A Gentle Introduction to Mathematical Fuzzy Logic

4. Łukasiewicz and Gödel–Dummett logic as logics of continuous t-norms

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Syntax

We consider primitive connectives $\mathcal{L} = \{\to, \wedge, \vee, \overline{0}\}$ and defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$:

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

Formulas are built from a fixed countable set of atoms using the connectives.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulas.

The semantics — classical logic

Definition 4.1

A $\mathbf{2}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $\{0, 1\}$ such that:

- $e(\overline{0}) = \overline{0}^{\mathbf{2}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\mathbf{2}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\mathbf{2}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{\mathbf{2}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ 0 & \text{otherwise.} \end{cases}$

Definition 4.2

A formula φ is a logical consequence of set of formulas Γ (in classical logic), $\Gamma \models_{\mathbf{2}} \varphi$, if for every $\mathbf{2}$-evaluation e:

$$\text{if } e(\gamma) = 1 \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) = 1.$$

The semantics — Gödel–Dummett logic

Definition 4.3

A $[0,1]_{\mathrm{G}}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $[0,1]$ such that:

- $e(\overline{0}) = \overline{0}^{[0,1]_{\mathrm{G}}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{\mathrm{G}}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{\mathrm{G}}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{\mathrm{G}}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ e(\psi) & \text{otherwise.} \end{cases}$

Definition 4.4

A formula φ is a logical consequence of set of formulas Γ (in Gödel–Dummett logic), $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$, if for every $[0,1]_{\mathrm{G}}$-evaluation e:

$$\text{if } e(\gamma) = 1 \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) = 1.$$

The semantics — Łukasiewicz logic

Definition 4.5

A $[0,1]_{Ł}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $[0,1]$; s.t.:

- $e(\overline{0}) = \overline{0}^{[0,1]_{Ł}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{Ł}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{Ł}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{Ł}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ 1 - e(\varphi) + e(\psi) & \text{otherwise} \end{cases}$

Definition 4.6

A formula φ is a logical consequence of set of formulas Γ (in Łukasiewicz logic), $\Gamma \models_{[0,1]_{Ł}} \varphi$, if for every $[0,1]_{Ł}$-evaluation e:

if $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$.

Changing the perspective

$$x \to_{\mathrm{G}} y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases} \qquad x \to_{\text{Ł}} y = \min\{1, 1 - x + y\}$$

$$x \mathbin{\&_{\mathrm{G}}} y = \min\{x, y\} \qquad x \mathbin{\&_{\text{Ł}}} y = \max\{0, x + y - 1\}$$

Exercise 20

Let T be either G or Ł. Prove that

- $x \mathbin{\&_T} y \leq z$ IFF $x \leq y \to_T z$
- $x \to_T y = \max\{z \mid x \mathbin{\&_T} z \leq y\}$
- $\min\{x, y\} = x \mathbin{\&_T} (x \to_T y)$
- $\max\{x, y\} = \min\{(x \to_T y) \to_T y, (y \to_T x) \to_T x\}$

Changing the language

We consider a new set of primitive connectives $\mathcal{L}_{\mathrm{HL}} = \{\overline{0}, \&, \to\}$ and defined now are connectives $\wedge$, $\vee$, $\neg$, $\overline{1}$, and $\leftrightarrow$:

$$\varphi \wedge \psi = \varphi \,\&\, (\varphi \to \psi) \qquad \varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

We keep the symbol $Fm_{\mathcal{L}}$ for the set of all formulass.

Changing the axioms – the original way

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
($\wedge$a)	$\varphi \wedge \psi \to \varphi$	
($\wedge$b)	$\varphi \wedge \psi \to \psi$	
($\wedge$c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	
($\vee$a)	$\varphi \to \varphi \vee \psi$	
($\vee$b)	$\psi \to \varphi \vee \psi$	
($\vee$c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$	
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Con)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom

Changing the axioms – an equivalent way

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
(Div)	$\varphi \mathbin{\&} (\varphi \to \psi) \to \psi \mathbin{\&} (\psi \to \varphi)$	divisibility
($\mathrm{Res_a}$)	$(\varphi \mathbin{\&} \psi \to \chi) \to (\varphi \to (\psi \to \chi))$	residuation
($\mathrm{Res_b}$)	$(\varphi \to (\psi \to \chi)) \to (\varphi \mathbin{\&} \psi \to \chi)$	residuation
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Con)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom

Exercise 21

(a) Prove that this new system without (Waj) is an axiomatic system of Gödel–Dummett logic (taking $\varphi \mathbin{\&} \psi = \varphi \wedge \psi$).

(b) Prove that this new system without (Con) is an axiomatic system of Łukasiewicz logic (taking $\varphi \mathbin{\&} \psi = \neg(\varphi \to \neg\psi)$).

Changing the axioms – an equivalent way

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)′	$\varphi \,\&\, \psi \to \varphi$	weakening
(Ex)′	$\varphi \,\&\, \psi \to \psi \,\&\, \varphi$	exchange
(Div)	$\varphi \,\&\, (\varphi \to \psi) \to \psi \,\&\, (\psi \to \varphi)$	divisibility
($\mathrm{Res_a}$)	$(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$	residuation
($\mathrm{Res_b}$)	$(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$	residuation
(Prl)′	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Con)′	$\varphi \to \varphi \,\&\, \varphi$	contraction
(Waj)′	$\neg\neg\varphi \to \varphi$	double negation law

Exercise 21

(c) Prove using only (Tr), ($\mathrm{Res_a}$), ($\mathrm{Res_b}$) and (MP) that axioms (We), (Ex), and (Con) prove their prime versions and vice versa.

(d) Prove that this new system without (Con)′ is an axiomatic system of Łukasiewicz logic (taking $\varphi \,\&\, \psi = \neg(\varphi \to \neg\psi)$).

Changing the axioms – an equivalent way

(HL1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(HL2)	$\varphi \mathbin{\&} \psi \to \varphi$
(HL3)	$\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$
(HL4)	$\varphi \mathbin{\&} (\varphi \to \psi) \to \psi \mathbin{\&} (\psi \to \varphi)$
(HL5a)	$(\varphi \mathbin{\&} \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(HL5b)	$(\varphi \to (\psi \to \chi)) \to (\varphi \mathbin{\&} \psi \to \chi)$
(HL6)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(HL7)	$\overline{0} \to \varphi$
(G)	$\varphi \to \varphi \mathbin{\&} \varphi$
(Ł)	$\neg\neg\varphi \to \varphi$

Petr Hájek's way

The logic HL

Axioms:

(HL1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(HL2)	$\varphi \mathbin{\&} \psi \to \varphi$
(HL3)	$\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$
(HL4)	$\varphi \mathbin{\&} (\varphi \to \psi) \to \psi \mathbin{\&} (\psi \to \varphi)$
(HL5a)	$(\varphi \mathbin{\&} \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(HL5b)	$(\varphi \to (\psi \to \chi)) \to (\varphi \mathbin{\&} \psi \to \chi)$
(HL6)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(HL7)	$\overline{0} \to \psi$

Inference rule: *modus ponens*.

We write $\Gamma \vdash_{\mathrm{HL}} \varphi$ if there is a proof of φ from Γ.

Note: Axioms $\mathrm{HL2}$ and $\mathrm{HL3}$ are redundant, the others are independent

Algebraic semantics — recall G-algebras

A Gödel algebra (or just G-algebra) is a structure

$\boldsymbol{B} = \langle B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \rightarrow^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}} \rangle$ such that:

(1) $\langle B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}} \rangle$ is a bounded lattice

(2) $z \leq x \rightarrow^{\boldsymbol{B}} y$ iff $x \wedge^{\boldsymbol{B}} z \leq y$ (residuation)

(3) $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$ (prelinearity)

where $x \leq y$ is defined as $x \wedge y = x$ or (equivalently) as $x \rightarrow y = \overline{1}$.

We say that a G-algebra $\boldsymbol{B}$ is linearly ordered (or G-chain) if $\leq$ is a total order.

By $\mathbf{ALG}^*(\mathrm{G})$ (or $\mathbf{ALG}^\ell(\mathrm{G})$ resp.) we denote the class of all G-algebras (G-chains resp.)

Changing the semantics — HL-algebras

An HL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \overline{0}, \overline{1} \rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a commutative monoid,

(3) $z \leq x \to y$ iff $x \,\&\, z \leq y$, (residuation)

(4) $x \,\&\, (x \to y) = x \wedge y$ (divisibility)

(5) $(x \to y) \vee (y \to x) = \overline{1}$ (prelinearity)

We say that $\boldsymbol{B}$ is

- linearly ordered (or HL-chain) if $\leq$ is a total order $\mathbb{HL}_{\mathrm{lin}}$
- standard $B = [0, 1]$ and $\leq$ is the usual order on reals $\mathbb{HL}_{\mathrm{std}}$
- G-algebra if $x \,\&\, x = x$ and MV-algebra if $\neg\neg x = x$

Exercise 22

Prove that the newly defined G- and MV- algebras are *termwise equivalent* with those defined earlier in this course.

Semantical consequence

Definition 4.7

A $\boldsymbol{B}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to B such that:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \,\&\, \psi) = e(\varphi) \,\&^{\boldsymbol{B}}\, e(\psi)$

Definition 4.8

A formula φ is a logical consequence of a set of formulas Γ w.r.t. a class $\mathbb{K}$ of HL-algebras, $\Gamma \models_{\mathbb{K}} \varphi$, if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

if $e(\gamma) = \overline{1}$ for every $\gamma \in \Gamma$, then $e(\varphi) = \overline{1}$.

General/linear/standard completeness theorem

Theorem 4.9

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{HL}} \varphi$
2. $\Gamma \models_{\mathbb{HL}} \varphi$
3. $\Gamma \models_{\mathbb{HL}_{\mathrm{lin}}} \varphi$

If Γ *is finite we can add:*

4. $\Gamma \models_{\mathbb{HL}_{\mathrm{std}}} \varphi$

Exercise 23

Prove the equivalence of the first three claims.

Hájek's (1998) approach

Goal: Generalize bivalent classical logic to $[0,1]$

Strategy: Impose some reasonable constraints on the truth functions of propositional connectives to get a well-behaved logic

Implementation:

- As a design choice, we assume the truth-functionality of all connectives w.r.t. $[0,1]$
- We require some natural conditions of $\&$
- A truth function of $\&$ satisfying these constraints will determine the rest of propositional calculus

The requirements of the truth function of conjunction

Let us consider an operation $*\colon [0,1]^2 \to [0,1]$

Commutativity: $x * y = y * x$

- When asserting two propositions, it does not matter in which order we put them down
- The commutativity of classical conjunction, which holds for crisp propositions, seems to be unharmed by taking into account also fuzzy propositions
- Thus, by using a non-commutative conjunction we would generalize to fuzzy-tolerance, not the Boolean logic, but rather some other logic that models order-dependent assertions of propositions (e.g., some kind of temporal logic)

The requirements of the truth function of conjunction

Associativity: $(x * y) * z = x * (y * z)$

- When asserting three propositions, it is irrelevant which two of them we put down first (be they fuzzy or not)

Monotony: *if* $x \leq x'$, *then* $x * y \leq x' * y$

- Increasing the truth value of the conjuncts should not decrease the truth value of their conjunction

Classicality: $x * 1 = x$ *(thus also* $x * 0 = 0$*)*

- $0, 1$ represent the classical truth values for crisp propositions
- Conjunction with full truth should not change the truth value

Continuity: $*$ *is continuous*

- An infinitesimal change of the truth value of a conjunct should not radically change the truth value of the conjunction

The requirements of the truth function of conjunction

We could add further conditions on $\&$ (e.g., idempotence), but it has proved suitable to stop here, as it already yields a rich and interesting theory and further conditions would be too limiting.

Such functions have previously been studied in the theory of probabilistic metric spaces and called *triangular norms* or shortly *t-norms* (continuous, as we require continuity):

Definition 4.10

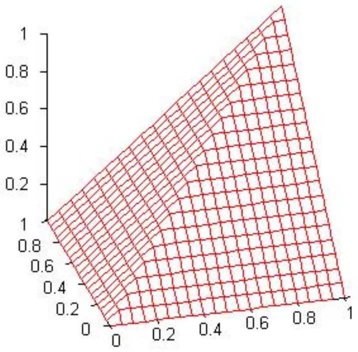
A binary function $*\colon [0,1]^2 \to [0,1]$ is a t-norm iff it is commutative, associative, monotone, and 1 is a neutral element.

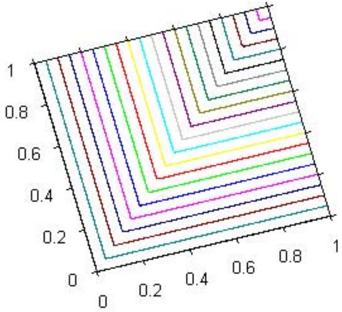
Lemma 4.11

A t-norm $*$ is continuous iff it is continuous in one variable, i.e., iff $f_x(y) = x * y$ is continuous for all $x \in [0,1]$ (analogously for left- and right-continuity).

Prominent examples of continuous t-norms (1)

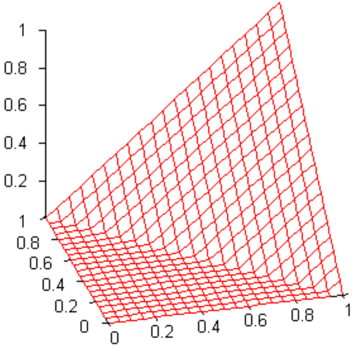
The minimum t-norm: $x *_G y = \min\{x, y\}$

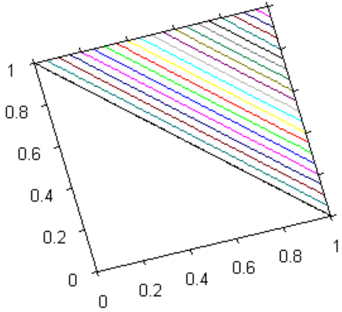




Prominent examples of continuous t-norms (2)

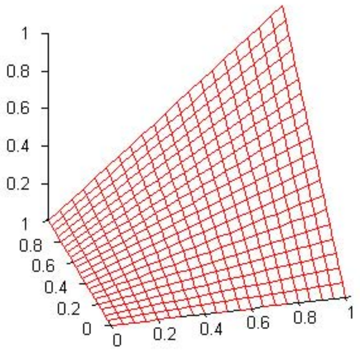
The Łukasiewicz t-norm: $x *_{Ł} y = \max\{0, x + y - 1\}$

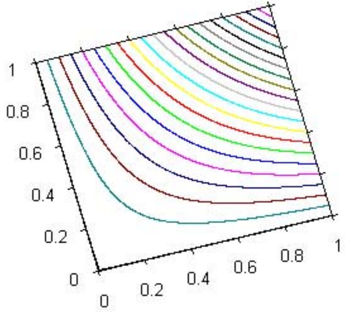




Prominent examples of continuous t-norms (3)

The product t-norm: $x *_{\Pi} y = x \cdot y$





Mostert–Shield's characterization

The idempotent elements (i.e., such x that $x * x = x$) of any continuous t-norm form a closed subset of $[0, 1]$.

Its complement is an (at most countable) union of open intervals.

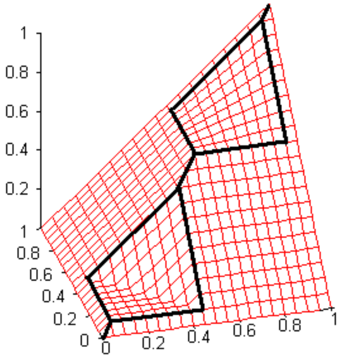
The restriction of $*$ to each of these intervals is isomorphic to $*_{Ł}$ (if it has nilpotent elements) or $*_{\Pi}$ (otherwise).

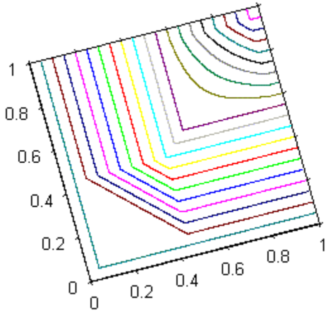
On the rest of $[0, 1]$ it coincides with $*_{G} = \min$.

All continuous t-norms are ordinal sums of isomorphic copies of $*_{Ł}, *_{\Pi}, *_{G}$.

Example

Ordinal sum of $*_{Ł}$ on $[0.05, 0.45]$, $*_{\Pi}$ on $[0.55, 0.95]$,
and the default $*_{G}$ elsewhere





Residua of left-continuous t-norms

Theorem 4.12

The following are equivalent for any t-norm $*$:

- $*$ *is left-continuous*
- *For each* x, y *there exist* $\max\{z \mid z * x \leq y\}$
- *There is a unique operation* $\Rightarrow_*$ *s.t.* $z * x \leq y$ *iff* $z \leq x \Rightarrow_* y$

Proof.

$1. \to 2$ via picture; $2. \to 3$ existence is easy $x \Rightarrow y = \max\{z \mid z * x \leq y\}$, uniqueness:

$$x \Rightarrow' y \leq x \Rightarrow' y \quad \text{IFF} \quad x * (x \Rightarrow y) \leq y \quad \text{IFF} \quad x \Rightarrow' y \leq x \Rightarrow y$$

□

Residua of left-continuous t-norms

Theorem 4.12

The following are equivalent for any t-norm $*$*:*

- $*$ *is left-continuous*
- *For each* x, y *there exist* $\max\{z \mid z * x \leq y\}$
- *There is a unique operation* $\Rightarrow_*$ *s.t.* $z * x \leq y$ *iff* $z \leq x \Rightarrow_* y$

Proof.

To prove $3. \to 1.$ it suffices to show

$$x * \sup Z = \sup_{z \in Z}(x * z) \quad \text{for each } x, y \text{ and a set } Z$$

Clearly $x * \sup Z \geq x * z$ (for $z \in Z$) ergo $x * \sup Z \geq \sup_{z \in Z}(x * z)$

From $z * x \leq \sup_{z \in Z}(x * z)$ (for $z \in Z$) get $z \leq x \Rightarrow \sup_{z \in Z}(x * z)$.

Thus $\sup Z \leq x \Rightarrow \sup_{z \in Z}(x * z)$ and so $x * \sup Z \leq \sup_{z \in Z}(x * z)$. □

Residua of left-continuous t-norms

Theorem 4.12

The following are equivalent for any t-norm $*$*:*

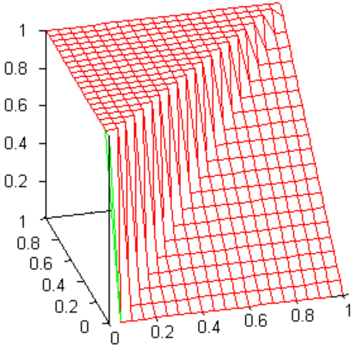
- $*$ *is left-continuous*
- *For each* x, y *there exist* $\max\{z \mid z * x \leq y\}$
- *There is a unique operation* $\Rightarrow_*$ *s.t.* $z * x \leq y$ *iff* $z \leq x \Rightarrow_* y$

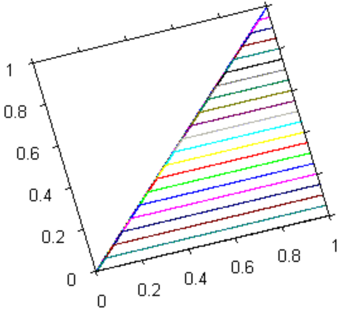
Definition 4.13

The operation $\Rightarrow_*$ is called the residuum of a t-norm $*$.

Residua of prominent continuous t-norms (1)

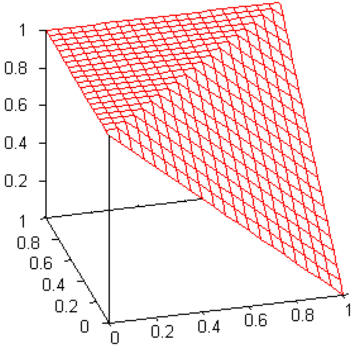
The residuum of $*_G$: Gödel implication $x \Rightarrow_G y = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise} \end{cases}$

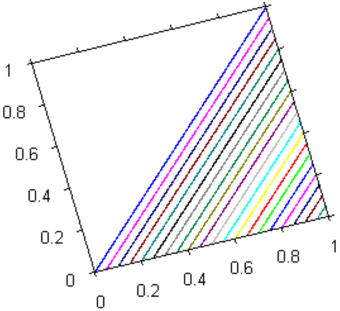




Residua of prominent continuous t-norms (2)

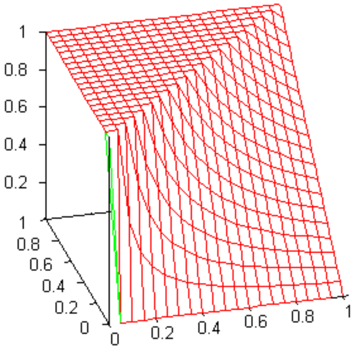
The residuum of $*_Ł$: Łukasiewicz implication $x \Rightarrow_Ł y = \min\{1, 1 - x + y\}$

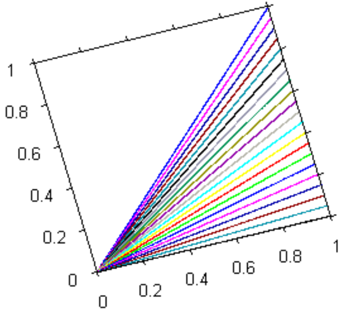




Residua of prominent continuous t-norms (3)

The residuum of $*_\Pi$: Goguen implication $x \Rightarrow_\Pi y = \begin{cases} \frac{y}{x} & \text{if } x > y \\ 1 & \text{if } x \leq y \end{cases}$





Basic properties of the residua of t-norms

Exercise 24

Prove that for each left-continuous t-norm $*$ the following holds:

- $(x \Rightarrow y) = 1$ iff $x \leq y$
- $(1 \Rightarrow y) = y$
- $\max\{x, y\} = \min\{(x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x\}$

Basic properties of the residua of t-norms

Theorem 4.14

Let $*$ be a left-continuous t-norm and $\Rightarrow$ its residuum. Then $*$ is right-continuous iff $\min\{x, y\} = x * (x \Rightarrow y)$.

Proof.

Recall that $*$ is right-continuous iff $x * \inf Z = \inf_{z \in Z}(x * z)$ for each x, y and a set Z. Left-to-right direction: using a picture; the converse one: clearly $x * \inf Z \leq \inf_{z \in Z}(x * z)$. Assume that $x * \inf Z < y < \inf_{z \in Z}(x * z)$.

Note that $y < x$ and so $y = x * (x \Rightarrow y)$.

Assume that $x \Rightarrow y \leq \inf Z$ so $y = x * (x \Rightarrow y) \leq x * \inf Z$ a contradiction.

Thus $\inf Z < x \Rightarrow y$, i.e., there is $z \in Z$ such that $z \leq x \Rightarrow y$.

Thus $\inf_{z \in Z}(x * z) \leq z * x \leq y$ a contradiction. □

Recall the HL-algebras

A HL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \overline{0}, \overline{1}\rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1}\rangle$ is a commutative monoid

(3) $z \leq x \to y$ iff $x \,\&\, z \leq y$, (residuation)

(4) $x \,\&\, (x \to y) = x \wedge y$ (divisibility)

(5) $(x \to y) \vee (y \to x) = \overline{1}$ (prelinearity)

We say that $\boldsymbol{B}$ is

- linearly ordered (or HL-chain) if $\leq$ is a total order. $\mathbb{HL}_{\mathrm{lin}}$
- standard $B = [0, 1]$ and $\leq$ is the usual order on reals. $\mathbb{HL}_{\mathrm{std}}$
- G-algebra if $x \,\&\, x = x$
- MV-algebra if $\neg\neg x = x$

Some properties of HL-algebras

Lemma 4.15

Let $\boldsymbol{B}$ *be an* HL*-algebra.*

1. $x \leq y$ *IFF* $x \to y = 1$
2. $x \leq y$ *implies* $x \,\&\, z \leq y \,\&\, z$ *ergo* $x \,\&\, z \leq z$
3. $x \,\&\, (y \vee z) = (x \,\&\, y) \vee (x \,\&\, z)$

Proof.

1. Trivial.
2. Clearly $y \leq z \to y \,\&\, z$, thus $x \leq z \to y \,\&\, z$ and so $x \,\&\, z \leq y \,\&\, z$
3. $\leq$: $x \,\&\, y \leq (x \,\&\, y) \vee (x \,\&\, z)$ thus $y \leq x \to (x \,\&\, y) \vee (x \,\&\, z)$
   $x \,\&\, z \leq (x \,\&\, y) \vee (x \,\&\, z)$ thus $z \leq x \to (x \,\&\, y) \vee (x \,\&\, z)$
   Thus $y \vee z \leq x \to (x \,\&\, y) \vee (x \,\&\, z)$ and so $x \,\&\, (y \vee z) \leq (x \,\&\, y) \vee (x \,\&\, z)$
   $\geq$: $y \leq y \vee z$ thus $x \,\&\, y \leq x \,\&\, (y \vee z)$; analogously $x \,\&\, z \leq x \,\&\, (y \vee z)$
   Thus $(x \,\&\, y) \vee (x \,\&\, z) \leq x \,\&\, (y \vee z)$. □

HL-algebras and continuous t-norms

Theorem 4.16

A structure $\boldsymbol{B} = ([0,1], \min, \max, \&, \to, 0, 1)$ *is a* HL*-algebra IFF* $\&$ *is a continuous t-norm and* $\to$ *its residuum.*

Exercise 25

(a) Prove the theorem above.

(b) Prove that $\boldsymbol{B}$ is G-algebra iff $\&$ is Gödel t-norm.

(c) Prove that $\boldsymbol{B}$ is MV-algebra iff $\&$ is isomorphic to Łukasiewicz t-norm.

For a cont. t-norm $*$ we denote its corresponding HL-algebra as $[0,1]_*$.

General/linear/standard completeness theorem

Recall:

Theorem 4.17

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$:

1. $\Gamma \vdash_{\mathrm{HL}} \varphi$
2. $\Gamma \models_{\mathbb{HL}} \varphi$
3. $\Gamma \models_{\mathbb{HL}_{\mathrm{lin}}} \varphi$

If Γ is finite we can add:

4. $\Gamma \models_{\mathbb{HL}_{\mathrm{std}}} \varphi$

Hájek's basic fuzzy logic HL is the logic of all continuous t-norms

Propositional calculi of continuous t-norms $*$

Definition 4.18

Let $\mathbb{K}$ be a class of continuous t-norms. By $\mathrm{L}(\mathbb{K})$ we denote the logic axiomatized by adding to HL the set of $\mathbb{K}$-tautologies, i.e., formulas φ such that $\models_{[0,1]_*} \varphi$ for each $* \in \mathbb{K}$. The $\mathrm{L}(\mathbb{K})$-algebras are the HL-algebras satisfying the additional axioms, i.e., equations $\varphi = 1$.

Theorem 4.19

For each class of continuous t-norms $\mathbb{K}$, the following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$:

1. $\Gamma \vdash_{\mathrm{L}(\mathbb{K})} \varphi$
2. $\Gamma \models_{\mathbb{L}(\mathbb{K})} \varphi$
3. $\Gamma \models_{\mathbb{L}(\mathbb{K})_{\mathrm{lin}}} \varphi$

If Γ is finite we can add:

4. $\Gamma \models_{\mathbb{K}} \varphi$

Properties of $\mathrm{L}(\mathbb{K})$

Theorem 4.20

Let $\mathbb{K}$ be a class of continuous t-norms. Then

1. $T, \varphi \vdash_{\mathrm{L}(\mathbb{K})} \psi$ *iff* $T \vdash_{\mathrm{L}(\mathbb{K})} \varphi^n \to \psi$ *for some* n*; where* $\varphi^n = \underbrace{\varphi \,\&\, \dots \,\&\, \varphi}_{n\times}$
2. *The logic* $\mathrm{L}(\mathbb{K})$ *is finitely axiomatizable.*
3. *There is a finite* $\mathbb{K}'$ *such that* $\mathrm{L}(\mathbb{K}) = \mathrm{L}(\mathbb{K}')$*.*
4. *The set of* $\mathbb{K}$*-tautologies (theorems of* $\mathrm{L}(\mathbb{K})$*) is* **coNP***-complete.*

The following two properties of $\mathrm{L}(\mathbb{K})$ *are valid iff* $\mathbb{K} = \{*_{\mathrm{G}}\}$*:*

5. $T, \varphi \vdash_{\mathrm{L}(\mathbb{K})} \psi$ *iff* $T \vdash_{\mathrm{L}(\mathbb{K})} \varphi \to \psi$*.*
6. $\Gamma \vdash_{\mathrm{L}(\mathbb{K})} \varphi$ *iff* $\Gamma \models_{\mathbb{K}} \varphi$*.*

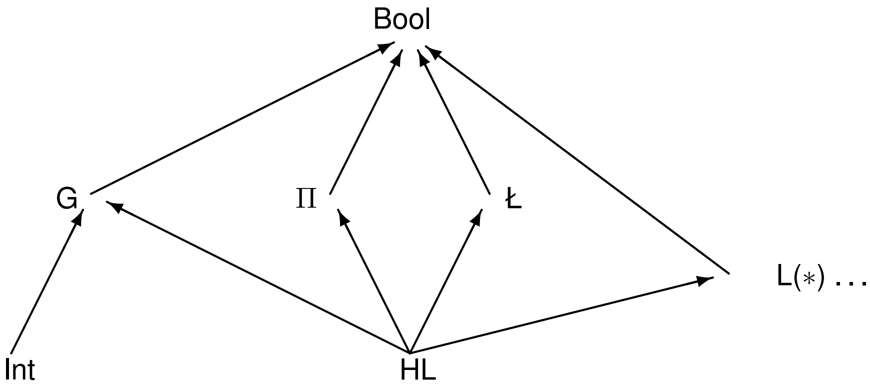
Theorem 4.21

There is a t-norm $*$ *such that* $\mathrm{HL} = \mathrm{L}(\{*\})$*.*

Notable $\mathrm{L}(\mathbb{K})$s

- Ł = $\mathrm{L}(*_{Ł})$, axiomatized by $\mathrm{HL}+$ $\neg\neg\varphi \to \varphi$
- Gödel–Dummett logic G = $\mathrm{L}(*_{\mathrm{G}})$, axiomatized by $\mathrm{HL}+$ $\varphi \to \varphi \,\&\, \varphi$
- SHL = $\mathrm{L}(\mathbb{K})$, where $\mathbb{K}$ are t-norms with $\neg x = 0$ for each $x > 0$, axiomatized by $\mathrm{HL}+$ $\varphi \wedge \neg\varphi \to \overline{0}$
- Product logic Π = $\mathrm{L}(*_{\Pi})$, axiomatized by $\mathrm{HL}+$ $\neg\neg\varphi \to ((\varphi \to \varphi \,\&\, \psi) \to \psi \,\&\, \neg\neg\psi)$ or equivalently as $\mathrm{SHL} + \neg\neg\chi \to ((\varphi \,\&\, \chi \to \psi \,\&\, \chi) \to (\varphi \to \psi))$; it cannot be axiomatized using one variable only.

Main t-norm fuzzy logics (as of 1998)



Standard epistemic logic

Modality $\mathsf{K}A$ = "the agent knows that A"

The principle of logical rationality of the agent

= the assumption that the agent can make inference steps
$\Rightarrow$ the axiom (K) of propositional epistemic logic:

$$\mathsf{K}A \mathbin{\&} \mathsf{K}(A \to B) \to \mathsf{K}B$$

The axiom is adopted in standard accounts of epistemic logic
Standard epistemic logic = the logic of *logically rational* agents

The logical omniscience paradox

An unwanted consequence of the logical rationality principle:

the agent's knowledge is closed under *modus ponens*

$\Rightarrow$ under the propositional consequence relation

$\Rightarrow$ the agent knows all propositional tautologies, once he/she/it knows the axioms of CL

= an extremely implausible assumption on real-world agents (consider, eg, a non-trivial tautology with 10^9 variables)

Three kinds of knowledge

Actual knowledge . . . the modality "is *known*"
= knowledge immediately available to the agent
(eg, the contents of its memory)

Potential knowledge . . . the modality "is *knowable*"
= knowledge in principle derivable from the actual knowledge
(by logical inference)

Feasible knowledge . . . the modality "is *realistically* knowable"
= knowledge effectively derivable from the actual knowledge
(taking the agent's physical restrictions into account)

The scope of the logical omniscience paradox

The logical omniscience paradox only affects feasible knowledge:

Actual knowledge is not closed under inference steps
$\Rightarrow$ the axiom (K) is not plausible for actual knowledge

Potential knowledge is indeed closed under logical consequence
$\Rightarrow$ no paradox there

Feasible knowledge, however, seems to be:

- closed under single inference steps (the agent *can* make them)
- yet not closed under the consequence relation as a whole
(the agent cannot feasibly know all logical truths)

Logical omniscience as an instance of the Sorites

The problem with feasible knowledge is that the agent

- can always make a next step of inference, but
- cannot make an arbitrarily large number of inference steps

Ie, if the agent can make n steps, so it can make $n+1$ steps,
and the agent can make 0 steps.
Yet it is not the case that for each N,
the agent can make N steps of inference

= An instance of the sorites paradox for the predicate
$P(n) \equiv$ "the agent can make at least n inference steps"

$\Rightarrow$ Every solution to the sorites paradox generates
a solution to the logical omniscience paradox

Why a degree-theoretical solution?

There have been many objections against degree-theoretical solutions to the sorites

However, a degree-theoretical solution is particularly suitable to the logical omniscience instances of the sorites, since

- the degrees have a clear interpretation (in terms of costs of the feasible task)
- and can be manipulated by suitable many-valued logics
- the (implausible) existence of a sharp breaking point in the number of steps the agent can perform is not presupposed

Resource-aware reasoning about knowledge

What limits the agent's ability to infer knowledge is
the agent's limited resources (time, memory, . . .)

$\Rightarrow$ Resource-aware reasoning about the agent's knowledge needed

Several models of resource-aware reasoning are available
(eg, in dynamic or linear logics)

Fuzzy logics are applicable to resource-aware reasoning, too, capturing moreover the gradual nature of feasibility
(some tasks are more feasible than others)

Resource-based interpretation of Łukasiewicz logic

Cost assignment: $c\colon Fm_{\mathcal{L}} \to [0,1]$ s.t. $1 - c(x)$ is an evaluation

Intuitively: instead 'p is true' we read 'p is cheap'.

The connectives then represent natural operations with costs:

- $\top$ = any 'costless task' $c(\top) = 0$
- $\bot$ = any 'unaffordable task' $c(\bot) = 1$
- Conjunction = bounded sum of the costs
  $$c(\varphi \mathbin{\&} \psi) = \min\{1, c(\varphi) + c(\psi)\}$$
- Implication = the 'surcharge' for ψ, given the cost of φ
  $$c(\varphi \to \psi) = \max\{0, c(\psi) - c(\varphi)\}$$

Tautologies of the form $A_1 \mathbin{\&} \ldots \mathbin{\&} A_n \to B$ represent cost-preserving rules of inference

(the cost of B is at most the sum of the costs of A_i)

Combination of costs in basic t-norm logics

Łukasiewicz logic:

$\&$ = bounded addition of costs (via a linear function)

0 = the maximal (or unaffordable) cost

Gödel logic:

$\&$ = the maximum of costs

natural, eg, in space complexity (erase temporary memory)

Product logic:

$\&$ = addition of costs (via the logarithm)

0 = the infinite cost

Other t-norm logics:

$\&$ = certain other ways of cost combination

(eg, additive up to some bound, then maxitive)

Feasibility in t-norm logics

Atomic formulas of t-norm logics can thus be understood as standing under the implicit graded modality *is affordable,* or *is feasible*

The degree of feasibility is inversely proportional (via a suitable normalization function) to the cost of realization (eg, the number of processor cycles)

Logical connectives then express natural operations with costs

Tautologies express degree/cost-preserving rules of inference

Feasible knowledge in fuzzy logic

Given the degrees of (the feasibility of) KA and K$(A \to B)$, the degree of KB (inferred by the agent) needs to make allowance for the (small) cost of performing the inference step of *modus ponens* by the agent (denote it by the atom (MP))

The plausible axiom of logical rationality for feasible knowledge in fuzzy logics thus becomes:

$$\mathsf{K}A \mathbin{\&} \mathsf{K}(A \to B) \mathbin{\&} (\mathrm{MP}) \to \mathsf{K}B$$

Logical omniscience in fuzzy logics

Since the degree of (MP) is slightly less than 1
(as the cost of performing *modus ponens* is small, but non-zero),
it decreases slightly the degree of the inferred knowledge KB

For longer derivations (of B from $A_1, \ldots, A_k$) that require n inference steps, the axiom only yields (where $A^n \equiv A \,\&\, .\overset{n}{.}. \,\&\, A$)

$$\mathsf{K}A_1 \,\&\, \ldots \,\&\, \mathsf{K}A_k \,\&\, (\mathrm{MP})^n \to \mathsf{K}B$$

Since $\&$ is non-idempotent, the degree of $(\mathrm{MP})^n$
(and so the guaranteed degree of KB) decreases, reaching eventually 0
(the resources are limited)

Elimination of the paradox in fuzzy logics

Thus in models over fuzzy logics,

- The feasibility of knowledge decreases with long derivations (as it intuitively should)
- The closure of feasible knowledge under logical consequence is only gradual (fading with the increasing difficulty of derivation),
- Yet the agents are still perfectly logically rational (able to perform each inference step, at appropriate costs)

$\Rightarrow$ No paradox under suitable fuzzy logics