Fuzzy Logic

5. The growing family of fuzzy logics

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Outline

1. Changing the perspective
2. Logic(s) of continuous t-norms
3. 15 years of development of MFL: A montage
4. Core semilinear logics
5. Logics in expanded languages
6. Application: Fuzzy Epistemic Logic

Syntax

We consider primitive connectives $\mathcal{L} = \{\rightarrow, \wedge, \vee, \overline{0}\}$ and defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$:

$$\neg\varphi = \varphi \rightarrow \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$

Formulas are built from a fixed countable set of atoms using the connectives.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulas.

The semantics — classical logic

Definition 5.1

A $\mathbf{2}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $\{0, 1\}$ such that:

- $e(\overline{0}) = \overline{0}^{\mathbf{2}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\mathbf{2}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\mathbf{2}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{\mathbf{2}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ 0 & \text{otherwise.} \end{cases}$

Definition 5.2

A formula φ is a logical consequence of set of formulas Γ (in classical logic), $\Gamma \models_{\mathbf{2}} \varphi$, if for every $\mathbf{2}$-evaluation e:

if $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$.

The semantics — Gödel–Dummett logic

Definition 5.3

A $[0,1]_{\mathrm{G}}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $[0,1]$ such that:

- $e(\overline{0}) = \overline{0}^{[0,1]_{\mathrm{G}}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{\mathrm{G}}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{\mathrm{G}}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{\mathrm{G}}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ e(\psi) & \text{otherwise.} \end{cases}$

Definition 5.4

A formula φ is a logical consequence of set of formulas Γ (in Gödel–Dummett logic), $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$, if for every $[0,1]_{\mathrm{G}}$-evaluation e:

if $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$.

The semantics — Łukasiewicz logic

Definition 5.5

A $[0,1]_Ł$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $[0,1]$; s.t.:

- $e(\overline{0}) = \overline{0}^{[0,1]_Ł} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_Ł} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_Ł} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_Ł} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ 1 - e(\varphi) + e(\psi) & \text{otherwise} \end{cases}$

Definition 5.6

A formula φ is a logical consequence of set of formulas Γ (in Łukasiewicz logic), $\Gamma \models_{[0,1]_Ł} \varphi$, if for every $[0,1]_Ł$-evaluation e:

if $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$.

Changing the perspective

$$x \to_{\mathrm{G}} y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases} \qquad x \to_{\text{Ł}} y = \min\{1, 1 - x + y\}$$

$$x \,\&_{\mathrm{G}}\, y = \min\{x, y\} \qquad x \,\&_{\text{Ł}}\, y = \max\{0, x + y - 1\}$$

Exercise 20

Let T be either G or Ł. Prove that

- $x \,\&_T\, y \leq z$ IFF $x \leq y \to_T z$
- $x \to_T y = \max\{z \mid x \,\&_T\, z \leq y\}$
- $\min\{x, y\} = x \,\&_T\, (x \to_T y)$
- $\max\{x, y\} = \min\{(x \to_T y) \to_T y, (y \to_T x) \to_T x\}$

Changing the language

We consider a new set of primitive connectives $\mathcal{L}_{\mathrm{MTL}} = \{\overline{0}, \&, \wedge, \to\}$ and defined now are connectives $\vee$, $\neg$, $\overline{1}$, and $\leftrightarrow$:

$$\varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

We keep the symbol $Fm_{\mathcal{L}}$ for the set of all formulas.

Changing the axioms – the original way

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
($\wedge$a)	$\varphi \wedge \psi \to \varphi$	
($\wedge$b)	$\varphi \wedge \psi \to \psi$	
($\wedge$c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	
($\vee$a)	$\varphi \to \varphi \vee \psi$	
($\vee$b)	$\psi \to \varphi \vee \psi$	
($\vee$c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$	
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Con)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom

Changing the axioms – an equivalent way 1

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	transitivity
(We)	$\varphi \to (\psi \to \varphi)$	weakening
(Ex)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	exchange
($\wedge$a)	$\varphi \wedge \psi \to \varphi$	
($\wedge$b)	$\varphi \wedge \psi \to \psi$	
($\wedge$c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	
($\mathrm{Res_a}$)	$(\varphi \mathbin{\&} \psi \to \chi) \to (\varphi \to (\psi \to \chi))$	residuation
($\mathrm{Res_b}$)	$(\varphi \to (\psi \to \chi)) \to (\varphi \mathbin{\&} \psi \to \chi)$	residuation
(Prl)	$(\varphi \to \psi) \vee (\psi \to \varphi)$	prelinearity
(EFQ)	$\overline{0} \to \varphi$	*Ex falso quodlibet*
(Con)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction
(Waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom

Exercise 21

(a) Prove that this new system without (Waj) is an axiomatic system of Gödel–Dummett logic (taking $\varphi \mathbin{\&} \psi = \varphi \wedge \psi$).

(b) Prove that this new system without (Con) is an axiomatic system of Łukasiewicz logic (taking $\varphi \mathbin{\&} \psi = \neg(\varphi \to \neg\psi)$).

Changing the axioms – an equivalent way 2

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(We)′	$\varphi \,\&\, \psi \to \varphi$
(Ex)′	$\varphi \,\&\, \psi \to \psi \,\&\, \varphi$
(∧a)	$\varphi \wedge \psi \to \varphi$
(∧b)	$\varphi \wedge \psi \to \psi$
(∧c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
$(\mathrm{Res_a})$	$(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
$(\mathrm{Res_b})$	$(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$
(Prl)′	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(EFQ)	$\overline{0} \to \varphi$

Exercise 22

Prove that axioms (We), (Ex), and (Prl) prove their prime versions and vice-versa. (Hint: the first two can be done using (Tr), $(\mathrm{Res_a})$, $(\mathrm{Res_b})$ and (MP) only.)

The logic MTL

Axioms:

(Tr)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	(MTL1)
(We)$'$	$\varphi \,\&\, \psi \to \varphi$	(MTL2)
(Ex)$'$	$\varphi \,\&\, \psi \to \psi \,\&\, \varphi$	(MTL3)
($\wedge$a)	$\varphi \wedge \psi \to \varphi$	(MTL4a)
($\wedge$b)	$\varphi \wedge \psi \to \psi$	(MTL4b)
($\wedge$c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$	(MTL4c)
(Res$_a$)	$(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$	(MTL5a)
(Res$_b$)	$(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$	(MTL5b)
(Prl)$'$	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$	(MTL6)
(EFQ)	$\overline{0} \to \varphi$	(MTL7)

Inference rule: *modus ponens*.

We write $\Gamma \vdash_{\mathrm{MTL}} \varphi$ if there is a proof of φ from Γ.

Note: axioms $(\mathrm{We})'$ and $(\mathrm{Ex})'$ are redundant, the others are independent.

Notable axiomatic extensions of MTL

- Hájek's Basic fuzzy Logic HL axiomatized as $\mathrm{MTL}+$
  $$\varphi \mathbin{\&} (\varphi \to \psi) \to \psi \mathbin{\&} (\psi \to \varphi)$$
- Łukasiewicz logic Ł axiomatized as $\mathrm{MTL} + (\mathrm{Waj})$ or $\mathrm{HL} + \neg\neg\varphi \to \varphi$
- Gödel–Dummett logic G axiomatized by $\mathrm{MTL} + (\mathrm{Con})$
- Product logic Π, axiomatized by $\mathrm{HL} +$
  $$\neg\neg\varphi \to ((\varphi \to \varphi \mathbin{\&} \psi) \to \psi \mathbin{\&} \neg\neg\psi)$$

Syntactical properties

Theorem 5.7

Let L *be an axiomatic extension of* MTL. *Prove that:*

- $T, \varphi \vdash_{\mathrm{L}} \psi$ *iff there is* n *such that* $T \vdash_{\mathrm{L}} \varphi^n \to \psi$
  (Local Deduction Theorem)
- *If* $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ *and* $\Gamma, \psi \vdash_{\mathrm{L}} \chi$, *then* $\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi$.
  (Proof by Cases Property)
- *If* $\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi$ *and* $\Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi$, *then* $\Gamma \vdash_{\mathrm{L}} \chi$.
  (Semilinearity Property)
- *If* $\Gamma \nvdash_{\mathrm{L}} \varphi$, *then there is a linear* $\Gamma' \supseteq \Gamma$ *such that* $\Gamma' \nvdash_{\mathrm{L}} \varphi$.
  (Linear Extension Property)

Exercise 23

Prove it!

Algebraic semantics — recall G-algebras

A Gödel algebra (or just G-algebra) is a structure

$\boldsymbol{B} = \langle B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \rightarrow^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}} \rangle$ such that:

(1) $\langle B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}} \rangle$ is a bounded lattice

(2) $z \leq x \rightarrow^{\boldsymbol{B}} y$ iff $x \wedge^{\boldsymbol{B}} z \leq y$ (residuation)

(3) $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$ (prelinearity)

where $x \leq y$ is defined as $x \wedge y = x$ or (equivalently) as $x \rightarrow y = \overline{1}$.

We say that a G-algebra $\boldsymbol{B}$ is linearly ordered (or G-chain) if $\leq$ is a total order.

By $\mathbb{G}$ (or $\mathbb{G}_{\text{lin}}$ resp.) we denote the class of all G-algebras (G-chains resp.)

Changing the semantics — MTL-algebras

An MTL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \overline{0}, \overline{1}\rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1}\rangle$ is a commutative monoid,

(3) $z \leq x \to y$ iff $x \,\&\, z \leq y$, (residuation)

(4) $(x \to y) \vee (y \to x) = \overline{1}$ (prelinearity)

We say that $\boldsymbol{B}$ is

- linearly ordered (or MTL-chain) if $\leq$ is a total order. $\mathbb{MTL}_{\text{lin}}$
- standard $B = [0, 1]$ and $\leq$ is the usual order on reals. $\mathbb{MTL}_{\text{std}}$
- HL-algebra if $x \,\&\, (x \to y) = x \wedge y$ (divisibility)
- G-algebra if $x \,\&\, x = x$
- MV-algebra if it is both HL and $\neg\neg x = x$.

Some properties of MTL-algebras

Lemma 5.8

Let $\boldsymbol{B}$ *be an* MTL*-algebra.*

1. $x \leq y$ *iff* $x \to y = 1$
2. $x \leq y$ *implies* $x \& z \leq y \& z$ *ergo* $x \& z \leq z$
3. $x \& (y \vee z) = (x \& y) \vee (x \& z)$

Proof.

1. trivial
2. Clearly $y \leq z \to y \& z$, thus $x \leq z \to y \& z$ and so $x \& z \leq y \& z$
3. $\leq$: $x \& y \leq (x \& y) \vee (x \& z)$ thus $y \leq x \to (x \& y) \vee (x \& z)$
   $x \& z \leq (x \& y) \vee (x \& z)$ thus $z \leq x \to (x \& y) \vee (x \& z)$
   Thus $y \vee z \leq x \to (x \& y) \vee (x \& z)$ and so $x \& (y \vee z) \leq (x \& y) \vee (x \& z)$
   $\geq$: $y \leq y \vee z$ thus $x \& y \leq x \& (y \vee z)$; analogously $x \& z \leq x \& (y \vee z)$
   Thus $(x \& y) \vee (x \& z) \leq x \& (y \vee z)$. □

Some properties of MTL-algebras

Lemma 5.8

Let $\boldsymbol{B}$ *be an* MTL*-algebra.*

1. $x \leq y$ *iff* $x \to y = 1$
2. $x \leq y$ *implies* $x \,\&\, z \leq y \,\&\, z$ *ergo* $x \,\&\, z \leq z$
3. $x \,\&\, (y \vee z) = (x \,\&\, y) \vee (x \,\&\, z)$

Exercise 25

Prove that the newly defined G- and MV- algebras are *termwise equivalent* with those defined earlier in this course.

Semantical consequence

Definition 5.9

A $\boldsymbol{B}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to B such that:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \,\&\, \psi) = e(\varphi) \,\&^{\boldsymbol{B}}\, e(\psi)$

Definition 5.10

A formula φ is a logical consequence of a set of formulas Γ w.r.t. a class $\mathbb{K}$ of MTL-algebras, $\Gamma \models_{\mathbb{K}} \varphi$, if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

if $e(\gamma) = \overline{1}$ for every $\gamma \in \Gamma$, then $e(\varphi) = \overline{1}$.

L-algebras

Definition 5.11

Let $\boldsymbol{A}$ be an MTL-algebra and L and axiomatic extension of MTL. We say that $\boldsymbol{A}$ is and L-algebra if $e(\varphi) = \overline{1}^{\boldsymbol{A}}$ for each $\boldsymbol{A}$-evaluation e and each additional axiom φ of L.

Exercise 26

Prove that just defined HL-, G-, and MV-algebras coincide with those defined above.

Let us by $\mathbb{L}$ denote the class of all L-algebras and use subscripts lin and std to denote the linear and standard ones.

General/linear completeness theorem

Theorem 5.12

Let L *be an axiomatic extension of* MTL*. Then the following are equivalent for every set of formulas* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{L}} \varphi$
2. $\Gamma \models_{\mathbb{L}} \varphi$
3. $\Gamma \models_{\mathbb{L}_{\mathrm{lin}}} \varphi$

Exercise 27

Prove it!

Outline

Hájek's (1998) approach

Goal: Generalize bivalent classical logic to $[0, 1]$

Strategy: Impose some reasonable constraints on the truth functions of propositional connectives to get a well-behaved logic

Implementation:

- As a design choice, we assume the truth-functionality of all connectives w.r.t. $[0, 1]$
- We require some natural conditions of $\&$
- A truth function of $\&$ satisfying these constraints will determine the rest of propositional calculus

The requirements of the truth function of conjunction

Let us consider an operation $*\colon [0,1]^2 \to [0,1]$

Commutativity: $x * y = y * x$

- When asserting two propositions, it does not matter in which order we put them down
- The commutativity of classical conjunction, which holds for crisp propositions, seems to be unharmed by taking into account also fuzzy propositions
- Thus, by using a non-commutative conjunction we would generalize to fuzzy-tolerance, not the Boolean logic, but rather some other logic that models order-dependent assertions of propositions (e.g., some kind of temporal logic)

The requirements of the truth function of conjunction

Associativity: $(x * y) * z = x * (y * z)$

- When asserting three propositions, it is irrelevant which two of them we put down first (be they fuzzy or not)

Monotony: *if* $x \leq x'$, *then* $x * y \leq x' * y$

- Increasing the truth value of the conjuncts should not decrease the truth value of their conjunction

Classicality: $x * 1 = x$ *(thus also* $x * 0 = 0$*)*

- $0, 1$ represent the classical truth values for crisp propositions
- Conjunction with full truth should not change the truth value

Continuity: $*$ *is continuous*

- An infinitesimal change of the truth value of a conjunct should not radically change the truth value of the conjunction

The requirements of the truth function of conjunction

We could add further conditions on $\&$ (e.g., idempotence), but it has proved suitable to stop here, as it already yields a rich and interesting theory and further conditions would be too limiting.

Such functions have previously been studied in the theory of probabilistic metric spaces and called *triangular norms* or shortly *t-norms* (continuous, as we require continuity):

Definition 5.13

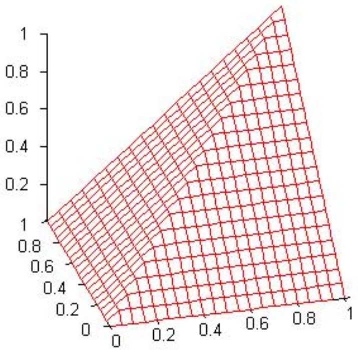
A binary function $*\colon [0,1]^2 \to [0,1]$ is a t-norm iff it is commutative, associative, monotone, and 1 is a neutral element.

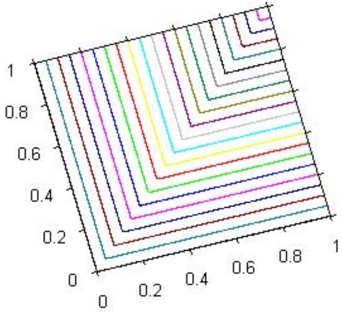
Lemma 5.14

A t-norm $*$ is continuous iff it is continuous in one variable, i.e., iff $f_x(y) = x * y$ is continuous for all $x \in [0,1]$ (analogously for left- and right-continuity).

Prominent examples of continuous t-norms (1)

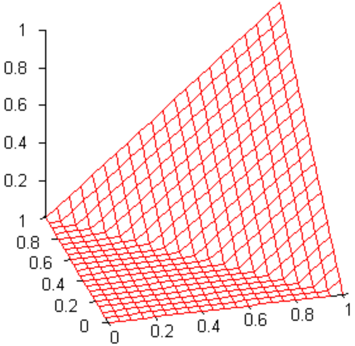
The minimum t-norm: $x *_G y = \min\{x, y\}$

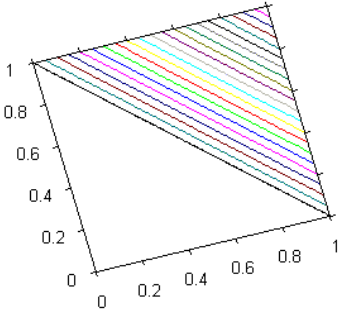




Prominent examples of continuous t-norms (2)

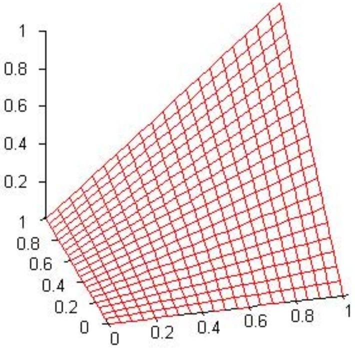
The Łukasiewicz t-norm: $x *_{Ł} y = \max\{0, x + y - 1\}$

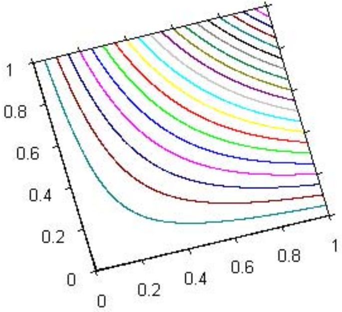




Prominent examples of continuous t-norms (3)

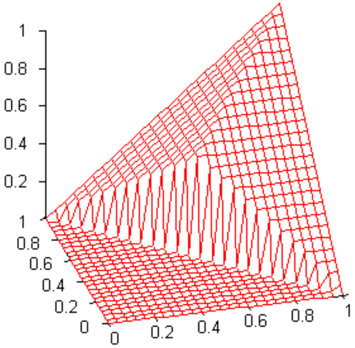
The product t-norm: $x *_\Pi y = x \cdot y$

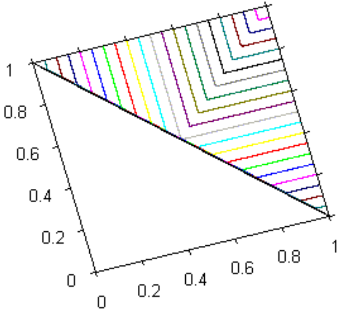




Prominent example of only left-continuous t-norms

The nilpotent minimum: $x *_{\mathrm{NM}} y = \begin{cases} \min\{x, y\} & x + y > 1, \\ 0 & \text{otherwise} \end{cases}$





Mostert–Shield's characterization

The idempotent elements (i.e., such x that $x * x = x$) of any continuous t-norm form a closed subset of $[0, 1]$.

Its complement is an (at most countable) union of open intervals.

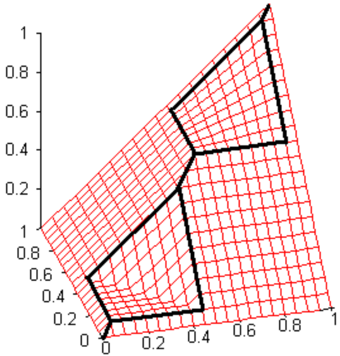
The restriction of $*$ to each of these intervals is isomorphic to $*_{Ł}$ (if it has nilpotent elements) or $*_{\Pi}$ (otherwise).

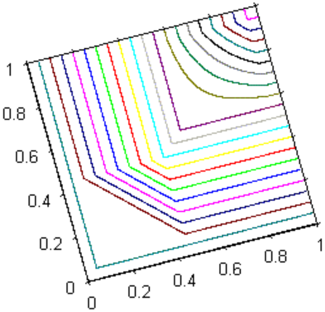
On the rest of $[0, 1]$ it coincides with $*_{G} = \min$.

All continuous t-norms are ordinal sums of isomorphic copies of $*_{Ł}, *_{\Pi}, *_{G}$.

Example

Ordinal sum of $*_{Ł}$ on $[0.05, 0.45]$, $*_{\Pi}$ on $[0.55, 0.95]$,
and the default $*_{G}$ elsewhere





Residua of left-continuous t-norms

Theorem 5.15

The following are equivalent for any t-norm $*$:

- $*$ *is left-continuous*
- *For each* x, y *there exist* $\max\{z \mid z * x \leq y\}$
- *There is a unique operation* $\Rightarrow_*$ *s.t.* $z * x \leq y$ *iff* $z \leq x \Rightarrow_* y$

Proof.

$1. \to 2$ via picture; $2. \to 3$ existence is easy $x \Rightarrow y = \max\{z \mid z * x \leq y\}$, uniqueness:

$$x \Rightarrow' y \leq x \Rightarrow' y \quad \text{iff} \quad x * (x \Rightarrow y) \leq y \quad \text{iff} \quad x \Rightarrow' y \leq x \Rightarrow y$$

□

Residua of left-continuous t-norms

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- $*$ *is left-continuous*
- *For each* x, y *there exist* $\max\{z \mid z * x \leq y\}$
- *There is a unique operation* $\Rightarrow_*$ *s.t.* $z * x \leq y$ *iff* $z \leq x \Rightarrow_* y$

Proof.

To prove $3. \to 1.$ it suffices to show

$$x * \sup Z = \sup_{z \in Z}(x * z) \quad \text{for each } x, y \text{ and a set } Z$$

Clearly $x * \sup Z \geq x * z$ (for $z \in Z$) ergo $x * \sup Z \geq \sup_{z \in Z}(x * z)$
From $z * x \leq \sup_{z \in Z}(x * z)$ (for $z \in Z$) get $z \leq x \Rightarrow \sup_{z \in Z}(x * z)$.
Thus $\sup Z \leq x \Rightarrow \sup_{z \in Z}(x * z)$ and so $x * \sup Z \leq \sup_{z \in Z}(x * z)$. □

Residua of left-continuous t-norms

Theorem 5.15

The following are equivalent for any t-norm $*$*:*

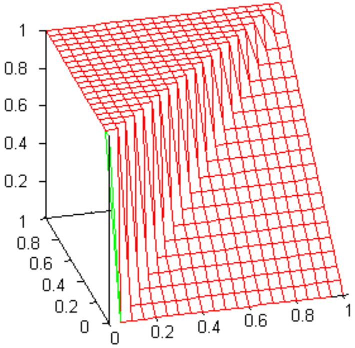
- $*$ *is left-continuous*
- *For each* x, y *there exist* $\max\{z \mid z * x \leq y\}$
- *There is a unique operation* $\Rightarrow_*$ *s.t.* $z * x \leq y$ *iff* $z \leq x \Rightarrow_* y$

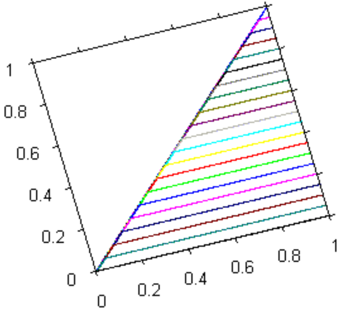
Definition 5.16

The operation $\Rightarrow_*$ is called the residuum of a t-norm $*$.

Residua of prominent continuous t-norms (1)

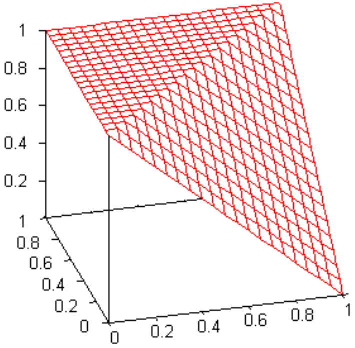
The residuum of $*_G$: Gödel implication $x \Rightarrow_G y = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise} \end{cases}$

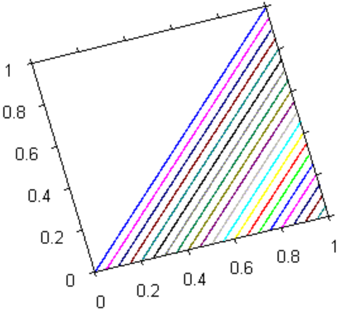




Residua of prominent continuous t-norms (2)

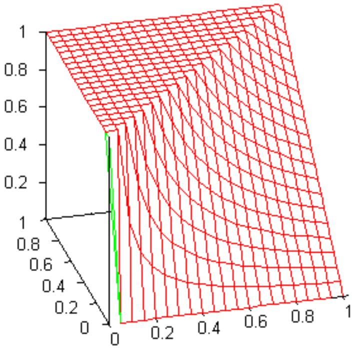
The residuum of $*_Ł$: Łukasiewicz implication $x \Rightarrow_Ł y = \min\{1, 1 - x + y\}$

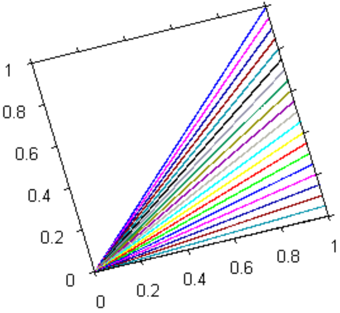




Residua of prominent continuous t-norms (3)

The residuum of $*_\Pi$: Goguen implication $x \Rightarrow_\Pi y = \begin{cases} \frac{y}{x} & \text{if } x > y \\ 1 & \text{if } x \leq y \end{cases}$





Basic properties of the residua of t-norms

Exercise 28

Prove that for each left-continuous t-norm $*$ the following holds:

- $(x \Rightarrow y) = 1$ iff $x \leq y$
- $(1 \Rightarrow y) = y$
- $\max\{x, y\} = \min\{(x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x\}$

Basic properties of the residua of t-norms

Theorem 5.17

Let $*$ be a left-continuous t-norm and $\Rightarrow$ its residuum. Then $*$ is right-continuous iff $\min\{x, y\} = x * (x \Rightarrow y)$.

Proof.

Recall that $*$ is right-continuous iff $x * \inf Z = \inf_{z \in Z}(x * z)$ for each x, y and a set Z. Left-to-right direction: using a picture; the converse one: clearly $x * \inf Z \leq \inf_{z \in Z}(x * z)$. Assume that $x * \inf Z < y < \inf_{z \in Z}(x * z)$.

Note that $y < x$ and so $y = x * (x \Rightarrow y)$.

Assume that $x \Rightarrow y \leq \inf Z$ so $y = x * (x \Rightarrow y) \leq x * \inf Z$ a contradiction.

Thus $\inf Z < x \Rightarrow y$, i.e., there is $z \in Z$ such that $z \leq x \Rightarrow y$.

Thus $\inf_{z \in Z}(x * z) \leq z * x \leq y$ a contradiction. □

MTL-algebras and (left-)continuous t-norms

Theorem 5.18

- *A structure* $\boldsymbol{B} = ([0,1], \min, \max, \&, \to, 0, 1)$ *is a* MTL*-algebra IFF* $\&$ *is a left-continuous t-norm and* $\to$ *its residuum.*
- *A structure* $\boldsymbol{B} = ([0,1], \min, \max, \&, \to, 0, 1)$ *is a* HL*-algebra IFF* $\&$ *is a continuous t-norm and* $\to$ *its residuum.*

Exercise 29

(a) Prove the theorem above.

(b) Prove that $\boldsymbol{B}$ is G-algebra iff $\&$ is Gödel t-norm.

(c) Prove that $\boldsymbol{B}$ is MV-algebra iff $\&$ is isomorphic to Łukasiewicz t-norm.

(d) Prove that $\boldsymbol{B}$ is Π-algebra iff $\&$ is isomorphic to product t-norm.

Standard completeness theorem for MTL

Theorem 5.19

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{MTL}} \varphi$
2. $\Gamma \models_{\mathbb{MTL}} \varphi$
3. $\Gamma \models_{\mathbb{MTL}_{\mathrm{lin}}} \varphi$
4. $\Gamma \models_{\mathbb{MTL}_{\mathrm{std}}} \varphi$

The logic MTL is the logic of all left-continuous t-norms.

Standard completeness theorem for HL

Theorem 5.20

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{HL}} \varphi$
2. $\Gamma \models_{\mathbb{HL}} \varphi$
3. $\Gamma \models_{\mathbb{HL}_{\mathrm{lin}}} \varphi$

If Γ *is finite we can add:*

4. $\Gamma \models_{\mathbb{HL}_{\mathrm{std}}} \varphi$

Hájek's basic fuzzy logic HL is the logic of all continuous t-norms

Outline

Three stages of development of an area of logic

Chagrov (*K voprosu ob obratnoi matematike modal'noi logiki,*
Online Journal Logical Studies, 2001)

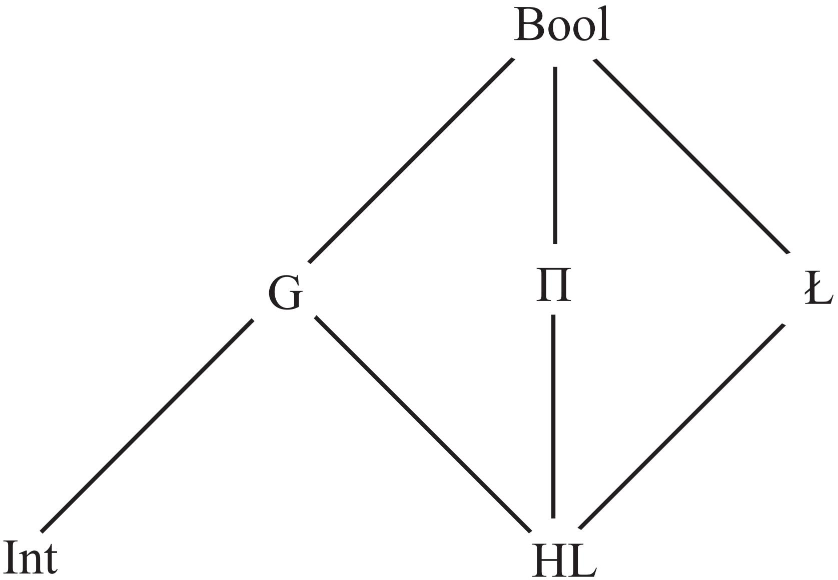
distinguishes three stages in the development of a field in logic.

Three stages of development of MFL

First stage: Emerging of the area (since 1965)

- 1965: Zadeh's fuzzy sets, 1968: 'fuzzy logic' (Goguen)
- 1970s: systems of fuzzy 'logic' lacking a good metatheory
- 1970s–1980s: first 'real' logics (Pavelka, Takeuti–Titani, . . .),
  discussion of many-valued logics in the fuzzy context

'Culminated' in Hájek's monograph (1998): G, Ł, HL, Π


Bool
G
Π
Ł
Int
HL

Three stages of development of MFL

Second stage: development of particular logics and introduction of many new ones (since the 1990s)

- New logics: MTL, SHL, UL, $\Pi_{\sim}$, ŁΠ, ...
- Algebraic semantics, proof theory, complexity
  Kripke-style and game-theoretic semantics, ...
- First-order, higher-order, and modal fuzzy logics

Systematic treatment of particular fuzzy logics

Basic fuzzy logic?

Hájek called the logic HL the Basic fuzzy Logic BL

HL was *basic* in the following two senses:

1. *it could not be made weaker without losing essential properties*
2. *it provided a base for the study of all fuzzy logics.*

Basic fuzzy logic?

Hájek called the logic HL the Basic fuzzy Logic BL

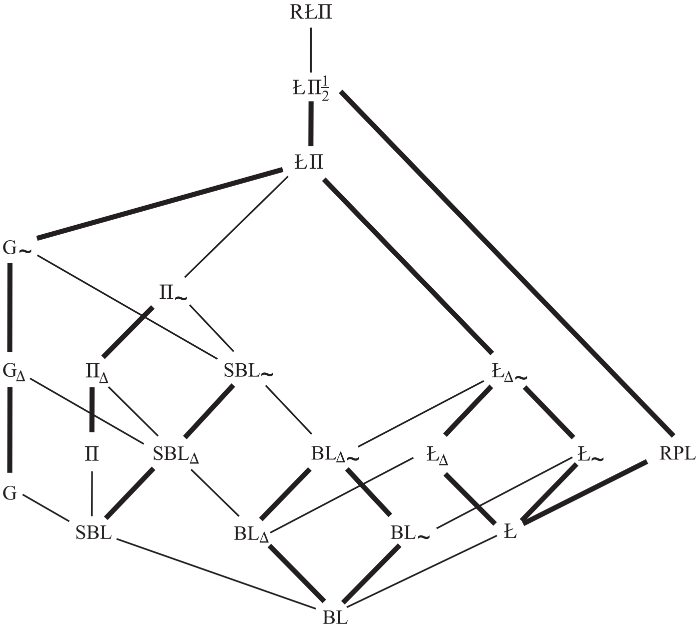
HL was *basic* in the following two senses:

1. *it could not be made weaker without losing essential properties*
2. *it provided a base for the study of all fuzzy logics.*

Because:

- HL is complete w.r.t. the semantics given by *all* continuous t-norms
- All then known fuzzy logics were expansions of HL. The methods to introduce, algebraize, and study HL could be modified for all expansions of HL.

fuzzy logics = expansions of HL


RŁΠ
ŁΠ½
ŁΠ
G∼
Π∼
G∆
Π∆
SBL∼
Ł∆∼
Π
SBL∆
BL∆∼
Ł∆
Ł∼
RPL
G
SBL
BL∆
BL∼
Ł
BL

“Removing legs from the flea”

In the 3rd EUSFLAT (Zittau, Germany, September 2003) Petr Hájek started his lecture *Fleas and fuzzy logic: a survey* with a joke.

“Removing legs from the flea”

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A group of scientists decide to investigate the ability of a flea can jump in relationship to how many legs it has.

They put the flea on a desk and said ’jump!’ The flea jumped and they noted:
“the flea with 6 legs can jump.”

They remove a leg, repeated the command, the flea jumped and they noted:
“the flea with 5 legs can jump.”

⋮

Finally, they removed the last legs repeated the command but
the flea didn’t move.

“Removing legs from the flea”

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“the flea with 6 legs can jump.”

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“the flea with 5 legs can jump.”

⋮

Finally, they removed the last legs repeated the command but
the flea didn’t move.

So they concluded:

“Upon removing all its legs the flea loses sense of hearing.”

7 Gödel logic

A G-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1} \rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a commutative monoid

(3) $z \leq x \rightarrow y$ iff $x \,\&\, z \leq y$, (residuation)

(4) $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$ (prelinearity)

(5) $x \,\&\, (x \rightarrow y) = x \wedge y$ (divisibility)

(6) $x \,\&\, y = x \wedge y$

6 Hájek's logic

An HL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1} \rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a commutative monoid

(3) $z \leq x \rightarrow y$ iff $x \,\&\, z \leq y$, (residuation)

(4) $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$ (prelinearity)

(5) $x \,\&\, (x \rightarrow y) = x \wedge y$ (divisibility)

Hájek logic HL is the logic of continuous t-norms

(well designed to jump)

5 Monoidal t-norm logic MTL

An MTL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1} \rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a commutative monoid

(3) $z \leq x \rightarrow y$ iff $x \& z \leq y$, (residuation)

(4) $(x \rightarrow y) \vee (y \rightarrow x) = \overline{1}$ (prelinearity)

MTL is the logic of left-continuous of t-norms

(designed to jump even further)

4 Uninorm logic: the non-integral case

A UL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1}, \bot, \top \rangle$ such that:

(1) $\langle B, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a commutative monoid

(3) $z \leq x \rightarrow y$ iff $x \& z \leq y$, (residuation)

(4) $((x \rightarrow y) \wedge \overline{1}) \vee ((y \rightarrow x) \wedge \overline{1}) = \overline{1}$ (prelinearity)

UL is the logic of residuated uninorms

(designed to jump even further in one direction)

3 psMTL^r: the non commutative case

A psMTL^r-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \rightsquigarrow, \overline{0}, \overline{1}\rangle$ such that:

(1) $\langle B, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1}\rangle$ is a monoid,

(3) $z \leq x \rightarrow y$ iff $x \,\&\, z \leq y$ iff $x \leq z \rightsquigarrow y$, (residuation)

(4) something ugly (prelinearity)

psMTL^r is the logic of residuated pseudo t-norms

(designed to jump even further in other direction)

2 psUL: the non commutative and non integral case

A psUL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \rightarrow, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top \rangle$ s.t.:

(1) $\langle B, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a monoid,

(3) $z \leq x \rightarrow y$ iff $x \,\&\, z \leq y$ iff $x \leq z \rightsquigarrow y$, (residuation)

(4) something even uglier (prelinearity)

psUL is NOT the logic of residuated pseudo uninorms

(lost all sense of hearing?)

1 SL^ℓ: the non associative case

An SL^ℓ-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top \rangle$ s.t.:

(1) $\langle B, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,

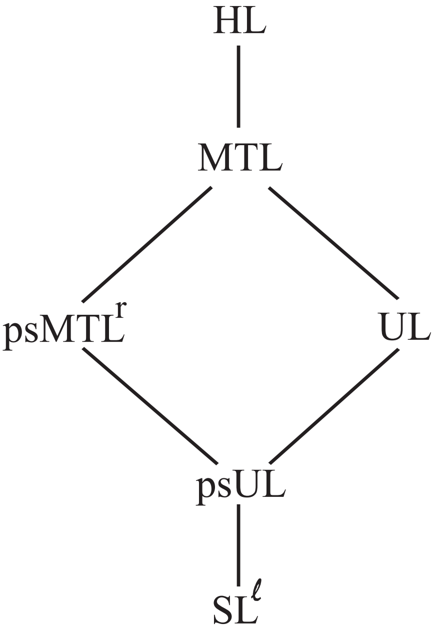
(2) $\langle B, \&, \overline{1} \rangle$ is a unital groupoid,

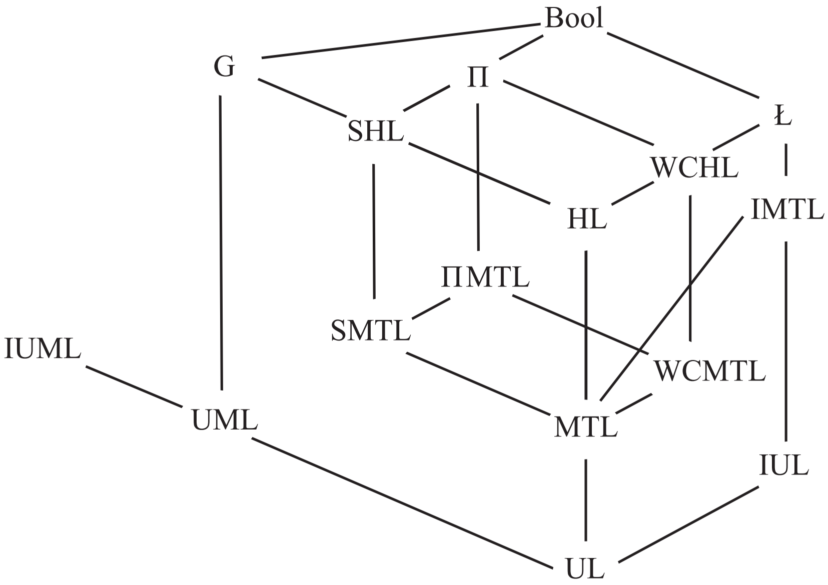
(3) $z \leq x \to y$ iff $x \,\&\, z \leq y$ iff $x \leq z \rightsquigarrow y$, (residuation)

(4) the ugliest thing possible (prelinearity)

SL^ℓ is the logic of residuated unital grupoids on [0,1]

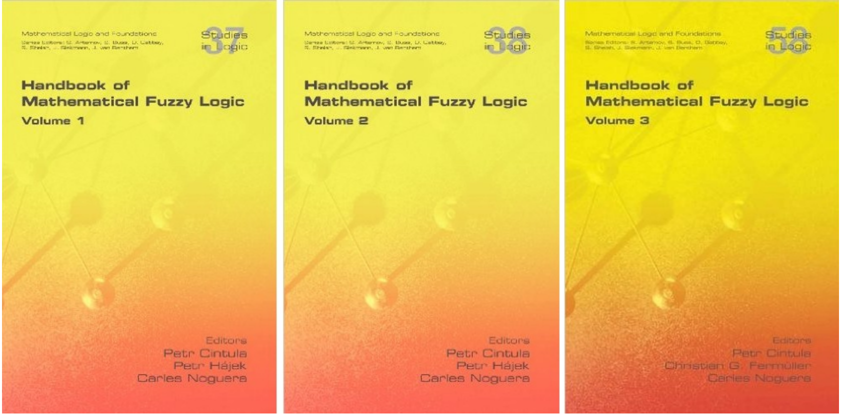
it jumps again!


HL
MTL
psMTLr
UL
psUL
SLℓ


Bool
G
Π
Ł
SHL
WCHL
HL
IMTL
ΠMTL
SMTL
IUML
WCMTL
UML
MTL
IUL
UL

Three stages of development of MFL

The second stage is still ongoing; the state of the art is summarized in:



P. Cintula, C. Fermüller, P. Hájek, C. Noguera (editors). Vol. 37, 38, and 58 of *Studies in Logic: Math. Logic and Foundations*. College Publications, 2011, 2015.

Three stages of development of MFL

Third stage: universal methods (since $\sim$2006)

- General methods to prove metamathematical properties
- Classification of existing fuzzy logics
- Systematic treatment of classes of fuzzy logics
- Determining the position of fuzzy logics in the logical landscape

Outline

Changing the language

We consider a new set of primitive connectives
$\mathcal{L}_{\mathrm{SL}} = \{\overline{0}, \overline{1}, \bot, \top, \&, \to, \rightsquigarrow, \vee, \wedge\}$, and a defined connective $\leftrightarrow$:

$$\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

We keep the symbol $Fm_{\mathcal{L}}$ for the set of formulas.

The 'minimal' algebraic semantics

Definition 5.21

An SL-*algebra* is a structure $\boldsymbol{B} = \langle B, \wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top \rangle$ such that:

(1) $\langle B, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a unital groupoid,

(3) $z \leq x \to y$ iff $x \,\&\, z \leq y$ iff $x \leq z \rightsquigarrow y$, (residuation)

Hilbert-system for SL – axioms

$(\mathrm{Adj}_{\&})$	$\varphi \to (\psi \to \psi \mathbin{\&} \varphi)$
$(\mathrm{Adj}_{\&\rightsquigarrow})$	$\varphi \to (\psi \rightsquigarrow \varphi \mathbin{\&} \psi)$
$(\&\wedge)$	$(\varphi \wedge \overline{1}) \mathbin{\&} (\psi \wedge \overline{1}) \to \varphi \wedge \psi$
$(\wedge 1)$	$\varphi \wedge \psi \to \varphi$
$(\wedge 2)$	$\varphi \wedge \psi \to \psi$
$(\wedge 3)$	$(\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$
$(\vee 1)$	$\varphi \to \varphi \vee \psi$
$(\vee 2)$	$\psi \to \varphi \vee \psi$
$(\vee 3)$	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
(Push)	$\varphi \to (\overline{1} \to \varphi)$
(Pop)	$(\overline{1} \to \varphi) \to \varphi$
(Res')	$\psi \mathbin{\&} (\varphi \mathbin{\&} (\varphi \to (\psi \to \chi))) \to \chi$
$(\mathrm{Res}'_{\rightsquigarrow})$	$(\varphi \mathbin{\&} (\varphi \to (\psi \rightsquigarrow \chi))) \mathbin{\&} \psi \to \chi$
(T')	$(\varphi \to (\varphi \mathbin{\&} (\varphi \to \psi)) \mathbin{\&} (\psi \to \chi)) \to (\varphi \to \chi)$
$(\mathrm{T}'_{\rightsquigarrow})$	$(\varphi \rightsquigarrow ((\varphi \rightsquigarrow \psi) \mathbin{\&} \varphi) \mathbin{\&} (\psi \to \chi)) \to (\varphi \rightsquigarrow \chi)$

Hilbert-system for SL – rules

$$
\begin{array}{rl}
\textbf{(MP)} & \varphi, \varphi \to \psi \vdash \psi \\
(\mathbf{Adj_u}) & \varphi \vdash \varphi \wedge \overline{1} \\
(\alpha) & \varphi \vdash \delta \mathbin{\&} \varepsilon \to \delta \mathbin{\&} (\varepsilon \mathbin{\&} \varphi) \\
(\alpha') & \varphi \vdash \delta \mathbin{\&} \varepsilon \to (\delta \mathbin{\&} \varphi) \mathbin{\&} \varepsilon \\
(\beta) & \varphi \vdash \delta \to (\varepsilon \to (\varepsilon \mathbin{\&} \delta) \mathbin{\&} \varphi) \\
(\beta') & \varphi \vdash \delta \to (\varepsilon \rightsquigarrow (\delta \mathbin{\&} \varepsilon) \mathbin{\&} \varphi)
\end{array}
$$

Convention

Convention

A logic is a provability relation on formulas in a language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$ s.t.

- it is axiomatized by adding axioms Ax and finitary rules (R) to the logic SL
- for each n-ary connective $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$, $\mathcal{L}$-formulas $\varphi, \psi, \chi_1, \ldots, \chi_n$, and each $i \leq n$ the following holds:

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n) \leftrightarrow c(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n)$$

Convention

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A logic is a provability relation on formulas in a language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$ s.t.

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- for each n-ary connective $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$, $\mathcal{L}$-formulas $\varphi, \psi, \chi_1, \ldots, \chi_n$, and each $i \leq n$ the following holds:

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n) \leftrightarrow c(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n)$$

Let us fix a logic L in language $\mathcal{L}$ which is the expansion of SL by axioms Ax and rules R.

Algebraic semantics for arbitrary logic L

Definition 5.22

Let $\boldsymbol{B}$ be an $\mathcal{L}$-algebra. A $\boldsymbol{B}$-evaluation is a mapping $e\colon Fm_{\mathcal{L}} \to B$ s.t.

- $e(*) = *^{\boldsymbol{B}}$ for truth constant $*$
- $e(\circ(\varphi_1, \dots, \varphi_n)) = \circ^{\boldsymbol{B}}(e(\varphi_1), \dots, e(\varphi_n))$ for each $n-$ary $\circ \in \mathcal{L}$

Definition 5.23

An $\mathcal{L}$-algebra $\boldsymbol{A}$ is an L-algebra, $\boldsymbol{A} \in \mathbb{L}$, if

- its reduct $\boldsymbol{A}_{\mathrm{SL}} = \langle A, \wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top \rangle$ is an SL-algebra,
- for each $\varphi \in Ax$, $\boldsymbol{A}$ satisfies the identity $\varphi \wedge \overline{1} = \overline{1}$,
- for each $\langle \{\psi_1, \dots, \psi_n\}, \varphi \rangle \in R$, $\boldsymbol{A}$ satisfies the quasi-identity

$$\texttt{If}\ \ \psi_1 \wedge \overline{1} = \overline{1}\ \ \texttt{and}\ \ \cdots\ \ \texttt{and}\ \ \psi_n \wedge \overline{1} = \overline{1}\ \ \texttt{then}\ \ \varphi \wedge \overline{1} = \overline{1}$$

$\boldsymbol{A}$ is a linearly ordered (or L-*chain*), $\boldsymbol{A} \in \mathbb{L}_{\mathrm{lin}}$, if its lattice order is total.

Logical consequence w.r.t. a class of algebras

Definition 5.24

A formula φ is a logical consequence of set of formulas Γ w.r.t. a class $\mathbb{K}$ of L-algebras, $\Gamma \models_{\mathbb{K}} \varphi$, if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

$$\text{if } e(\gamma) \geq \overline{1} \text{ for every } \gamma \in \Gamma, \text{ then } e(\varphi) \geq \overline{1}.$$

Observation

1. An $\mathcal{L}$-algebra $\boldsymbol{A}$ is an L-algebra iff
   - its reduct $\boldsymbol{A}_{\mathrm{SL}} = \langle A, \wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top \rangle$ is an SL-algebra,
   - if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Gamma \models_{\boldsymbol{A}} \varphi$.
2. $\mathbb{L}$ is the largest class $\mathbb{K}$ of $\mathcal{L}$-algebras such that $\vdash_{\mathrm{L}} \subseteq \models_{\mathbb{K}}$

General completeness theorem

Theorem 5.25 (Completeness theorem)

For every set of formulas Γ *and every formula* φ *we have:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \text{ if, and only if, } \Gamma \models_{\mathbb{L}} \varphi.$$

Each L is an algebraizable logic and $\mathbb{L}$ is its equivalent algebraic semantics with translations:

$$E(p,q) = \{p \leftrightarrow q\} \text{ and } \mathcal{E}(p) = \{p \wedge \overline{1} \approx \overline{1}\}.$$

Indeed, all we have to do is to prove:

$$p \vdash p \wedge \overline{1} \leftrightarrow \overline{1} \qquad \text{and} \qquad p \wedge \overline{1} \leftrightarrow \overline{1} \vdash p$$

Core semilinear logics

Definition 5.26

A logic L is core semilinear logic whenever it is complete w.r.t. linearly ordered L-algebras, i.e.,

$$T \vdash_{\mathrm{L}} \varphi \quad \text{iff} \quad T \models_{\mathbb{L}_{\mathrm{lin}}} \varphi$$

Core semilinear logics — syntactic characterization

Theorem 5.27 (Syntactic characterization)

Let L *be axiomatized by axioms* Ax *and rules* R. *TFAE:*

1. L *is a core semilinear logic*
2. $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ *and if* $\langle \Gamma, \varphi \rangle \in R$, *then* $\Gamma \vee \chi \vdash_{\mathrm{L}} \varphi \vee \chi$ *for every* χ
3. $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ *and if* $\Gamma \vdash_{\mathrm{L}} \varphi$, *then* $\Gamma \vee \chi \vdash_{\mathrm{L}} \varphi \vee \chi$ *for every* χ
4. $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ *and for every set of formulas* $\Gamma \cup \{\varphi, \psi, \chi\}$:
$$\Gamma, \varphi \vdash_{\mathrm{L}} \chi \quad \textit{and} \quad \Gamma, \psi \vdash_{\mathrm{L}} \chi \quad \textit{imply} \quad \Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi.$$
5. *For every set of formulas* $\Gamma \cup \{\varphi, \psi, \chi\}$:
$$\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \quad \textit{and} \quad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi \quad \textit{imply} \quad \Gamma \vdash_{\mathrm{L}} \chi.$$
6. *If* $\Gamma \nvdash_{\mathrm{L}} \varphi$ *then there is a linear theory* $\Gamma' \supseteq \Gamma$ *s.t.* $\Gamma \nvdash_{\mathrm{L}} \varphi$

Core semilinear logics — semantic characterization

Theorem 5.28 (Semantic characterization)

Let L *be a logic. TFAE:*

1. L *is a core semilinear logic*
2. *finitely relatively subdirectly irreducible* L*-algebras are exactly the* L*-chains*
3. *relatively subdirectly irreducible* L*-algebras are linearly ordered*

Weakest semilinear extension

Definition 5.29

By L^{ℓ} we denote the least core semilinear logic extending L.

Lemma 5.30

(a) *The previous definition is sound because that the class of core semilinear logics is closed under arbitrary intersections.*

(b) ${\mathbb{L}^{\ell}}_{\mathrm{lin}} = \mathbb{L}_{\mathrm{lin}}$.

Theorem 5.31

If L is axiomatized by rules R, then L^{ℓ} is axiomatized by adding axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$ and rules: $\langle \Gamma \vee \chi, \varphi \vee \chi \rangle$ for each $\langle \Gamma, \varphi \rangle \in R$.

In many cases we can prove that L^{ℓ} is an axiomatic extension of L.

Hilbert-system for SL^{ℓ} – axioms

To the axioms of SL we add

$$
\begin{array}{rl}
(\mathrm{PRL}\alpha) & [(\varphi \to \psi) \wedge \overline{1}] \vee (\delta \mathbin{\&} \varepsilon \to \delta \mathbin{\&} (\varepsilon \mathbin{\&} [(\psi \to \varphi) \wedge \overline{1}]) \\
(\mathrm{PRL}\alpha') & [(\varphi \to \psi) \wedge \overline{1}] \vee (\delta \mathbin{\&} \varepsilon \to (\delta \mathbin{\&} [(\psi \to \varphi) \wedge \overline{1}]) \mathbin{\&} \varepsilon) \\
(\mathrm{PRL}\beta) & [(\varphi \to \psi) \wedge \overline{1}] \vee (\delta \to (\varepsilon \to (\varepsilon \mathbin{\&} \delta) \mathbin{\&} [(\psi \to \varphi) \wedge \overline{1}])) \\
(\mathrm{PRL}\beta') & [(\varphi \to \psi) \wedge \overline{1}] \vee (\delta \to (\varepsilon \rightsquigarrow (\delta \mathbin{\&} \varepsilon) \mathbin{\&} [(\psi \to \varphi) \wedge \overline{1}]))
\end{array}
$$

A linear/standard completeness theorem of SL^{ℓ}

Let us by $\mathbb{SL}^{\ell}_{\mathrm{std}}$ denote the class of SL-algebras with the domain $[0,1]$ and the usual order.

Theorem 5.32 (Standard completeness theorem of SL^{ℓ})

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*:*

1. $\Gamma \vdash^{\ell}_{\mathrm{SL}} \varphi$
2. $\Gamma \models_{\mathbb{SL}^{\ell}} \varphi$
3. $\Gamma \models_{\mathbb{SL}^{\ell}_{\mathrm{lin}}} \varphi$
4. $\Gamma \models_{\mathbb{SL}^{\ell}_{\mathrm{std}}} \varphi$

Is SL^ℓ the new basic fuzzy logic?

We need to show that it is *basic* in the following two senses:

1. *it cannot be made weaker without losing essential properties and*
2. *it provides a base for the study of all fuzzy logics.*

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1. SL^ℓ is complete w.r.t. a hardly-to-be-made-weaker semantics over real numbers.
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fuzzy logics = core semilinear logics

Outline

Adding Baaz delta

Let L be an axiomatic extension of MTL.

We add a unary connective $\triangle$ known as Baaz delta or 0–1 projector.

The logic $\mathrm{L}_\triangle$ is the extension of L by the axioms:

$$\begin{array}{l}
\triangle\varphi \vee \neg\triangle\varphi, \\
\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi), \\
\triangle\varphi \to \varphi, \\
\triangle\varphi \to \triangle\triangle\varphi, \\
\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi).
\end{array}$$

and the rule of $\triangle$-necessitation: from φ infer $\triangle\varphi$.

Adding Baaz delta: syntactic properties

Lemma 5.33

$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}_\triangle} \triangle\varphi \leftrightarrow \triangle\psi \qquad \varphi \vee \chi \vdash_{\mathrm{L}_\triangle} \triangle\varphi \vee \chi$

Theorem 5.34

- $T, \varphi \vdash_{\mathrm{L}_\triangle} \psi$ *iff* $T \vdash_{\mathrm{L}_\triangle} \triangle\varphi \to \psi$ *(Delta Deduction Theorem)*
- *If* $\Gamma, \varphi \vdash_{\mathrm{L}_\triangle} \chi$ *and* $\Gamma, \psi \vdash_{\mathrm{L}_\triangle} \chi$, *then* $\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}_\triangle} \chi$. *(Proof by Cases Property)*
- *If* $\Gamma, \varphi \to \psi \vdash_{\mathrm{L}_\triangle} \chi$ *and* $\Gamma, \psi \to \varphi \vdash_{\mathrm{L}_\triangle} \chi$, *then* $\Gamma \vdash_{\mathrm{L}_\triangle} \chi$. *(Semilinearity Property)*
- *If* $\Gamma \nvdash_{\mathrm{L}_\triangle} \varphi$, *then there is a linear* $\Gamma' \supseteq \Gamma$ *such that* $\Gamma' \nvdash_{\mathrm{L}_\triangle} \varphi$. *(Linear Extension Property)*

Exercise 30

Prove this lemma and theorem.

Adding Baaz delta: semantics and completeness

An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1}, \triangle\rangle$ is an $\mathrm{L}_\triangle$-algebra if:

(0) $\langle A, \wedge, \vee, \&, \rightarrow, \overline{0}, \overline{1}\rangle$ is an L-algebra,

(1) $\triangle x \vee (\triangle x \rightarrow 0) = \overline{1}$,

(2) $\triangle(x \vee y) \leq (\triangle x \vee \triangle y)$

(3) $\triangle x \leq x$

(4) $\triangle x \leq \triangle\triangle x$

(5) $\triangle(x \rightarrow y) \leq \triangle x \rightarrow \triangle y$

(6) $\triangle\overline{1} = \overline{1}$.

Let $\boldsymbol{A}$ be an $\mathrm{L}_\triangle$-chain. Then for every $x \in A$, $\triangle x = \begin{cases} \overline{1} & \text{if } x = \overline{1} \\ \overline{0} & \text{otherwise.} \end{cases}$

Theorem 5.35

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$:

1. $\Gamma \vdash_{\mathrm{L}_\triangle} \varphi$
2. $\Gamma \models_{(\mathbb{L}_\triangle)_{lin}} \varphi$

Adding an involutive negation

Let $\mathrm{L}_\sim$ be $\mathrm{L}_\triangle$ plus a new unary connective $\sim$ and the following axioms:

(~ 1) $\sim\sim\varphi \leftrightarrow \varphi$,

(~ 2) $\triangle(\varphi \to \psi) \to (\sim\psi \to \sim\varphi)$.

An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \to, \overline{0}, \overline{1}, \triangle, \sim\rangle$ is a $\mathrm{L}_\sim$-algebra if:

(0) $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \to, \overline{0}, \overline{1}, \triangle\rangle$ is an $\mathrm{L}_\triangle$-algebra,

(1) $x = \sim\sim x$,

(2) $\triangle(x \to y) \le \sim y \to \sim x$,

Theorem 5.36

$\mathrm{L}_\sim$ is complete w.r.t. $\mathrm{L}_\sim$-chains and w.r.t. standard L chains expanded with $\triangle$ and some involutive negation.
Furthermore $\mathrm{G}_\sim$ is complete w.r.t. $\mathrm{G}_\sim$-chains and w.r.t. $[0,1]_{\mathrm{G}_\triangle}$ expanded with the involutive negation $1 - x$.

Adding multiplication

We add a binary connective $\odot$ and define the Product Lukasiewicz logic $\mathrm{PŁ}$ by adding the following axioms to Ł:

(P1)	$(\chi \odot \varphi) \ominus (\chi \odot \psi) \leftrightarrow \chi \odot (\varphi \ominus \psi)$	(distributivity)
(P2)	$\varphi \odot (\psi \odot \chi) \leftrightarrow (\varphi \odot \psi) \odot \chi$	(associativity)
(P3)	$\varphi \to \varphi \odot \overline{1}$	(neutral element)
(P4)	$\varphi \odot \psi \to \varphi$	(monotonicity)
(P5)	$\varphi \odot \psi \to \psi \odot \varphi$	(commutativity)

$\mathrm{PŁ}'$ is the extension of $\mathrm{PŁ}$ with a new rule: (ZD) from $\neg(\varphi \odot \varphi)$ infer $\neg\varphi$.

Lemma 5.37

$\varphi \leftrightarrow \psi \vdash_{\mathrm{PŁ}} \varphi \odot \chi \leftrightarrow \psi \odot \chi$ $\qquad$ $\neg(\varphi \odot \varphi) \vee \chi \vdash_{\mathrm{PŁ}} \neg\varphi \vee \chi$

$\varphi \leftrightarrow \psi \vdash_{\mathrm{PŁ}'} \varphi \odot \chi \leftrightarrow \psi \odot \chi$

Theorem 5.38 (Deduction theorem)

$\Gamma, \varphi \vdash_{\mathrm{PŁ}} \psi$ *iff there is* n *such that* $\Gamma \vdash_{\mathrm{PŁ}} \varphi^n \to \psi$. *does not hold for* $\mathrm{PŁ}'$.

PŁ-algebras and PŁ′-algebras:

A PŁ-algebra is a structure $\boldsymbol{A} = \langle A, \oplus, \neg, \odot, \overline{0}, \overline{1}\rangle$ such that $\langle A, \oplus, \neg, \overline{0}\rangle$ is an MV-algebra and the following equations hold:

(1)	$(x \odot y) \ominus (x \odot z) \approx x \odot (y \ominus z)$	(distributivity)
(2)	$x \odot (y \odot z) \approx (x \odot y) \odot z$	(associativity)
(3)	$x \odot \overline{1} \approx x$	(neutral element)
(4)	$x \odot y \approx y \odot x$	(commutativity)

A PŁ′-algebra is a PŁ-algebra where the following quasiequation holds:

(5) $x \odot x \approx \overline{0} \Rightarrow x \approx \overline{0}$ (domain of integrity)

$[0,1]_{\text{PŁ}} = \langle [0,1], \oplus, \neg, \odot, 0, 1\rangle$ (where $\odot$ is the usual algebraic product) is both the standard PŁ and PŁ′-algebra

Both logics enjoy the completeness w.r.t. their chains but only PŁ′ enjoys the standard completeness.

Adding truth constants: Rational Pavelka Logic

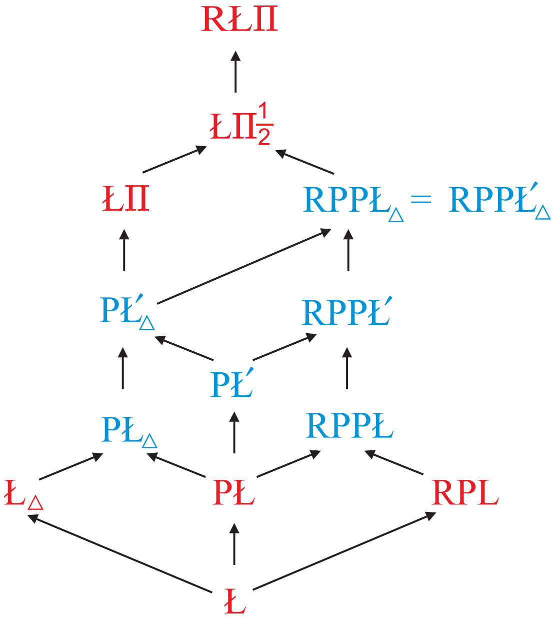
RPL is the expansion of Ł with a constant $\overline{r}$ for each $r \in [0,1] \cap \mathbb{Q}$ and axioms: $\overline{r} \oplus \overline{s} \leftrightarrow \overline{\min\{1, r+s\}}$ and $\neg\overline{r} \leftrightarrow \overline{1-r}$.

We define:

- The truth degree of φ over T is $||\varphi||_T = \inf\{e(\varphi) \mid e[T] \subseteq \{1\}\}$
- The provability degree of φ over T is $|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{RPL}} \overline{r} \to \varphi\}$.

Theorem 5.39 (Pavelka style completeness)

$||\varphi||_T = |\varphi|_T$, *for each set of formulas* $T \cup \{\varphi\}$.


RŁΠ
ŁΠ½
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RPPŁ△ = RPPŁ′△
PŁ′△
RPPŁ′
PŁ′
PŁ△
RPPŁ
Ł△
PŁ
RPL
Ł

$\text{Ł}\Pi$ and $\text{Ł}\Pi\frac{1}{2}$ logics: connectives

Logic $\text{Ł}\Pi$ has the following basic connectives:

$\overline{0}$	0	truth constant falsum
$\varphi \to_{\text{Ł}} \psi$	$x \to_{\text{Ł}} y = \min(1, 1 - x + y)$	Łukasiewicz implication
$\varphi \to_{\Pi} \psi$	$x \to_{\Pi} y = \min(1, \frac{x}{y})$	product implication
$\varphi \odot \psi$	$x \odot y = x \cdot y$	product conjunction

Logic $\text{Ł}\Pi\frac{1}{2}$ has an additional truth constant $\overline{\frac{1}{2}}$ with std. semantics $\frac{1}{2}$.

We define the following derived connectives:

$\neg_{\text{Ł}}\varphi$	is	$\varphi \to_{\text{Ł}} \overline{0}$	$\neg_{\text{Ł}} x = 1 - x$
$\neg_{\Pi}\varphi$	is	$\varphi \to_{\Pi} \overline{0}$	$\neg_{\text{Ł}} x = \frac{0}{x}$
$\triangle\varphi$	is	$\neg_{\Pi}\neg_{\text{Ł}}\varphi$	$\triangle 1 = 1$; $\triangle x = 0$ otherwise
$\varphi \,\&\, \psi$	is	$\neg_{\text{Ł}}(\varphi \to_{\text{Ł}} \neg_{\text{Ł}}\psi)$	$x \,\&\, y = \max(0, x + y - 1)$
$\varphi \oplus \psi$	is	$\neg_{\text{Ł}}\varphi \to_{\text{Ł}} \psi$	$x \oplus y = \min(1, x + y)$
$\varphi \ominus \psi$	is	$\varphi \,\&\, \neg_{\text{Ł}}\psi$	$x \ominus y = \max(0, x - y)$
$\varphi \wedge \psi$	is	$\varphi \,\&\, (\varphi \to_{\text{Ł}} \psi)$	$x \wedge y = \min(x, y)$
$\varphi \vee \psi$	is	$(\varphi \to_{\text{Ł}} \psi) \to_{\text{Ł}} \psi$	$x \vee y = \max(x, y)$
$\varphi \to_{G} \psi$	is	$\triangle(\varphi \to_{\text{Ł}} \psi) \vee \psi$	$x \to_{G} y = 1$ if $x \leq y$, otherwise y

$Ł\Pi$ and $Ł\Pi\frac{1}{2}$ logics: axiomatic system

Logic $Ł\Pi$ is given by the following axioms:

(Ł)	Axioms of Łukasiewicz logic,
(Π)	Axioms of product logic,
(Ł$\triangle$)	$\triangle(\varphi \to_{Ł} \psi) \to_{Ł} (\varphi \to_{\Pi} \psi)$,
($\Pi\triangle$)	$\triangle(\varphi \to_{\Pi} \psi) \to_{Ł} (\varphi \to_{Ł} \psi)$,
(Dist)	$\varphi \odot (\chi \ominus \psi) \leftrightarrow_{Ł} (\varphi \odot \chi) \ominus (\varphi \odot \psi)$.

The deduction rules are modus ponens and $\triangle$-necessitation (from φ infer $\triangle\varphi$).

The logic $Ł\Pi\frac{1}{2}$ results from the logic $Ł\Pi$ by adding axiom $\overline{\frac{1}{2}} \leftrightarrow \neg_{Ł}\overline{\frac{1}{2}}$.

Alternative axiomatization (in the language of $\mathrm{L}_{\sim}$)

(Π) axioms and deduction rules of $\Pi_{\sim}$,

(A) $(\varphi \to_{Ł} \psi) \to_{Ł} ((\psi \to_{Ł} \chi) \to_{Ł} (\varphi \to_{Ł} \chi))$,

where $\varphi \to_{Ł} \psi$ is defined as $\sim(\varphi \,\&\, \sim(\varphi \to \psi))$.

$Ł\Pi$ and $Ł\Pi\frac{1}{2}$ logics: algebras

An $Ł\Pi$-algebra is a structure: $\boldsymbol{A} = (A, \oplus, \sim, \to_\Pi, \odot, \overline{0}, \overline{1})$

(1) $(A, \oplus, \neg, \odot, 0)$ is a PŁ-algebra

(2) $z \leq (x \to_\Pi y)$ iff $x \odot z \leq y$

OR

$(1'')$ $(A, \oplus, \sim, 0)$ is an MV-algebra

$(2'')$ $(A, \to_\Pi, \odot, \wedge, \vee, 0, 1)$ is a Π-algebra

$(3'')$ $x \odot (y \ominus z) = (x \odot y) \ominus (x \odot z)$

$(4'')$ $\triangle(x \to_Ł y) \to_Ł (x \to_\Pi y) = 1$

OR

$(1')$ $(A, \odot, \to_\Pi, \wedge, \vee, \sim, 0, 1)$ is $\Pi_\sim$-algebra

$(2')$ $(x \to_Ł y) \leq ((y \to_Ł z) \to_Ł (x \to_Ł z))$

$(3')$ $x \to_Ł y = \sim(x \odot \sim(x \to_\Pi y))$

Some theorems about $Ł\Pi$ and $Ł\Pi\frac{1}{2}$ logics

- Both logics $Ł\Pi$ and $Ł\Pi\frac{1}{2}$ have
  - $\triangle$-deduction theorem
  - Proof by Cases Property
  - Semilinearity Property
  - Linear Extension Property
  - general/linaer completeness
  - finite standard completeness
- In $Ł\Pi\frac{1}{2}$ we can define truth constants for each rational from [0,1]
- Let $*$ be a continuous t-norm s.t. $*$ is finite ordinal sum (it the sense of Mostert–Shields Theorem). Then the logic $L(*)$ is interpretable in $Ł\Pi\frac{1}{2}$

Outline

Standard epistemic logic

Modality $\mathsf{K}A$ = “the agent knows that A”

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The principle of logical rationality of the agent

= the assumption that the agent can make inference steps $\Rightarrow$ the axiom (K) of propositional epistemic logic:

$$\mathsf{K}A \mathbin{\&} \mathsf{K}(A \to B) \to \mathsf{K}B$$

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= the assumption that the agent can make inference steps $\Rightarrow$ the axiom (K) of propositional epistemic logic:

$$\mathsf{K}A \mathbin{\&} \mathsf{K}(A \to B) \to \mathsf{K}B$$

The axiom is adopted in standard accounts of epistemic logic

Standard epistemic logic = the logic of *logically rational* agents

The logical omniscience paradox

An unwanted consequence of the logical rationality principle:

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the agent's knowledge is closed under *modus ponens*

$\Rightarrow$ under the propositional consequence relation

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$\Rightarrow$ the agent knows all propositional tautologies, once he/she/it knows the axioms of CL

The logical omniscience paradox

An unwanted consequence of the logical rationality principle:

the agent's knowledge is closed under *modus ponens*

$\Rightarrow$ under the propositional consequence relation

$\Rightarrow$ the agent knows all propositional tautologies, once he/she/it knows the axioms of CL

= an extremely implausible assumption on real-world agents (consider, eg, a non-trivial tautology with 10^9 variables)

Three kinds of knowledge

Actual knowledge . . . the modality "is *known*" = knowledge immediately available to the agent

(eg, the contents of its memory)

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(by logical inference)

Feasible knowledge . . . the modality "is *realistically* knowable" = knowledge effectively derivable from the actual knowledge

(taking the agent's physical restrictions into account)

The scope of the logical omniscience paradox

The logical omniscience paradox only affects feasible knowledge:

Actual knowledge is not closed under inference steps
$\Rightarrow$ the axiom (K) is not plausible for actual knowledge

Potential knowledge is indeed closed under logical consequence
$\Rightarrow$ no paradox there

Feasible knowledge, however, seems to be:

- closed under single inference steps (the agent *can* make them)
- yet not closed under the consequence relation as a whole
(the agent cannot feasibly know all logical truths)

Logical omniscience as an instance of the Sorites

The problem with feasible knowledge is that the agent

- can always make a next step of inference, but
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Yet it is not the case that for each N,
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= An instance of the sorites paradox for the predicate
$P(n) \equiv$ “the agent can make at least n inference steps”

$\Rightarrow$ Every solution to the sorites paradox generates
a solution to the logical omniscience paradox

Why a degree-theoretical solution?

There have been many objections against degree-theoretical
solutions to the sorites

However, a degree-theoretical solution is particularly suitable to the logical omniscience instances of the sorites, since

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- and can be manipulated by suitable many-valued logics

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However, a degree-theoretical solution is particularly suitable to the logical omniscience instances of the sorites, since

- the degrees have a clear interpretation (in terms of costs of the feasible task)
- and can be manipulated by suitable many-valued logics
- the (implausible) existence of a sharp breaking point in the number of steps the agent can perform is not presupposed

Resource-aware reasoning about knowledge

What limits the agent's ability to infer knowledge is
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Several models of resource-aware reasoning are available
(eg, in dynamic or linear logics)

Resource-aware reasoning about knowledge

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⇒ Resource-aware reasoning about the agent's knowledge needed

Several models of resource-aware reasoning are available
(eg, in dynamic or linear logics)

Fuzzy logics are applicable to resource-aware reasoning, too,
capturing moreover the gradual nature of feasibility
(some tasks are more feasible than others)

Resource-based interpretation of Łukasiewicz logic

Cost assignment: $c\colon Fm_{\mathcal{L}} \to [0,1]$ s.t. $1 - c(x)$ is an evaluation

Intuitively: instead 'p is true' we read 'p is cheap'.

The connectives then represent natural operations with costs:

- $\top$ = any 'costless task' $\quad c(\top) = 0$
- $\bot$ = any 'unaffordable task' $\quad c(\bot) = 1$
- Conjunction = bounded sum of the costs
  $$c(\varphi \mathbin{\&} \psi) = \min\{1, c(\varphi) + c(\psi)\}$$
- Implication = the 'surcharge' for ψ, given the cost of φ
  $$c(\varphi \to \psi) = \max\{0, c(\psi) - c(\varphi)\}$$

Tautologies of the form $A_1 \mathbin{\&} \ldots \mathbin{\&} A_n \to B$ represent
cost-preserving rules of inference
(the cost of B is at most the sum of the costs of A_i)

Combination of costs in basic t-norm logics

Łukasiewicz logic: $\&$ = bounded addition of costs (via a linear function) 0 = the maximal (or unaffordable) cost

Gödel logic: $\&$ = the maximum of costs natural, eg, in space complexity (erase temporary memory)

Product logic: $\&$ = addition of costs (via the logarithm) 0 = the infinite cost

Other t-norm logics: $\&$ = certain other ways of cost combination (eg, additive up to some bound, then maxitive)

Feasibility in t-norm logics

Atomic formulas of t-norm logics can thus be understood as standing under the implicit graded modality *is affordable,* or *is feasible*

The degree of feasibility is inversely proportional (via a suitable normalization function) to the cost of realization (eg, the number of processor cycles)

Logical connectives then express natural operations with costs

Tautologies express degree/cost-preserving rules of inference

Feasible knowledge in fuzzy logic

Given the degrees of (the feasibility of) KA and K$(A \to B)$, the degree of KB (inferred by the agent) needs to make allowance for the (small) cost of performing the inference step of *modus ponens* by the agent (denote it by the atom (MP))

Feasible knowledge in fuzzy logic

Given the degrees of (the feasibility of) $\mathsf{K}A$ and $\mathsf{K}(A \to B)$, the degree of $\mathsf{K}B$ (inferred by the agent) needs to make allowance for the (small) cost of performing the inference step of *modus ponens* by the agent (denote it by the atom (MP))

The plausible axiom of logical rationality for feasible knowledge in fuzzy logics thus becomes:

$$\mathsf{K}A \mathbin{\&} \mathsf{K}(A \to B) \mathbin{\&} (\mathrm{MP}) \to \mathsf{K}B$$

Logical omniscience in fuzzy logics

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(as the cost of performing *modus ponens* is small, but non-zero),
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Since $\&$ is non-idempotent, the degree of $(\mathrm{MP})^n$
(and so the guaranteed degree of KB) decreases, reaching eventually 0
(the resources are limited)

Elimination of the paradox in fuzzy logics

Thus in models over fuzzy logics,

- The feasibility of knowledge decreases with long derivations
  (as it intuitively should)
- The closure of feasible knowledge under logical consequence is only gradual (fading with the increasing difficulty of derivation),
- Yet the agents are still perfectly logically rational
  (able to perform each inference step, at appropriate costs)

$\Rightarrow$ No paradox under suitable fuzzy logics