A Gentle Introduction to Mathematical Fuzzy Logic

4. Łukasiewicz and Gödel–Dummett logic as logics of continuous t-norms

Petr Cintula\(^1\) and Carles Noguera\(^2\)

\(^1\)Institute of Computer Science, Czech Academy of Sciences, Prague, Czech Republic

\(^2\)Institute of Information Theory and Automation, Czech Academy of Sciences, Prague, Czech Republic

www.cs.cas.cz/cintula/MFL
We consider primitive connectives $\mathcal{L} = \{\rightarrow, \land, \lor, \overline{0}\}$ and defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$:

\[
\neg \varphi = \varphi \rightarrow \overline{0} \quad \overline{1} = \neg \overline{0} \quad \varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)
\]

Formulas are built from a fixed countable set of atoms using the connectives.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulas.
The semantics — classical logic

**Definition 4.1**

A *2-evaluation* is a mapping \( e \) from \( Fm_L \) to \( \{0, 1\} \) such that:

1. \( e(\overline{0}) = \overline{0}^2 = 0 \)
2. \( e(\varphi \land \psi) = e(\varphi) \land^2 e(\psi) = \min\{e(\varphi), e(\psi)\} \)
3. \( e(\varphi \lor \psi) = e(\varphi) \lor^2 e(\psi) = \max\{e(\varphi), e(\psi)\} \)
4. \( e(\varphi \rightarrow \psi) = e(\varphi) \rightarrow^2 e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ 0 & \text{otherwise.} \end{cases} \)

**Definition 4.2**

A formula \( \varphi \) is a *logical consequence* of set of formulas \( \Gamma \) (in classical logic), \( \Gamma \models_2 \varphi \), if for every 2-evaluation \( e \):

\[
\text{if } e(\gamma) = 1 \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) = 1.
\]
Definition 4.3

A $[0, 1]_G$-evaluation is a mapping $e$ from $Fm_L$ to $[0, 1]$ such that:

- $e(0) = 0^{[0,1]}_G = 0$
- $e(\varphi \land \psi) = e(\varphi) \land^{[0,1]}_G e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \lor \psi) = e(\varphi) \lor^{[0,1]}_G e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \rightarrow \psi) = e(\varphi) \rightarrow^{[0,1]}_G e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ e(\psi) & \text{otherwise.} \end{cases}$

Definition 4.4

A formula $\varphi$ is a **logical consequence** of set of formulas $\Gamma$ (in Gödel–Dummett logic), $\Gamma \models^{[0,1]}_G \varphi$, if for every $[0, 1]_G$-evaluation $e$:

If $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$. 
The semantics — Łukasiewicz logic

**Definition 4.5**

A $[0, 1]_L$-evaluation is a mapping $e$ from $Fm_L$ to $[0, 1]$; s.t.:

- $e(0) = 0^{[0,1]_L} = 0$
- $e(\varphi \land \psi) = e(\varphi) \land [0,1]_L e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \lor \psi) = e(\varphi) \lor [0,1]_L e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \rightarrow \psi) = e(\varphi) \rightarrow [0,1]_L e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ 1-e(\varphi)+e(\psi) & \text{otherwise} \end{cases}$

**Definition 4.6**

A formula $\varphi$ is a **logical consequence** of set of formulas $\Gamma$ (in Łukasiewicz logic), $\Gamma \models [0,1]_L \varphi$, if for every $[0, 1]_L$-evaluation $e$:

if $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$. 

Changing the perspective

\[ x \rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases} \]

\[ x \&_G y = \min\{x, y\} \]

\[ x \rightarrow_L y = \min\{1, 1 - x + y\} \]

\[ x \&_L y = \max\{0, x + y - 1\} \]

**Exercise 20**

Let \( T \) be either \( G \) or \( L \). Prove that

- \( x \&_T y \leq z \iff x \leq y \rightarrow_T z \)
- \( x \rightarrow_T y = \max\{z \mid x \&_T z \leq y\} \)
- \( \min\{x, y\} = x \&_T (x \rightarrow_T y) \)
- \( \max\{x, y\} = \min\{(x \rightarrow_T y) \rightarrow_T y, (y \rightarrow_T x) \rightarrow_T x\} \)
We consider a new set of primitive connectives $L_{HL} = \{0, \&, \to\}$ and defined now are connectives $\land$, $\lor$, $\neg$, $\top$, and $\leftrightarrow$: 

$$\phi \land \psi = \phi \& (\phi \to \psi) \quad \phi \lor \psi = ((\phi \to \psi) \to \psi) \land ((\psi \to \phi) \to \phi)$$
$$\neg \phi = \phi \to \overline{0} \quad \top = \neg \overline{0} \quad \phi \leftrightarrow \psi = (\phi \to \psi) \land (\psi \to \phi)$$

We keep the symbol $F_{mL}$ for the set of all formulass.
Changing the axioms – the original way

(Tr) \((\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))\)  
transitivity

(We) \(\varphi \to (\psi \to \varphi)\)  
weakening

(Ex) \((\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))\)  
exchange

(\&a) \(\varphi \land \psi \to \varphi\)

(\&b) \(\varphi \land \psi \to \psi\)

(\&c) \((\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \land \psi))\)

(\lor a) \(\varphi \to \varphi \lor \psi\)

(\lor b) \(\psi \to \varphi \lor \psi\)

(\lor c) \((\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \lor \psi \to \chi))\)

(Prl) \((\varphi \to \psi) \lor (\psi \to \varphi)\)  
prelinearity

(EFQ) \(\overline{0} \to \varphi\)  
Ex falso quodlibet

(Con) \((\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)\)  
contraction

(Waj) \(((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)\)  
Wajsberg axiom
Changing the axioms – an equivalent way

(Tr) \((\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))\) transitivity

(We) \(\varphi \rightarrow (\psi \rightarrow \varphi)\) weakening

(Ex) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))\) exchange

(Div) \(\varphi \land (\varphi \rightarrow \psi) \rightarrow \psi \land (\psi \rightarrow \varphi)\) divisibility

(Res_a) \((\varphi \land \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))\) residuation

(Res_b) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi \rightarrow \chi)\) residuation

(Prl) \((\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)\) prelinearity

(EFQ) \(0 \rightarrow \varphi\) Ex falso quodlibet

(Con) \((\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)\) contraction

(Waj) \(((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \varphi)\) Wajsberg axiom

Exercise 21

(a) Prove that this new system without (Waj) is an axiomatic system of Gödel–Dummett logic (taking \(\varphi \land \psi = \varphi \land \psi\)).

(b) Prove that this new system without (Con) is an axiomatic system of Łukasiewicz logic (taking \(\varphi \land \psi = \neg(\varphi \rightarrow \neg\psi)\)).
Changing the axioms – an equivalent way

(Tr) \((\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))\) \hspace{1cm} \text{transitivity}
(We)’ \(\varphi \& \psi \rightarrow \varphi\) \hspace{1cm} \text{weakening}
(Ex)’ \(\varphi \& \psi \rightarrow \psi \& \varphi\) \hspace{1cm} \text{exchange}
(Div) \(\varphi \& (\varphi \rightarrow \psi) \rightarrow \psi \& (\psi \rightarrow \varphi)\) \hspace{1cm} \text{divisibility}
(Res_a) \((\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))\) \hspace{1cm} \text{residuation}
(Res_b) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)\) \hspace{1cm} \text{residuation}
(Prl)’ \(((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)\) \hspace{1cm} \text{prelinearity}
(EFQ) \(\overline{0} \rightarrow \varphi\) \hspace{1cm} \text{Ex falso quodlibet}
(Con)’ \(\varphi \rightarrow \varphi \& \varphi\) \hspace{1cm} \text{contraction}
(Waj)’ \(\neg\neg \varphi \rightarrow \varphi\) \hspace{1cm} \text{double negation law}

Exercise 21

(c) Prove using only (Tr), (Res_a), (Res_b) and (MP) that axioms (We), (Ex), and (Con) prove their prime versions and vice versa.

(d) Prove that this new system without (Con)’ is an axiomatic system of Łukasiewicz logic (taking \(\varphi \& \psi = \neg (\varphi \rightarrow \neg \psi)\)).
Changing the axioms – an equivalent way

(\text{HL1}) \quad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))

(\text{HL2}) \quad \varphi \land \psi \to \varphi

(\text{HL3}) \quad \varphi \land \psi \to \psi \land \varphi

(\text{HL4}) \quad \varphi \land (\varphi \to \psi) \to \psi \land (\psi \to \varphi)

(\text{HL5a}) \quad (\varphi \land \psi \to \chi) \to (\varphi \to (\psi \to \chi))

(\text{HL5b}) \quad (\varphi \to (\psi \to \chi)) \to (\varphi \land \psi \to \chi)

(\text{HL6}) \quad ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)

(\text{HL7}) \quad 0 \to \varphi

(\text{G}) \quad \varphi \to \varphi \land \varphi

(\text{Ł}) \quad \neg \neg \varphi \to \varphi

Petr Hájek’s way
The logic HL

Axioms:

(\text{HL1}) \quad (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))

(\text{HL2}) \quad \varphi \land \psi \rightarrow \varphi

(\text{HL3}) \quad \varphi \land \psi \rightarrow \psi \land \varphi

(\text{HL4}) \quad \varphi \land (\varphi \rightarrow \psi) \rightarrow \psi \land (\psi \rightarrow \varphi)

(\text{HL5a}) \quad (\varphi \land \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))

(\text{HL5b}) \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi \rightarrow \chi)

(\text{HL6}) \quad (((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)

(\text{HL7}) \quad 0 \rightarrow \psi

Inference rule: \textit{modus ponens}.

We write \( \Gamma \vdash_{\text{HL}} \varphi \) if there is a proof of \( \varphi \) from \( \Gamma \).

Note: Axioms HL2 and HL3 are redundant, the others are independent.
Algebraic semantics — recall G-algebras

A Gödel algebra (or just G-algebra) is a structure
\[ B = \langle B, \wedge^B, \vee^B, \rightarrow^B, 0^B, 1^B \rangle \]
such that:

1. \( \langle B, \wedge^B, \vee^B, 0^B, 1^B \rangle \) is a bounded lattice
2. \( z \leq x \rightarrow^B y \iff x \wedge^B z \leq y \) (residuation)
3. \( (x \rightarrow y) \vee (y \rightarrow x) = 1 \) (prelinearity)

where \( x \leq y \) is defined as \( x \wedge y = x \) or (equivalently) as \( x \rightarrow y = 1 \).

We say that a G-algebra \( B \) is linearly ordered (or G-chain) if \( \leq \) is a total order.

By \( \text{ALG}^*(G) \) (or \( \text{ALG}^\ell(G) \) resp.) we denote the class of all G-algebras (G-chains resp.)
Changing the semantics — HL-algebras

An **HL-algebra** is a structure $B = \langle B, \land, \lor, \&, \rightarrow, 0, 1 \rangle$ such that:

1. $\langle B, \land, \lor, 0, 1 \rangle$ is a bounded lattice,
2. $\langle B, \&, 1 \rangle$ is a commutative monoid,
3. $z \leq x \rightarrow y$ iff $x \& z \leq y$, \hspace{1cm} (residuation)
4. $x \& (x \rightarrow y) = x \land y$ \hspace{1cm} (divisibility)
5. $(x \rightarrow y) \lor (y \rightarrow x) = 1$ \hspace{1cm} (prelinearity)

We say that $B$ is
- **linearly ordered** (or **HL-chain**) if $\leq$ is a total order
- **standard** $B = [0, 1]$ and $\leq$ is the usual order on reals
- **G-algebra** if $x \& x = x$ and **MV-algebra** if $\neg\neg x = x$

**Exercise 22**

Prove that the newly defined G- and MV- algebras are *termwise equivalent* with those defined earlier in this course.
Semantical consequence

Definition 4.7

A $B$-evaluation is a mapping $e$ from $Fm_L$ to $B$ such that:

- $e(0) = \overline{0}^B$
- $e(\varphi \to \psi) = e(\varphi) \to^B e(\psi)$
- $e(\varphi \& \psi) = e(\varphi) \&^B e(\psi)$

Definition 4.8

A formula $\varphi$ is a logical consequence of a set of formulas $\Gamma$ w.r.t. a class $\mathcal{K}$ of $HL$-algebras, $\Gamma \models^\mathcal{K} \varphi$, if for every $B \in \mathcal{K}$ and every $B$-evaluation $e$:

if $e(\gamma) = \overline{1}$ for every $\gamma \in \Gamma$, then $e(\varphi) = \overline{1}$. 
Theorem 4.9

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_L$:

1. $\Gamma \vdash_{HL} \varphi$
2. $\Gamma \models_{HL} \varphi$
3. $\Gamma \models_{HL_{lin}} \varphi$

If $\Gamma$ is finite we can add:

4. $\Gamma \models_{HL_{std}} \varphi$

Exercise 23

Prove the equivalence of the first three claims.
Hájek’s (1998) approach

**Goal:** Generalize bivalent classical logic to $[0, 1]$

**Strategy:** Impose some reasonable constraints on the truth functions of propositional connectives to get a well-behaved logic

**Implementation:**
- As a design choice, we assume the truth-functionality of all connectives w.r.t. $[0, 1]$
- We require some natural conditions of $\&$
- A truth function of $\&$ satisfying these constraints will determine the rest of propositional calculus
The requirements of the truth function of conjunction

Let us consider an operation $\ast : [0, 1]^2 \rightarrow [0, 1]$

**Commutativity:** $x \ast y = y \ast x$

- When asserting two propositions, it does not matter in which order we put them down.
- The commutativity of classical conjunction, which holds for crisp propositions, seems to be unharmed by taking into account also fuzzy propositions.
- Thus, by using a non-commutative conjunction we would generalize to fuzzy-tolerance, not the Boolean logic, but rather some other logic that models order-dependent assertions of propositions (e.g., some kind of temporal logic).
The requirements of the truth function of conjunction

**Associativity:** \((x \ast y) \ast z = x \ast (y \ast z)\)
- When asserting three propositions, it is irrelevant which two of them we put down first (be they fuzzy or not)

**Monotony:** \(\text{if } x \leq x', \text{ then } x \ast y \leq x' \ast y\)
- Increasing the truth value of the conjuncts should not decrease the truth value of their conjunction

**Classicality:** \(x \ast 1 = x \) (thus also \(x \ast 0 = 0\))
- 0, 1 represent the classical truth values for crisp propositions
- Conjunction with full truth should not change the truth value

**Continuity:** \(* \text{ is continuous}\)
- An infinitesimal change of the truth value of a conjunct should not radically change the truth value of the conjunction
The requirements of the truth function of conjunction

We could add further conditions on \& (e.g., idempotence), but it has proved suitable to stop here, as it already yields a rich and interesting theory and further conditions would be too limiting.

Such functions have previously been studied in the theory of probabilistic metric spaces and called triangular norms or shortly t-norms (continuous, as we require continuity):

**Definition 4.10**

A binary function \( * : [0, 1]^2 \rightarrow [0, 1] \) is a t-norm iff it is commutative, associative, monotone, and 1 is a neutral element.

**Lemma 4.11**

A t-norm \( * \) is continuous iff it is continuous in one variable, i.e., iff \( f_x(y) = x * y \) is continuous for all \( x \in [0, 1] \) (analogously for left- and right-continuity).
Prominent examples of continuous t-norms (1)

The **minimum t-norm**: \( x \ast_G y = \min\{x, y\} \)
Prominent examples of continuous t-norms (2)

The Łukasiewicz t-norm: $x \star_L y = \max\{0, x + y - 1\}$
Prominent examples of continuous t-norms (3)

The product t-norm: \( x \times_\Pi y = x \cdot y \)
Mostert–Shield’s characterization

The idempotent elements (i.e., such $x$ that $x \ast x = x$) of any continuous t-norm form a closed subset of $[0, 1]$. Its complement is an (at most countable) union of open intervals.

The restriction of $\ast$ to each of these intervals is isomorphic to $\ast_L$ (if it has nilpotent elements) or $\ast_\Pi$ (otherwise).

On the rest of $[0, 1]$ it coincides with $\ast_G = \min$.

All continuous t-norms are ordinal sums of isomorphic copies of $\ast_L, \ast_\Pi, \ast_G$. 
Example

Ordinal sum of $*_L$ on $[0.05, 0.45]$, $*_{\Pi}$ on $[0.55, 0.95]$, and the default $*_G$ elsewhere.
Residua of left-continuous t-norms

**Theorem 4.12**

The following are equivalent for any t-norm \( \ast \):

- \( \ast \) is left-continuous
- For each \( x, y \) there exist \( \max\{z \mid z \ast x \leq y\} \)
- There is a unique operation \( \Rightarrow \ast \) s.t. \( z \ast x \leq y \iff z \leq x \Rightarrow \ast y \)

**Proof.**

1. \( \rightarrow 2 \) via picture; 2. \( \rightarrow 3 \) existence is easy \( x \Rightarrow y = \max\{z \mid z \ast x \leq y\} \), uniqueness:

\[
x \Rightarrow' y \leq x \Rightarrow' y \quad \text{IFF} \quad x \ast (x \Rightarrow y) \leq y \quad \text{IFF} \quad x \Rightarrow' y \leq x \Rightarrow y
\]
Residua of left-continuous t-norms

Theorem 4.12

The following are equivalent for any t-norm $*$:

- $*$ is left-continuous
- For each $x, y$ there exist $\max\{z \mid z \ast x \leq y\}$
- There is a unique operation $\Rightarrow$ s.t. $z \ast x \leq y$ iff $z \leq x \Rightarrow z \ast y$

Proof.

To prove 3. $\Rightarrow$ 1. it suffices to show

$$x \ast \sup Z = \sup_{z \in Z} (x \ast z) \quad \text{for each } x, y \text{ and a set } Z$$

Clearly $x \ast \sup Z \geq x \ast z$ (for $z \in Z$) ergo $x \ast \sup Z \geq \sup_{z \in Z} (x \ast z)$

From $z \ast x \leq \sup_{z \in Z} (x \ast z)$ (for $z \in Z$) get $z \leq x \Rightarrow \sup_{z \in Z} (x \ast z)$.

Thus $\sup Z \leq x \Rightarrow \sup_{z \in Z} (x \ast z)$ and so $x \ast \sup Z \leq \sup_{z \in Z} (x \ast z)$.

$\square$
The following are equivalent for any t-norm $\ast$:

- $\ast$ is left-continuous
- For each $x, y$ there exist $\max\{z \mid z \ast x \leq y\}$
- There is a unique operation $\Rightarrow_\ast$ s.t. $z \ast x \leq y$ iff $z \leq x \Rightarrow_\ast y$

**Definition 4.13**

The operation $\Rightarrow_\ast$ is called the residuum of a t-norm $\ast$. 
Residua of prominent continuous t-norms (1)

The residuum of $*_G$: Gödel implication $x \Rightarrow_G y = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise} \end{cases}$
Residua of prominent continuous t-norms (2)

The residuum of $*_L$: Łukasiewicz implication $x \Rightarrow_L y = \min\{1, 1 - x + y\}$
Residua of prominent continuous t-norms (3)

The residuum of $*_{\Pi}$: Goguen implication $x \Rightarrow_{\Pi} y = \begin{cases} \frac{y}{x} & \text{if } x > y \\ 1 & \text{if } x \leq y \end{cases}$
Exercise 24

Prove that for each left-continuous t-norm \( * \) the following holds:

1. \((x \Rightarrow y) = 1 \text{ iff } x \leq y\)
2. \((1 \Rightarrow y) = y\)
3. \(\max\{x, y\} = \min\{(x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x\}\)
Basic properties of the residua of t-norms

**Theorem 4.14**

Let $\ast$ be a left-continuous t-norm and $\Rightarrow$ its residuum. Then $\ast$ is right-continuous iff $\min\{x, y\} = x \ast (x \Rightarrow y)$.

**Proof.**

Recall that $\ast$ is right-continuous iff $x \ast \inf Z = \inf_{z \in Z}(x \ast z)$ for each $x, y$ and a set $Z$. Left-to-right direction: using a picture; the converse one: clearly $x \ast \inf Z \leq \inf_{z \in Z}(x \ast z)$. Assume that $x \ast \inf Z < y < \inf_{z \in Z}(x \ast z)$.

Note that $y < x$ and so $y = x \ast (x \Rightarrow y)$.

Assume that $x \Rightarrow y \leq \inf Z$ so $y = x \ast (x \Rightarrow y) \leq x \ast \inf Z$ a contradiction.

Thus $\inf Z < x \Rightarrow y$, i.e., there is $z \in Z$ such that $z \leq x \Rightarrow y$.

Thus $\inf_{z \in Z}(x \ast z) \leq z \ast x \leq y$ a contradiction.
Recall the HL-algebras

A **HL-algebra** is a structure $B = \langle B, \land, \lor, \&, \rightarrow, 0, 1 \rangle$ such that:

1. $\langle B, \land, \lor, 0, 1 \rangle$ is a bounded lattice,
2. $\langle B, \&, 1 \rangle$ is a commutative monoid
3. $z \leq x \rightarrow y$ iff $x \& z \leq y,$ (residuation)
4. $x \& (x \rightarrow y) = x \land y$ (divisibility)
5. $(x \rightarrow y) \lor (y \rightarrow x) = 1$ (prelinearity)

We say that $B$ is

- linearly ordered (or HL-chain) if $\leq$ is a total order.
- standard $B = [0, 1]$ and $\leq$ is the usual order on reals.
- G-algebra if $x \& x = x$
- MV-algebra if $\neg\neg x = x$
Some properties of HL-algebras

Lemma 4.15

Let $B$ be an HL-algebra.

1. $x \leq y$ IFF $x \to y = 1$
2. $x \leq y$ implies $x \& z \leq y \& z$ ergo $x \& z \leq z$
3. $x \& (y \lor z) = (x \& y) \lor (x \& z)$

Proof.

1. Trivial.
2. Clearly $y \leq z \to y \& z$, thus $x \leq z \to y \& z$ and so $x \& z \leq y \& z$
3. $\leq: x \& y \leq (x \& y) \lor (x \& z)$ thus $y \leq x \to (x \& y) \lor (x \& z)$
   $x \& z \leq (x \& y) \lor (x \& z)$ thus $z \leq x \to (x \& y) \lor (x \& z)$
   Thus $y \lor z \leq x \to (x \& y) \lor (x \& z)$ and so $x \& (y \lor z) \leq (x \& y) \lor (x \& z)$
   $\geq: y \leq y \lor z$ thus $x \& y \leq x \& (y \lor z)$; analogously $x \& z \leq x \& (y \lor z)$
   Thus $(x \& y) \lor (x \& z) \leq x \& (y \lor z)$.
Theorem 4.16

A structure $B = ([0, 1], \min, \max, \& , \rightarrow, 0, 1)$ is a HL-algebra IFF $\&$ is a continuous t-norm and $\rightarrow$ its residuum.

Exercise 25

(a) Prove the theorem above.
(b) Prove that $B$ is G-algebra iff $\&$ is Gödel t-norm.
(c) Prove that $B$ is MV-algebra iff $\&$ is isomorphic to Łukasiewicz t-norm.

For a cont. t-norm $\ast$ we denote its corresponding HL-algebra as $[0, 1] \ast$. 
Recall:

**Theorem 4.17**

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_L$:

1. $\Gamma \vdash_{HL} \varphi$
2. $\Gamma \models_{HL} \varphi$
3. $\Gamma \models_{HL_{lin}} \varphi$

If $\Gamma$ is finite we can add:

4. $\Gamma \models_{HL_{std}} \varphi$

Hájek's basic fuzzy logic $HL$ is the logic of all continuous t-norms.
Propositional calculi of continuous t-norms

Definition 4.18

Let $\mathcal{K}$ be a class of continuous t-norms. By $L(\mathcal{K})$ we denote the logic axiomatized by adding to HL the set of $\mathcal{K}$-tautologies, i.e., formulas $\varphi$ such that $\models_{[0,1]} \varphi$ for each $\ast \in \mathcal{K}$. The $L(\mathcal{K})$-algebras are the HL-algebras satisfying the additional axioms, i.e., equations $\varphi = 1$.

Theorem 4.19

For each class of continuous t-norms $\mathcal{K}$, the following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_L$:

1. $\Gamma \vdash_{L(\mathcal{K})} \varphi$
2. $\Gamma \models_{L(\mathcal{K})} \varphi$
3. $\Gamma \models_{L(\mathcal{K})_{\text{lin}}} \varphi$

If $\Gamma$ is finite we can add:

4. $\Gamma \models_{\mathcal{K}} \varphi$
Properties of $L(K)$

**Theorem 4.20**

Let $K$ be a class of continuous t-norms. Then

1. $T, \varphi \vdash_{L(K)} \psi$ iff $T \vdash_{L(K)} \varphi^n \rightarrow \psi$ for some $n$; where $\varphi^n = \underbrace{\varphi \& \ldots \& \varphi}_{n \times}$

2. The logic $L(K)$ is finitely axiomatizable.

3. There is a finite $K'$ such that $L(K) = L(K')$.

4. The set of $K$-tautologies (theorems of $L(K)$) is coNP-complete.

The following two properties of $L(K)$ are valid iff $K = \{*_{G}\}$:

5. $T, \varphi \vdash_{L(K)} \psi$ iff $T \vdash_{L(K)} \varphi \rightarrow \psi$.

6. $\Gamma \vdash_{L(K)} \varphi$ iff $\Gamma \models_{K} \varphi$.

**Theorem 4.21**

There is a t-norm * such that $HL = L(\{*\})$. 
Notable $L(K)$s

- $Ł = L(∗Ł)$, axiomatized by $HL+ ¬¬φ → φ$
- Gödel–Dummett logic $G = L(∗G)$, axiomatized by $HL+ φ → φ \& φ$
- $SHL = L(K)$, where $K$ are t-norms with $¬x = 0$ for each $x > 0$, axiomatized by $HL+ φ \& ¬φ → 0$
- Product logic $Π = L(∗Π)$, axiomatized by $HL+$
  $¬¬φ → ((φ → φ \& ψ) → ψ \& ¬¬ψ)$ or equivalently as
  $SHL + ¬¬χ → ((φ \& χ → ψ \& χ) → (φ → ψ))$; it cannot be axiomatized using one variable only.
Main t-norm fuzzy logics (as of 1998)
Standard epistemic logic

Modality $KA = \text{“the agent knows that } A\text{”}$

The principle of logical rationality of the agent

$= \text{the assumption that the agent can make inference steps}$

$\Rightarrow \text{the axiom (K) of propositional epistemic logic:}$

$$KA \& K(A \rightarrow B) \rightarrow KB$$

The axiom is adopted in standard accounts of epistemic logic

Standard epistemic logic $= \text{the logic of logically rational agents}$
The logical omniscience paradox

An unwanted consequence of the logical rationality principle:

the agent’s knowledge is closed under *modus ponens*

⇒ under the propositional consequence relation

⇒ the agent knows *all propositional tautologies*, once

he/she/it knows the axioms of CL

= an extremely implausible assumption on real-world agents

(consider, eg, a non-trivial tautology with $10^9$ variables)
Three kinds of knowledge

**Actual knowledge** . . . the modality “is known”
   = knowledge immediately available to the agent
   (eg, the contents of its memory)

**Potential knowledge** . . . the modality “is knowable”
   = knowledge in principle derivable from the actual knowledge
   (by logical inference)

**Feasible knowledge** . . . the modality “is realistically knowable”
   = knowledge effectively derivable from the actual knowledge
   (taking the agent’s physical restrictions into account)
The scope of the logical omniscience paradox

The logical omniscience paradox only affects feasible knowledge:

Actual knowledge is not closed under inference steps
\[\Rightarrow\] the axiom (K) is not plausible for actual knowledge

Potential knowledge is indeed closed under logical consequence
\[\Rightarrow\] no paradox there

Feasible knowledge, however, seems to be:

- closed under single inference steps (the agent \textit{can} make them)
- yet not closed under the consequence relation as a whole (the agent cannot feasibly know all logical truths)
Logical omniscience as an instance of the Sorites

The problem with feasible knowledge is that the agent
- can always make a next step of inference, but
- cannot make an arbitrarily large number of inference steps

I.e., if the agent can make $n$ steps, so it can make $n + 1$ steps,
and the agent can make 0 steps.
Yet it is not the case that for each $N$,
the agent can make $N$ steps of inference

= An instance of the sorites paradox for the predicate
$P(n) \equiv \text{“the agent can make at least } n \text{ inference steps”}$

$\Rightarrow$ Every solution to the sorites paradox generates
a solution to the logical omniscience paradox
Why a degree-theoretical solution?

There have been many objections against degree-theoretical solutions to the sorites. However, a degree-theoretical solution is particularly suitable to the logical omniscience instances of the sorites, since

- the degrees have a clear interpretation (in terms of costs of the feasible task)
- and can be manipulated by suitable many-valued logics
- the (implausible) existence of a sharp breaking point in the number of steps the agent can perform is not presupposed
Resource-aware reasoning about knowledge

What limits the agent’s ability to infer knowledge is
the agent’s limited resources (time, memory, . . . )

⇒ Resource-aware reasoning about the agent’s knowledge needed

Several models of resource-aware reasoning are available
(eg, in dynamic or linear logics)

Fuzzy logics are applicable to resource-aware reasoning, too,
capturing moreover the gradual nature of feasibility
(some tasks are more feasible than others)
Resource-based interpretation of Łukasiewicz logic

Cost assignment: \( c : Fm_L \rightarrow [0, 1] \) s.t. \( 1 - c(x) \) is an evaluation

Intuitively: instead ‘\( p \) is true’ we read ‘\( p \) is cheap’.

The connectives then represent natural operations with costs:

- \( \top = \) any ‘costless task’ \( c(\top) = 0 \)
- \( \bot = \) any ‘unaffordable task’ \( c(\bot) = 1 \)
- Conjunction = bounded sum of the costs
  \[
  c(\varphi \& \psi) = \min\{1, c(\varphi) + c(\psi)\}
  \]
- Implication = the ‘surcharge’ for \( \psi \), given the cost of \( \varphi \)
  \[
  c(\varphi \rightarrow \psi) = \max\{0, c(\psi) - c(\varphi)\}
  \]

Tautologies of the form \( A_1 \& \ldots \& A_n \rightarrow B \) represent

\textbf{cost-preserving} rules of inference

(the cost of \( B \) is at most the sum of the costs of \( A_i \))
Combination of costs in basic t-norm logics

Łukasiewicz logic:
\& = bounded addition of costs (via a linear function)
0 = the maximal (or unaffordable) cost

Gödel logic:
\& = the maximum of costs
natural, eg, in space complexity (erase temporary memory)

Product logic:
\& = addition of costs (via the logarithm)
0 = the infinite cost

Other t-norm logics:
\& = certain other ways of cost combination
(eg, additive up to some bound, then maxitive)
Feasibility in t-norm logics

Atomic formulas of t-norm logics can thus be understood as standing under the implicit graded modality is affordable, or is feasible.

The degree of feasibility is inversely proportional (via a suitable normalization function) to the cost of realization (eg, the number of processor cycles).

Logical connectives then express natural operations with costs.

Tautologies express degree/cost-preserving rules of inference.
Feasible knowledge in fuzzy logic

Given the degrees of (the feasibility of) $K_A$ and $K(A \rightarrow B)$, the degree of $K_B$ (inferred by the agent) needs to make allowance for the (small) cost of performing the inference step of *modus ponens* by the agent (denote it by the atom $(MP)$).

The plausible axiom of logical rationality for feasible knowledge in fuzzy logics thus becomes:

$$K_A \& K(A \rightarrow B) \& (MP) \rightarrow K_B$$
Logical omniscience in fuzzy logics

Since the degree of (MP) is slightly less than 1 (as the cost of performing modus ponens is small, but non-zero), it decreases slightly the degree of the inferred knowledge KB.

For longer derivations (of B from $A_1, \ldots, A_k$) that require $n$ inference steps, the axiom only yields (where $A^n \equiv A \& \ldots \& A$)

$$KA_1 \& \ldots \& KA_k \& (MP)^n \rightarrow KB$$

Since $\&$ is non-idempotent, the degree of $(MP)^n$ (and so the guaranteed degree of KB) decreases, reaching eventually 0 (the resources are limited).
Elimination of the paradox in fuzzy logics

Thus in models over fuzzy logics,

- The feasibility of knowledge decreases with long derivations (as it intuitively should)
- The closure of feasible knowledge under logical consequence is only gradual (fading with the increasing difficulty of derivation),
- Yet the agents are still perfectly logically rational (able to perform each inference step, at appropriate costs)

⇒ No paradox under suitable fuzzy logics