, x \in [z]_R \to [z]_R \subseteq [y]_R$ $\qquad$ analogously, $\mathrm{Trans}^2(R), \mathrm{Sym}(R)$
5. $x \in [y]_R \,\&\, x \in [z]_R \to [z]_R \approx [y]_R$ $\qquad$ 3., 4.
6. $\exists x\,(x \in [y]_R \,\&\, x \in [z]_R) \to [y]_R \approx [z]_R$ $\qquad$ 5.

From similarities to partitions

Definitions

$$[x]_R = \{y \mid Ryx\}.$$

$$\mathrm{V}/R \ = \ \{A \mid \exists x\,(A = [x]_R)\}$$

Results

- $\mathrm{Crisp}(\mathrm{V}/R)$
- $\triangle\,\mathrm{Refl}(R) \to \mathrm{Cover}(\mathrm{V}/R)\ \&\ \mathrm{NormM}(\mathrm{V}/R)$ $\qquad \mathrm{Refl}(R) \to \exists x\,(x \in [x]_R)$
- $\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R) \to \mathrm{Disj}(\mathrm{V}/R)$ $\qquad \mathrm{Trans}(R) \to \exists xy\,(Rxy \to [x]_R \subseteq [y]_R)$

So together we proved:

$$\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R)\ \&\ \triangle\,\mathrm{Refl}(R) \to \mathrm{Part}(\mathrm{V}/R)$$

$$\triangle\,\mathrm{Sim}(R) \to \triangle\,\mathrm{Part}(\mathrm{V}/R)$$

Semantical content

$$\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R) \to \mathrm{Disj}(\mathrm{V}/R)$$

This says that for any fuzzy relation $R\colon U^2 \to [0,1]$ we have:

$$(\inf_{x,y,z\in U}(Rxy\ \&\ Ryz \to Rxz))^2\ \&\ \inf_{x,y\in U}(Rxy \to Ryx) \leq$$
$$\inf_{A,B\in \mathrm{V}/R}(\sup_{z\in U}(z\in A\ \&\ z\in B) \to \inf_{z\in U}(z\in A \leftrightarrow z\in B))$$

where

$$x\ \&\ y = \max\{x+y-1, 0\} \qquad x \to y = \min\{1-x+y, 1\}$$

Consider relation such that $\|\,\mathrm{Trans}(R)\| = \|\,\mathrm{Sym}(R)\| = 0.9$. Then we know that

$$0.7 \leq \|\,\mathrm{Disj}(\mathrm{V}/R)\|$$

From partitions to similarities

Definition

$$R^{\mathcal{A}} \quad = \quad \{\langle x, y\rangle \mid (\exists A \in \mathcal{A})(x \in A \mathbin{\&} y \in A)\}$$

Results

- $\mathrm{Sym}(R^{\mathcal{A}})$
- $\mathrm{Crisp}(\mathcal{A}) \mathbin{\&} \mathrm{Cover}(\mathcal{A}) \to \triangle\,\mathrm{Refl}(R^{\mathcal{A}})$
- $\mathrm{Disj}(\mathcal{A}) \to \mathrm{Trans}(R^{\mathcal{A}})$

1. $(y \in X \mathbin{\&} y \in Y) \to X \approx Y$ — $\mathrm{Disj}\,(\mathcal{X})$
2. $(\exists X \in \mathcal{X})(x \in X \mathbin{\&} y \in X)$ — $R^{\mathcal{X}}xy$
3. $(\exists Y \in \mathcal{X})(y \in Y \mathbin{\&} z \in Y)$ — $R^{\mathcal{X}}yz$
4. $(\exists X, Y \in \mathcal{X})(x \in X \mathbin{\&} y \in X \mathbin{\&} y \in Y \mathbin{\&} z \in Y)$ — 2.,3.
5. $(\exists X, Y \in \mathcal{X})(x \in X \mathbin{\&} X \approx Y \mathbin{\&} z \in Y)$ — 1., 4.
6. $(\exists X, Y \in \mathcal{X})(x \in Y \mathbin{\&} z \in Y)$ — 5.

From partitions to similarities

Definition

$$R^{\mathcal{A}} \quad = \quad \{\langle x, y\rangle \mid (\exists A \in \mathcal{A})(x \in A \mathbin{\&} y \in A)\}$$

Results

- $\mathrm{Sym}(R^{\mathcal{A}})$
- $\mathrm{Crisp}(\mathcal{A}) \mathbin{\&} \mathrm{Cover}(\mathcal{A}) \to \triangle\,\mathrm{Refl}(R^{\mathcal{A}})$
- $\mathrm{Disj}(\mathcal{A}) \to \mathrm{Trans}(R^{\mathcal{A}})$

So together we proved:

$$\mathrm{Part}(\mathcal{A}) \to \triangle\,\mathrm{Sym}(R^{\mathcal{A}}) \mathbin{\&} \triangle\,\mathrm{Refl}(R^{\mathcal{A}}) \mathbin{\&} \mathrm{Trans}(R^{\mathcal{A}})$$

$$\mathrm{Part}(\mathcal{A}) \to \mathrm{Sim}(R^{\mathcal{A}})$$

$$\triangle\,\mathrm{Part}(\mathcal{A}) \to \triangle\,\mathrm{Sim}(R^{\mathcal{A}})$$

There and back again . . .

Results

$$\mathrm{Sim}(R) \to (R^{\mathrm{V}/R} \approx R)$$

$$\triangle\, \mathrm{Part}(\mathcal{A}) \to \mathrm{V}/R^{\mathcal{A}} = \mathcal{A}$$

Proof of $R \subseteq R^{\mathrm{V}/R}$:

1. Rxy
2. $[y]_R = [y]_R \,\&\, x \in [y]_R \,\&\, y \in [y]_R$ $\qquad \mathrm{Refl}(R)$
3. $\exists z\,([z]_R = [z]_R \,\&\, x \in [z]_R \,\&\, y \in [z]_R)$
4. $\exists Z\, \exists z\,([z]_R = Z \,\&\, x \in Z \,\&\, y \in Z)$
5. $\exists Z\,(\exists z\,([z]_R = Z) \,\&\, x \in Z \,\&\, y \in Z)$
6. $(\exists Z \in \mathrm{V}/R\,)(x \in Z \,\&\, y \in Z)$
7. $R^{\mathrm{V}/R}\, xy$

Counterfactual conditionals

Counterfactuals are conditionals with false antecedents:

If it were the case that A, it would be the case that C

Their logical analysis is notoriously problematic:

- If interpreted as material implications, they come out always true due to the false antecedent
- However, some counterfactuals are obviously false

$\Rightarrow$ a simple logical analysis does not work

Properties of counterfactuals

Counterfactual conditionals do not obey standard inference rules of the material implication:

Weakening: $$\frac{A \mathrel{\Box\!\!\rightarrow} C}{A \wedge B \mathrel{\Box\!\!\rightarrow} C}$$

If I won the lottery, I would go for a trip around the globe.
If I won the lottery and then WW3 started, I would go for a trip around the globe. (!)

Contraposition: $$\frac{A \mathrel{\Box\!\!\rightarrow} C}{\neg C \mathrel{\Box\!\!\rightarrow} \neg A}$$

If I won the lottery, I would still live in the Prague.
If I left Prague, I would not win the lottery (!)

Properties of counterfactuals

Transitivity: $$\frac{A \mathrel{\Box\!\!\rightarrow} B, B \mathrel{\Box\!\!\rightarrow} C}{A \mathrel{\Box\!\!\rightarrow} C}$$

If I quitted teaching in the university, I would try to teach in some high school.
If I became a millionaire, I would quit teaching in the university.
If I became a millionaire, I would try to teach in some high school. (!)

Lewis' semantics of counterfactuals

Lewis' semantics is based on a *similarity relation* which orders possible worlds with respect to their similarity to the actual world:

The counterfactual conditional $A \mathrel{\Box\!\!\rightarrow} C$ is true at a world w w.r.t. a similarity ordering if (very roughly) in the closest possible word to w where A holds also C holds.

Why a fuzzy semantics for counterfactuals?

Lewis' semantics is based on the notion of similarity of possible worlds

Similarity relations are prominently studied in fuzzy mathematics (formalized as axiomatic theories over fuzzy logic)

$\Rightarrow$ Let us see if fuzzy logic can provide a viable semantics for counterfactuals

Advantages and disadvantages

Advantages

- Automatic accommodation of gradual counterfactuals

  "If ants were *large,* they would be *heavy.*"

- Accommodation of graduality of counterfactuals (some counterfactual conditionals seem to hold "more" than others)

  "If ants were *large,* they would be *heavy*" vs.
  "If ants were *large,* they would *rule the earth*"

- Standard fuzzy handling of the similarity of worlds

Disadvantages

- Needs non-classical logic for semantic reasoning

  (but a well-developed one $\Rightarrow$ a low cost for experts)

Similarity relations = fuzzy equivalence relations

Axioms: $Sxx, \quad Sxy \to Syx, \quad Sxy \mathbin{\&} Syz \to Sxz$

(interpreted in fuzzy logic!)

Notice: Similarities are *transitive* (in the sense of fuzzy logic),
but avoid Poincaré's paradox:

$$x_1 \approx x_2 \approx x_3 \approx \cdots \approx x_n, \text{ though } x_1 \not\approx x_n,$$

since the degree of $x_1 \approx x_n$ can decrease with n,

due to the non-idempotent $\&$ of fuzzy logic

Ordering of worlds by similarity

Σxy ... the world x is similar to the world y

$x \preccurlyeq_w y$ ... x is *more or roughly as* similar to w as y

Define: $x \preccurlyeq_w y \;\equiv\; \Sigma wy \to \Sigma wx$

A trick: we identify a formula A with the set $\{w \mid \|A\|_w\}$

The closest A-worlds: $\mathrm{Min}_{\preccurlyeq_w} A = \{x \mid x \in A \wedge \forall a \in A\, (x \preccurlyeq_w a)\}$

(the properties of minima in fuzzy orderings are well known)

Define: $\|A \mathrel{\Box\!\!\rightarrow} B\|_w \;\equiv\; (\mathrm{Min}_{\preccurlyeq_w} A) \subseteq B$

... the closest A-worlds are B-worlds (fuzzily!)

Properties of fuzzy counterfactuals

Non-triviality: $(A \mathrel{\Box\!\!\rightarrow} B) = 1$ for all B only if $A = \emptyset$

Non-desirable properties are invalid:
$\not\models (A \mathrel{\Box\!\!\rightarrow} B) \mathbin{\&} (B \mathrel{\Box\!\!\rightarrow} C) \rightarrow (A \mathrel{\Box\!\!\rightarrow} C)$ $\quad \not\models (A \mathrel{\Box\!\!\rightarrow} C) \rightarrow (A \mathbin{\&} B \mathrel{\Box\!\!\rightarrow} C)$
$\not\models (A \mathrel{\Box\!\!\rightarrow} C) \rightarrow (\neg C \mathrel{\Box\!\!\rightarrow} \neg A)$

Desirable properties are valid, eg: $A \mathrel{\Box\!\!\rightarrow} B \models A \rightarrow B$ + many more theorems on $\mathrel{\Box\!\!\rightarrow}$ easily derivable

in higher-order fuzzy logic

However, some of Lewis' tautologies only hold for full degrees