A Gentle Introduction to Mathematical Fuzzy Logic

5. The growing family of fuzzy logics

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Outline
Adding Baaz delta

Let $L$ be a logic of continuous t-norm, i.e., $L = L(\mathbb{K})$ for some class $\mathbb{K}$ of continuous t-norms.

We add a unary connective $\triangle$ known as Baaz delta or 0–1 projector.

The logic $L_{\triangle}$ is the extension of $L$ by the axioms:

$$
\triangle \varphi \lor \neg \triangle \varphi,
\triangle (\varphi \lor \psi) \rightarrow (\triangle \varphi \lor \triangle \psi),
\triangle \varphi \rightarrow \varphi,
\triangle \varphi \rightarrow \triangle \triangle \varphi,
\triangle (\varphi \rightarrow \psi) \rightarrow (\triangle \varphi \rightarrow \triangle \psi).
$$

and the rule of $\triangle$-necessitation: from $\varphi$ infer $\triangle \varphi$. 
Adding Baaz delta: syntactic properties

Lemma 5.1
\[ \varphi \leftrightarrow \psi \vdash_{L_\Delta} \Delta \varphi \leftrightarrow \Delta \psi \quad \varphi \lor \chi \vdash_{L_\Delta} \Delta \varphi \lor \chi \]

Theorem 5.2
- \[ T, \varphi \vdash_{L_\Delta} \psi \text{ iff } T \vdash_{L_\Delta} \Delta \varphi \to \psi \] (Delta Deduction Theorem)
- \[ \text{If } \Gamma, \varphi \vdash_{L_\Delta} \chi \text{ and } \Gamma, \psi \vdash_{L_\Delta} \chi, \text{ then } \Gamma, \varphi \lor \psi \vdash_{L_\Delta} \chi. \] (Proof by Cases Property)
- \[ \text{If } \Gamma, \varphi \to \psi \vdash_{L_\Delta} \chi \text{ and } \Gamma, \psi \to \varphi \vdash_{L_\Delta} \chi, \text{ then } \Gamma \vdash_{L_\Delta} \chi. \] (Semilinearity Property)
- \[ \text{If } \Gamma \not\vdash_{L_\Delta} \varphi, \text{ then there is a linear } \Gamma' \supseteq \Gamma \text{ such that } \Gamma' \not\vdash_{L_\Delta} \varphi. \] (Linear Extension Property)

Exercise 26
Prove this lemma and theorem.
Adding Baaz delta: semantics and completeness

An algebra \( A = \langle A, \wedge, \vee, \& , \rightarrow, 0, 1, \triangle \rangle \) is an \( L_\triangle \)-algebra if:

1. \( \langle A, \wedge, \vee, \&, \rightarrow, 0, 1 \rangle \) is an \( L \)-algebra,
2. \( \triangle x \vee (\triangle x \rightarrow 0) = \overline{1} \),
3. \( \triangle (x \vee y) \leq (\triangle x \vee \triangle y) \),
4. \( \triangle x \leq \triangle \triangle x \),
5. \( \triangle (x \rightarrow y) \leq \triangle x \rightarrow \triangle y \),
6. \( \triangle \overline{1} = \overline{1} \).

Let \( A \) be an \( L_\triangle \)-chain. Then for every \( x \in A \), \( \triangle x = \begin{cases} \overline{1} & \text{if } x = \overline{1} \\ 0 & \text{otherwise} \end{cases} \).

**Theorem 5.3**

The following are equivalent for every set of formulas \( \Gamma \cup \{ \varphi \} \subseteq Fm_\mathcal{L} ::

1. \( \Gamma \vdash_{L_\triangle} \varphi \)
2. \( \Gamma \models (L_\triangle)_{lin} \varphi \)

If \( \Gamma \) is finite we can add:
3. \( \Gamma \models [0,1]_{*,\triangle} \varphi \) for any \( * \in K \)
Adding an involutive negation

Let $L\sim$ be $L\triangle$ plus a new unary connective $\sim$ and the following axioms:

\begin{align*}
(\sim 1) & \quad \sim\sim \varphi \leftrightarrow \varphi, \\
(\sim 2) & \quad \triangle (\varphi \to \psi) \to (\sim \psi \to \sim \varphi).
\end{align*}

An algebra $A = \langle A, \wedge, \vee, \& , \to, \overline{0}, \overline{1}, \triangle, \sim \rangle$ is a $L\sim$-algebra if:

1. $A = \langle A, \wedge, \vee, \& , \to, \overline{0}, \overline{1}, \triangle \rangle$ is an $L\triangle$-algebra,
2. $x = \sim\sim x$,
3. $\triangle (x \to y) \leq \sim y \to \sim x$,

**Theorem 5.4**

$L\sim$ is complete w.r.t. $L\sim$-chains and w.r.t. standard $L$ chains expanded with $\triangle$ and some involutive negation. Furthermore $G\sim$ is complete w.r.t. $G\sim$-chains and w.r.t. $[0, 1]_{G\triangle}$ expanded with the involutive negation $1 - x$. 
Adding multiplication

We add a binary connective \( \odot \) and define the Product Lukasiewicz logic \( \mathsf{PL} \) by adding the following axioms to \( \mathsf{L} \):

(P1) \( (\chi \odot \varphi) \odot (\chi \odot \psi) \leftrightarrow \chi \odot (\varphi \odot \psi) \) (distributivity)

(P2) \( \varphi \odot (\psi \odot \chi) \leftrightarrow (\varphi \odot \psi) \odot \chi \) (associativity)

(P3) \( \varphi \rightarrow \varphi \odot \top \) (neutral element)

(P4) \( \varphi \odot \psi \rightarrow \varphi \) (monotonicity)

(P5) \( \varphi \odot \psi \rightarrow \psi \odot \varphi \) (commutativity)

\( \mathsf{PL}' \) is the extension of \( \mathsf{PL} \) with a new rule: (ZD) from \( \neg (\varphi \odot \varphi) \) infer \( \neg \varphi \).

Lemma 5.5

\( \varphi \leftrightarrow \psi \vdash_{\mathsf{PL}} \varphi \odot \chi \leftrightarrow \psi \odot \chi \) \( \neg (\varphi \odot \varphi) \lor \chi \vdash_{\mathsf{PL}} \neg \varphi \lor \chi \)

\( \varphi \leftrightarrow \psi \vdash_{\mathsf{PL}'} \varphi \odot \chi \leftrightarrow \psi \odot \chi \)

Theorem 5.6 (Deduction theorem)

\( \Gamma, \varphi \vdash_{\mathsf{PL}} \psi \) iff there is \( n \) such that \( \Gamma \vdash_{\mathsf{PL}} \varphi^n \rightarrow \psi \). does not hold for \( \mathsf{PL}' \).
**PŁ-algebras and PŁ’-algebras:**

A **PŁ-algebra** is a structure $A = \langle A, \oplus, \neg, \odot, 0, 1 \rangle$ such that $\langle A, \oplus, \neg, 0 \rangle$ is an MV-algebra and the following equations hold:

1. $(x \odot y) \oplus (x \odot z) \approx x \odot (y \oplus z)$ (distributivity)
2. $x \odot (y \odot z) \approx (x \odot y) \odot z$ (associativity)
3. $x \odot 1 \approx x$ (neutral element)
4. $x \odot y \approx y \odot x$ (commutativity)

A **PŁ’-algebra** is a PŁ-algebra where the following quasiequation holds:

5. $x \odot x \approx 0 \Rightarrow x \approx 0$ (domain of integrity)

$[0, 1]_{PŁ} = \langle [0, 1], \oplus, \neg, \odot, 0, 1 \rangle$ (where $\odot$ is the usual algebraic product) is both the standard PŁ and PŁ’-algebra

Both logics enjoy the completeness w.r.t. their chains but only PŁ’ enjoys the standard completeness.
Adding truth constants: Rational Pavelka Logic

RPL is the expansion of $Ł$ with a constant $\bar{r}$ for each $r \in [0, 1] \cap \mathbb{Q}$ and axioms: $\bar{r} \oplus \bar{s} \leftrightarrow \min\{1, r + s\}$ and $\neg \bar{r} \leftrightarrow 1 - r$.

We define:

- The truth degree of $\varphi$ over $T$ is $||\varphi||_T = \inf\{e(\varphi) \mid e[T] \subseteq \{1\}\}$
- The provability degree of $\varphi$ over $T$ is $|\varphi|_T = \sup\{r \mid T \vdash_{\text{RPL}} \bar{r} \rightarrow \varphi\}$.

Theorem 5.7 (Pavelka style completeness)

$||\varphi||_T = |\varphi|_T$, for each set of formulas $T \cup \{\varphi\}$. 
ŁΠ and ŁΠ₁² logics: connectives

Logic ŁΠ has the following basic connectives:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{0}$</td>
<td>0</td>
</tr>
<tr>
<td>$\varphi \rightarrow_{L} \psi$</td>
<td>$x \rightarrow_{L} y = \min(1, 1 - x + y)$</td>
</tr>
<tr>
<td>$\varphi \rightarrow_{\Pi} \psi$</td>
<td>$x \rightarrow_{\Pi} y = \min(1, \frac{x}{y})$</td>
</tr>
<tr>
<td>$\varphi \otimes \psi$</td>
<td>$x \otimes y = x \cdot y$</td>
</tr>
</tbody>
</table>

Łukasiewicz implication

product implication

product conjunction

Logic ŁΠ₁² has an additional truth constant $\frac{1}{2}$ with std. semantics $\frac{1}{2}$.

We define the following derived connectives:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neg_{L} \varphi$</td>
<td>$\varphi \rightarrow_{L} \bar{0}$</td>
</tr>
<tr>
<td>$\neg_{\Pi} \varphi$</td>
<td>$\varphi \rightarrow_{\Pi} \bar{0}$</td>
</tr>
<tr>
<td>$\triangle \varphi$</td>
<td>$\neg_{\Pi} \neg_{L} \varphi$</td>
</tr>
<tr>
<td>$\varphi &amp; \psi$</td>
<td>$\neg_{L} (\varphi \rightarrow_{L} \neg_{L} \psi)$</td>
</tr>
<tr>
<td>$\varphi \oplus \psi$</td>
<td>$\neg_{L} \varphi \rightarrow_{L} \psi$</td>
</tr>
<tr>
<td>$\varphi \ominus \psi$</td>
<td>$\varphi &amp; \neg_{L} \psi$</td>
</tr>
<tr>
<td>$\varphi &amp; \psi$</td>
<td>$\varphi &amp; (\varphi \rightarrow_{L} \psi)$</td>
</tr>
<tr>
<td>$\varphi \lor \psi$</td>
<td>$(\varphi \rightarrow_{L} \psi) \rightarrow_{L} \psi$</td>
</tr>
<tr>
<td>$\varphi \rightarrow_{G} \psi$</td>
<td>$\triangle(\varphi \rightarrow_{L} \psi) \lor \psi$</td>
</tr>
</tbody>
</table>

truth constant falsum

Łukasiewicz implication

product implication

product conjunction

$\neg_{L} x = 1 - x$

$\neg_{L} x = \frac{0}{x}$

$\triangle 1 = 1$; $\triangle x = 0$ otherwise

$x \& y = \max(0, x + y - 1)$

$x \oplus y = \min(1, x + y)$

$x \ominus y = \max(0, x - y)$

$x \& y = \min(x, y)$

$x \lor y = \max(x, y)$

$x \rightarrow_{G} y = 1$ if $x \leq y$, otherwise $y$
ŁΠ and ŁΠ½ logics: axiomatic system

Logic ŁΠ is given by the following axioms:

(Ł) Axioms of Łukasiewicz logic,
(Π) Axioms of product logic,
(Ł△) △(φ →Ł ψ) →Ł (φ →Π ψ),
(Π△) △(φ →Π ψ) →Ł (φ →Ł ψ),
(Dist) φ ⊙ (χ ⊗ ψ) ↔Ł (φ ⊙ χ) ⊗ (φ ⊙ ψ).

The deduction rules are modus ponens and △-necessitation
(from φ infer △φ).

The logic ŁΠ½ results from the logic ŁΠ by adding axiom ½ ↔ ¬Ł ½.
Alternative axiomatization (in the language of $L_{\sim}$)

(Π) axioms and deduction rules of $\Pi_{\sim}$,

(A) $(\varphi \to_{L} \psi) \to_{L} ((\psi \to_{L} \chi) \to_{L} (\varphi \to_{L} \chi))$,

where $\varphi \to_{L} \psi$ is defined as $\sim(\varphi \& \sim(\varphi \to \psi))$. 
ŁΠ and ŁΠ₁² logics: algebras

An ŁΠ-algebra is a structure: $A = (A, \oplus, \sim, \to_\Pi, \odot, 0, 1)$

(1) $(A, \oplus, \neg, \odot, 0)$ is a PŁ-algebra

(2) $z \leq (x \to_\Pi y)$ iff $x \odot z \leq y$

OR

(1") $(A, \oplus, \sim, 0)$ is an MV-algebra

(2") $(A, \to_\Pi, \odot, \land, \lor, 0, 1)$ is a Π-algebra

(3") $x \odot (y \ominus z) = (x \odot y) \ominus (x \odot z)$

(4") $\triangle(x \to_L y) \to_L (x \to_\Pi y) = 1$

OR

(1') $(A, \odot, \to_\Pi, \land, \lor, \sim, 0, 1)$ is Π∼-algebra

(2') $(x \to_L y) \leq ((y \to_L z) \to_L (x \to_L z))$

(3') $x \to_L y = \sim (x \odot \sim (x \to_\Pi y))$
Some theorems about $ŁΠ$ and $ŁΠ_2^1$ logics

- Both logics $ŁΠ$ and $ŁΠ_2^1$ have
  - $\triangle$-deduction theorem
  - Proof by Cases Property
  - Semilinearity Property
  - Linear Extension Property
  - general/linaer completeness
  - finite standard completeness

- In $ŁΠ_2^1$ we can define truth constants for each rational from $[0,1]$.

- Let $*$ be a continuous t-norm s.t. $*$ is finite ordinal sum (in the sense of Mostert–Shields Theorem). Then the logic $L(\ast)$ is interpretable in $ŁΠ_2^1$. 

Monoidal t-norm logic MTL

The most prominent example of post-1998 fuzzy logics . . .

We know that left-continuity of $*$ is sufficient

for the residuum (ie, $\Rightarrow$ such that $z \ast x \leq y$ iff $z \leq x \Rightarrow y$ holds)

to be defined as $(x \Rightarrow y) = \sup\{z \mid z \ast x = y\}$

⇒ We can weaken the condition of the continuity of $*$

. . . MTL = the logic of left-continuous t-norms

(turns out to be even more important than HL)

Differences from HL:

• The minimum is no longer definable from $\ast, \Rightarrow, 0$

  ($\land$ has to be added as a primitive connective)

• The HL axiom $(\varphi \land (\varphi \rightarrow \psi)) \rightarrow (\psi \land (\psi \rightarrow \varphi))$ fails in MTL

  (it has to be replaced by three weaker axioms

  ensuring the lattice behavior of $\land$)
Example of left-, not right-continuous t-norm

\[
\ast_{\text{NM}} \text{ nilpotent minimum: } x \ast_{\text{NM}} y = \begin{cases} 
\min\{x, y\} & x + y > 1, \\
0 & \text{otherwise}
\end{cases} 
\] (Fodor 1995)

Its logic \(\text{NM} = \text{MTL+} \land \lnot \varphi \rightarrow \varphi + \lnot (\varphi \land \psi) \lor ((\varphi \land \psi) \rightarrow (\varphi \land \psi))\) (Wang 1997; Esteva & Godo 2001)
Changing the language

We consider a new set of primitive connectives $\mathcal{L}_{MTL} = \{\overline{0}, \& , \land, \rightarrow\}$ and defined now are connectives $\neg$, $\lor$, $\overline{1}$, and $\leftrightarrow$:

\[
\neg \varphi = \varphi \rightarrow \overline{0} \quad \overline{1} = \neg \overline{0} \quad \varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)
\]

\[
\varphi \lor \psi = ((\varphi \rightarrow \psi) \rightarrow \psi) \& ((\psi \rightarrow \varphi) \rightarrow \varphi)
\]

We keep the symbol $Fm_\mathcal{L}$ for the set of formulas.
Recall our axioms

The shared part

(Tr) \((\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))\)  
transitivity

(We)’ \(\varphi \& \psi \rightarrow \varphi\)  
weakening

(Ex)’ \(\varphi \& \psi \rightarrow \psi \& \varphi\)  
exchange

(Res\(_a\)) \((\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))\)  
residuation

(Res\(_b\)) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)\)  
residuation

(Prl)’ \(((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)\)  
prelinearity

(EFQ) \(\overline{0} \rightarrow \varphi\)  
Ex falso quodlibet

In HL we had

(Div) \(\varphi \& (\varphi \rightarrow \psi) \rightarrow \psi \& (\psi \rightarrow \varphi)\)  
divisibility

Recall that in the original systems we also had:

(\^a) \(\varphi \& \psi \rightarrow \varphi\)

(\^b) \(\varphi \& \psi \rightarrow \psi\)

(\^c) \((\chi \rightarrow \varphi) \rightarrow (((\chi \rightarrow \psi) \rightarrow (\chi \rightarrow \varphi \& \psi))\)

(\lor a) \(\varphi \rightarrow \varphi \lor \psi\)

(\lor b) \(\psi \rightarrow \varphi \lor \psi\)

(\lor c) \((\varphi \rightarrow \chi) \rightarrow (((\psi \rightarrow \chi) \rightarrow (\varphi \lor \psi \rightarrow \chi))\)
The logic MTL

Axioms:

(Tr) \((\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))\) \(\text{(MTL1)}\)
(We)\(\'\) \(\varphi \& \psi \to \varphi\) \(\text{(MTL2)}\)
(Ex)\(\'\) \(\varphi \& \psi \to \psi \& \varphi\) \(\text{(MTL3)}\)
(\(\wedge\)a) \(\varphi \wedge \psi \to \varphi\) \(\text{(MTL4a)}\)
(\(\wedge\)b) \(\varphi \wedge \psi \to \psi\) \(\text{(MTL4b)}\)
(\(\wedge\)c) \((\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))\) \(\text{(MTL4c)}\)
(Res\(_a\)) \((\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))\) \(\text{(MTL5a)}\)
(Res\(_b\)) \((\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)\) \(\text{(MTL5b)}\)
(Prl)\(\'\) \(((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)\) \(\text{(MTL6)}\)
(EFQ) \(\overline{0} \to \varphi\) \(\text{(MTL7)}\)

Inference rule: *modus ponens.*

We write \(\Gamma \vdash_{MTL} \varphi\) if there is a proof of \(\varphi\) from \(\Gamma\).

Note: axioms (MTL2) and (MTL3) are redundant, the others are independent.
Theorem 5.8

- \( T, \varphi \vdash_{\text{MTL}} \psi \) iff there is \( n \) such that \( T \vdash_{\text{MTL}} \varphi^n \rightarrow \psi \)
  
  *(Local Deduction Theorem)*

- If \( \Gamma, \varphi \vdash_{\text{MTL}} \chi \) and \( \Gamma, \psi \vdash_{\text{MTL}} \chi \), then \( \Gamma, \varphi \lor \psi \vdash_{\text{MTL}} \chi \).
  
  *(Proof by Cases Property)*

- If \( \Gamma, \varphi \rightarrow \psi \vdash_{\text{MTL}} \chi \) and \( \Gamma, \psi \rightarrow \varphi \vdash_{\text{MTL}} \chi \), then \( \Gamma \vdash_{\text{MTL}} \chi \).
  
  *(Semilinearity Property)*

- If \( \Gamma \not\vdash_{\text{MTL}} \varphi \), then there is a linear \( \Gamma' \supseteq \Gamma \) such that \( \Gamma' \not\vdash_{\text{MTL}} \varphi \).
  
  *(Linear Extension Property)*
Recall the **HL-algebras**

An **HL-algebra** is a structure $B = \langle B, \land, \lor, \& , \rightarrow, \overline{0}, \overline{1} \rangle$ such that:

1. $\langle B, \land, \lor, \overline{0}, \overline{1} \rangle$ is a bounded lattice,
2. $\langle B, \& , \overline{1} \rangle$ is a commutative monoid,
3. $z \leq x \rightarrow y$ iff $x \& z \leq y$, \hspace{1cm} (residuation)
4. $x \& (x \rightarrow y) = x \land y$ \hspace{1cm} (divisibility)
5. $(x \rightarrow y) \lor (y \rightarrow x) = \overline{1}$ \hspace{1cm} (prelinearity)

We say that $B$ is

- **linearly ordered** (or **HL-chain**) if $\leq$ is a total order.
- **standard** $B = [0, 1]$ and $\leq$ is the usual order on reals.
- **G-algebra** if $x \& x = x$
- **MV-algebra** if $\neg\neg x = x$
Introducing: MTL-algebras

An **MTL-algebra** is a structure $B = \langle B, \land, \lor, \& , \rightarrow , \bar{0}, \bar{1} \rangle$ such that:

1. $\langle B, \land, \lor, \bar{0}, \bar{1} \rangle$ is a bounded lattice,
2. $\langle B, \& , \bar{1} \rangle$ is a commutative monoid,
3. $z \leq x \rightarrow y$ iff $x \& z \leq y$, (residuation)
4. $(x \rightarrow y) \lor (y \rightarrow x) = \bar{1}$ (prelinearity)

We say that $B$ is

- **linearly ordered** (or MTL-chain) if $\leq$ is a total order. **$MTL_{\text{lin}}$**
- **standard** $B = [0, 1]$ and $\leq$ is the usual order on reals. **$MTL_{\text{std}}$**
- **IMTL-algebra** if $\neg \neg x = x$. 

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Mathematical Fuzzy Logic
www.cs.cas.cz/cintula/MFL
<table>
<thead>
<tr>
<th>Exercise 27</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Prove that ( \text{HL-algebras} ) are exactly ( \text{MTL-algebras} ) satisfying ( x &amp; (x \rightarrow y) \approx x \wedge y ).</td>
</tr>
<tr>
<td>(b) Prove that ( \text{G-algebras} ) are exactly ( \text{MTL-algebras} ) satisfying ( x &amp; x \approx x ).</td>
</tr>
<tr>
<td>(c) Prove that all ( \text{MV-algebras} ) are ( \text{IMTL-algebras} ) but not vice versa.</td>
</tr>
<tr>
<td>(d) Prove that a structure ( \mathcal{B} = \langle [0, 1], \min, \max, &amp; , \rightarrow, 0, 1 \rangle ) is an ( \text{MTL-algebra} ) IFF ( &amp; ) is a left-continuous t-norm and ( \rightarrow ) its residuum.</td>
</tr>
</tbody>
</table>
General/linear/standard completeness theorem

**Theorem 5.9**

The following are equivalent for every set of formulas \( \Gamma \cup \{ \varphi \} \subseteq Fm_L \):

1. \( \Gamma \vdash_{\text{MTL}} \varphi \)
2. \( \Gamma \models_{\text{MTL}} \varphi \)
3. \( \Gamma \models_{\text{MTL}_{\text{lin}}} \varphi \)
4. \( \Gamma \models_{\text{MTL}_{\text{std}}} \varphi \)

**Exercise 28**

Prove the equivalence of the first three claims.
Outline
Three stages of development of an area of logic

Chagrov (K voprosu ob obratnoi matematike modal’noi logiki,
Online Journal Logical Studies, 2001) distinguishes three stages in the development of a field in logic.
Three stages of development of MFL

First stage: Emerging of the area (since 1965)
- 1965: Zadeh’s fuzzy sets, 1968: ‘fuzzy logic’ (Goguen)
- 1970s: systems of fuzzy ‘logic’ lacking a good metatheory

Three stages of development of MFL

Second stage: development of particular logics and introduction of many new ones (since the 1990s)

- New logics: MTL, SHL, UL, $\Pi \sim$, $\Pi_\lambda$, $\ldots$
- Algebraic semantics, proof theory, complexity
  Kripke-style and game-theoretic semantics, $\ldots$
- First-order, higher-order, and modal fuzzy logics
  Systematic treatment of particular fuzzy logics
Hájek called the logic HL the Basic fuzzy Logic BL

HL was basic in the following two senses:

1. *it could not be made weaker without losing essential properties*
2. *it provided a base for the study of all fuzzy logics.*
Hajek called the logic $\text{HL}$ the Basic fuzzy Logic $\text{BL}$.

$\text{HL}$ was \textit{basic} in the following two senses:

1. \textit{it could not be made weaker without losing essential properties}
2. \textit{it provided a base for the study of all fuzzy logics.}

Because:

- $\text{HL}$ is complete w.r.t. the semantics given by \textit{all} continuous t-norms
- All then known fuzzy logics were expansions of $\text{HL}$. The methods to introduce, algebraize, and study $\text{HL}$ could be modified for all expansions of $\text{HL}$.

\textit{fuzzy logics = expansions of $\text{HL}$}
“Removing legs from the flea”

In the 3rd EUSFLAT (Zittau, Germany, September 2003) Petr Hájek started his lecture *Fleas and fuzzy logic: a survey* with a joke.

"Upon removing the last leg the flea loses sense of hearing."
“Removing legs from the flea”

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A **G-algebra** is a structure \( \mathcal{B} = \langle B, \wedge, \vee, \& , \rightarrow , \bar{0}, \bar{1} \rangle \) such that:

1. \( \langle B, \wedge, \vee, \bar{0}, \bar{1} \rangle \) is a bounded lattice,
2. \( \langle B, \& , \bar{1} \rangle \) is a commutative monoid
3. \( z \leq x \rightarrow y \text{ iff } x \& z \leq y \), \hspace{1cm} \text{(residuation)}
4. \( (x \rightarrow y) \vee (y \rightarrow x) = \bar{1} \), \hspace{1cm} \text{(prelinearity)}
5. \( x \& (x \rightarrow y) = x \wedge y \), \hspace{1cm} \text{(divisibility)}
6. \( x \& y = x \wedge y \)
An **HL-algebra** is a structure $B = \langle B, \wedge, \vee, \&, \rightarrow, 0, 1 \rangle$ such that:

1. $\langle B, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice,
2. $\langle B, \&, 1 \rangle$ is a commutative monoid
3. $z \leq x \rightarrow y$ iff $x \& z \leq y$, (residuation)
4. $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity)
5. $x \& (x \rightarrow y) = x \wedge y$ (divisibility)

**Hájek logic** HL is the logic of continuous t-norms (well designed to jump)
5 Monoidal t-norm logic MTL

An MTL-algebra is a structure $B = \langle B, \land, \lor, \& \text{,} \to, 0, 1 \rangle$ such that:

1. $\langle B, \land, \lor, 0, 1 \rangle$ is a bounded lattice,
2. $\langle B, \& \text{,} 1 \rangle$ is a commutative monoid
3. $z \leq x \to y$ iff $x \& z \leq y$, (residuation)
4. $(x \to y) \lor (y \to x) = 1$ (prelinearity)

MTL is the logic of left-continuous of t-norms

(designed to jump even further)
4 Uninorm logic: the non-integral case

A **UL-algebra** is a structure $B = \langle B, \land, \lor, \& , \rightarrow, 0, 1, \bot, \top \rangle$ such that:

1. $\langle B, \land, \lor, \bot, \top \rangle$ is a bounded lattice,
2. $\langle B, \& , 1 \rangle$ is a commutative monoid
3. $z \leq x \rightarrow y$ iff $x \& z \leq y$, \hspace{1cm} (residuation)
4. $((x \rightarrow y) \land \top) \lor ((y \rightarrow x) \land \top) = \top$ \hspace{1cm} (prelinearity)

UL is the logic of residuated uninorms

(Designed to jump even further in one direction)
3 psMTL’: the non commutative case

A psMTL’-algebra is a structure $B = \langle B, \wedge, \vee, \& , \rightarrow, \sim\sim, 0, 1 \rangle$ such that:

1. $\langle B, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice,
2. $\langle B, \& , 1 \rangle$ is a monoid,
3. $z \leq x \rightarrow y$ iff $x \& z \leq y$ iff $x \leq z \sim\sim y$, (residuation)
4. something ugly (prelinearity)

psMTL’ is the logic of residuated pseudo t-norms
(designed to jump even further in other direction)
psUL: the non commutative and non integral case

A psUL-algebra is a structure $B = \langle B, \wedge, \vee, \& , \rightarrow, \rightsquigarrow, \bar{0}, \bar{1}, \bot, \top \rangle$ s.t.:

1. $\langle B, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,
2. $\langle B, \& , \bar{1} \rangle$ is a monoid,
3. $z \leq x \rightarrow y$ iff $x \& z \leq y$ iff $x \leq z \rightsquigarrow y$, (residuation)
4. something even uglier (prelinearity)

psUL is NOT the logic of residuated pseudo uninorms

(lost all sense of hearing?)
An $\text{SL}^\ell$-algebra is a structure $B = \langle B, \wedge, \vee, \&\!, \rightarrow, \sim\sim\!, 0, 1, \bot, \top \rangle$ s.t.:

1. $\langle B, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,
2. $\langle B, \&\!, 1 \rangle$ is a unital groupoid,
3. $z \leq x \rightarrow y$ iff $x \& z \leq y$ iff $x \leq z \sim\sim\! y$, (residuation)
4. the ugliest thing possible (prelinearity)

$\text{SL}^\ell$ is the logic of residuated unital grupoids on $[0,1]$
Three stages of development of MFL

The second stage is still ongoing; the state of the art is summarized in:

Three stages of development of MFL

Third stage: universal methods (since ∼2006)

- General methods to prove metamathematical properties
- Classification of existing fuzzy logics
- Systematic treatment of classes of fuzzy logics
- Determining the position of fuzzy logics in the logical landscape
Outline
Changing the language

We consider a new set of primitive connectives
\[ \mathcal{L}_{SL} = \{0, 1, \bot, \top, \& , \rightarrow, \sim, \lor, \land\} \], and a defined connective \( \leftrightarrow \):

\[ \varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi) \]

We keep the symbol \( Fm_\mathcal{L} \) for the set of formulas.
The ‘minimal’ algebraic semantics

Definition 5.10
An **SL-algebra** is a structure $B = \langle B, \wedge, \vee, \& \rightarrow, \sim \rightarrow, \overline{0}, \overline{1}, \perp, \top \rangle$ such that:

1. $\langle B, \wedge, \vee, \perp, \top \rangle$ is a bounded lattice,
2. $\langle B, \&, \overline{1} \rangle$ is a unital groupoid,
3. $z \leq x \rightarrow y$ iff $x \& z \leq y$ iff $x \leq z \sim \rightarrow y$, (residuation)
Hilbert-system for SL – axioms

\((\text{Adj}_\&\&\)) \quad \varphi \rightarrow (\psi \rightarrow \varphi \& \varphi)

\((\text{Adj}_\&\&\sim\sim\)) \quad \varphi \rightarrow (\psi \sim \varphi \& \psi)

\((\&\&\wedge)\) \quad (\varphi \& \bar{\iota}) \& (\psi \& \bar{\iota}) \rightarrow \varphi \& \psi

\((\wedge 1)\) \quad \varphi \& \psi \rightarrow \varphi

\((\wedge 2)\) \quad \varphi \& \psi \rightarrow \psi

\((\wedge 3)\) \quad (\chi \rightarrow \varphi) \& (\chi \rightarrow \psi) \rightarrow (\chi \rightarrow \varphi \& \psi)

\((\vee 1)\) \quad \varphi \rightarrow \varphi \vee \psi

\((\vee 2)\) \quad \psi \rightarrow \varphi \vee \psi

\((\vee 3)\) \quad (\varphi \rightarrow \chi) \& (\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)

\((\text{Push})\) \quad \varphi \rightarrow (\bar{\iota} \rightarrow \varphi)

\((\text{Pop})\) \quad (\bar{\iota} \rightarrow \varphi) \rightarrow \varphi

\((\text{Res}')\) \quad \psi \& (\varphi \& (\varphi \rightarrow (\psi \rightarrow \chi))) \rightarrow \chi

\((\text{Res}'_{\sim\sim})\) \quad (\varphi \& (\varphi \rightarrow (\psi \sim \chi))) \& \psi \rightarrow \chi

\((\text{T}')\) \quad \varphi \rightarrow (\varphi \& (\varphi \rightarrow \psi)) \& (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)

\((\text{T}'_{\sim\sim})\) \quad (\varphi \sim (((\varphi \sim \psi) \& \varphi) \& (\psi \rightarrow \chi)) \rightarrow (\varphi \sim \chi)
Hilbert-system for SL – rules

\[(\text{MP}) \quad \varphi, \varphi \rightarrow \psi \vdash \psi\]

\[(\text{Adj}u) \quad \varphi \vdash \varphi \land \overline{1}\]

\[(\alpha) \quad \varphi \vdash \delta \land \varepsilon \rightarrow \delta \land (\varepsilon \land \varphi)\]

\[(\alpha') \quad \varphi \vdash \delta \land \varepsilon \rightarrow (\delta \land \varphi) \land \varepsilon\]

\[(\beta) \quad \varphi \vdash \delta \rightarrow (\varepsilon \rightarrow (\varepsilon \land \delta) \land \varphi)\]

\[(\beta') \quad \varphi \vdash \delta \rightarrow (\varepsilon \Rightarrow (\delta \land \varepsilon) \land \varphi)\]
Contribution

**Convention**

A logic is a provability relation on formulas in a language $\mathcal{L} \supseteq \mathcal{L}_{SL}$ s.t.

- it is axiomatized by adding axioms $Ax$ and finitary rules ($R$) to the logic $SL$
- for each $n$-ary connective $c \in \mathcal{L} \setminus \mathcal{L}_{SL}$, $\mathcal{L}$-formulas $\varphi, \psi, \chi_1, \ldots, \chi_n$, and each $i \leq n$ the following holds:

$$\varphi \leftrightarrow \psi \vdash_{\mathcal{L}} c(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n) \leftrightarrow c(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n)$$
A **logic** is a provability relation on formulas in a language $\mathcal{L} \supseteq \mathcal{L}_{SL}$ s.t.

- it is axiomatized by adding axioms $Ax$ and **finitary** rules $(R)$ to the logic $SL$
- for each $n$-ary connective $c \in \mathcal{L} \setminus \mathcal{L}_{SL}$, $\mathcal{L}$-formulas $\varphi, \psi, \chi_1, \ldots, \chi_n$, and each $i \leq n$ the following holds:

$$\varphi \leftrightarrow \psi \vdash_{L} c(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n) \iff c(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n)$$

Let us fix a logic $L$ in language $\mathcal{L}$ which is the expansion of $SL$ by axioms $Ax$ and rules $R$. 
Definition 5.11

Let $B$ be an $\mathcal{L}$-algebra. A $B$-evaluation is a mapping $e: Fm_\mathcal{L} \rightarrow B$ s.t.

- $e(\ast) = \ast^B$ for truth constant $\ast$
- $e(\circ(\varphi_1, \ldots, \varphi_n)) = \circ^B(e(\varphi_1), \ldots, e(\varphi_n))$ for each $n$-ary $\circ \in \mathcal{L}$

Definition 5.12

An $\mathcal{L}$-algebra $A$ is an $\mathcal{L}$-algebra, $A \in \mathbb{L}$, if

- its reduct $A_{SL} = \langle A, \wedge, \vee, \&; \rightarrow, \leadsto, \bar{0}, \bar{1}, \bot, \top \rangle$ is an $\mathcal{SL}$-algebra,
- for each $\varphi \in Ax$, $A$ satisfies the identity $\varphi \wedge \bar{1} = \bar{1}$,
- for each $\langle \{\psi_1, \ldots, \psi_n\}, \varphi \rangle \in R$, $A$ satisfies the quasi-identity

$$\text{If } \psi_1 \wedge \bar{1} = \bar{1} \text{ and } \ldots \text{ and } \psi_n \wedge \bar{1} = \bar{1} \text{ then } \varphi \wedge \bar{1} = \bar{1}$$

$A$ is a linearly ordered (or $L$-chain), $A \in \mathbb{L}_{lin}$, if its lattice order is total.
Logical consequence w.r.t. a class of algebras

**Definition 5.13**

A formula $\varphi$ is a **logical consequence** of set of formulas $\Gamma$ w.r.t. a class $K$ of $L$-algebras, $\Gamma \models_{K} \varphi$, if for every $B \in K$ and every $B$-evaluation $e$:

if $e(\gamma) \geq \bar{1}$ for every $\gamma \in \Gamma$, then $e(\varphi) \geq \bar{1}$.

**Observation**

1. An $L$-algebra $A$ is an $L$-algebra iff
   - its reduct $A_{SL} = \langle A, \land, \lor, \&, \rightarrow, \sim, \bar{0}, \bar{1}, \bot, \top \rangle$ is an $SL$-algebra,
   - if $\Gamma \vdash_{L} \varphi$, then $\Gamma \models_{A} \varphi$.

2. $L$ is the largest class $K$ of $L$-algebras such that $\vdash_{L} \subseteq \models_{K}$.
Theorem 5.14 (Completeness theorem)

For every set of formulas $\Gamma$ and every formula $\varphi$ we have:

$$\Gamma \vdash_L \varphi \text{ if, and only if, } \Gamma \models_L \varphi.$$ 

Each $L$ is an algebraizable logic and $L$ is its equivalent algebraic semantics with translations:

$$E(p, q) = \{p \leftrightarrow q\} \text{ and } T(p) = \{p \land \top \approx \top\}.$$ 

Indeed, all we have to do is to prove:

$$p \vdash p \land \top \leftrightarrow \top \quad \text{and} \quad p \land \top \leftrightarrow \top \vdash p.$$
Core semilinear logics

Definition 5.15

A logic $\mathcal{L}$ is **core semilinear logic** whenever it is complete w.r.t. linearly ordered $\mathcal{L}$-algebras, i.e.,

$$T \models_{\mathcal{L}} \varphi \iff T \models_{\mathcal{L}_{\text{lin}}} \varphi$$
Theorem 5.16 (Syntactic characterization)

Let $\mathcal{L}$ be axiomatized by axioms $\text{Ax}$ and rules $\mathcal{R}$. TFAE:

1. $\mathcal{L}$ is a core semilinear logic
2. $\vdash_{\mathcal{L}} (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)$ and if $\langle \Gamma, \varphi \rangle \in \mathcal{R}$, then $\Gamma \lor \chi \vdash_{\mathcal{L}} \varphi \lor \chi$ for every $\chi$
3. $\vdash_{\mathcal{L}} (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)$ and if $\Gamma \vdash_{\mathcal{L}} \varphi$, then $\Gamma \lor \chi \vdash_{\mathcal{L}} \varphi \lor \chi$ for every $\chi$
4. $\vdash_{\mathcal{L}} (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)$ and for every set of formulas $\Gamma \cup \{\varphi, \psi, \chi\}$:
   
   $\Gamma, \varphi \vdash_{\mathcal{L}} \chi$ and $\Gamma, \psi \vdash_{\mathcal{L}} \chi$ imply $\Gamma, \varphi \lor \psi \vdash_{\mathcal{L}} \chi$.
5. For every set of formulas $\Gamma \cup \{\varphi, \psi, \chi\}$:
   
   $\Gamma, \varphi \rightarrow \psi \vdash_{\mathcal{L}} \chi$ and $\Gamma, \psi \rightarrow \varphi \vdash_{\mathcal{L}} \chi$ imply $\Gamma \vdash_{\mathcal{L}} \chi$.
6. If $\Gamma \not\vdash_{\mathcal{L}} \varphi$ then there is a linear theory $\Gamma' \supseteq \Gamma$ s.t. $\Gamma \not\vdash_{\mathcal{L}} \varphi$.  

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Core semilinear logics — semantic characterization

Theorem 5.17 (Semantic characterization)

Let $L$ be a logic. TFAE:

1. $L$ is a core semilinear logic
2. finitely relatively subdirectly irreducible $L$-algebras are exactly the $L$-chains
3. relatively subdirectly irreducible $L$-algebras are linearly ordered
Weakest semilinear extension

**Definition 5.18**

By $L^\ell$ we denote the least core semilinear logic extending $L$.

**Exercise 29**

(a) Prove that the previous definition is sound (show that the class of core semilinear logics is closed under arbitrary intersections).

(b) Prove that $\mathbb{I}_{\text{lin}}^\ell = \mathbb{I}_{\text{lin}}$.

**Theorem 5.19**

*If $L$ is axiomatized by rules $R$, then $L^\ell$ is axiomatized by adding axiom $(\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)$ and rules: $\langle \Gamma \lor \chi, \varphi \lor \chi \rangle$ for each $\langle \Gamma, \varphi \rangle \in R$.*

In many cases we can prove that $L^\ell$ is an *axiomatic* extension of $L$. 
Hilbert-system for $\text{SL}^\ell$ – axioms

To the axioms of $\text{SL}$ we add

$$(\text{PRL}_\alpha) \quad (\phi \to \psi) \land \overline{1} \lor (\delta \land \varepsilon \to \delta \land (\varepsilon \land (\psi \to \phi) \land \overline{1}))$$

$$(\text{PRL}_{\alpha'}') \quad (\phi \to \psi) \land \overline{1} \lor (\delta \land \varepsilon \to (\delta \land (\psi \to \phi) \land \overline{1}) \land \varepsilon)$$

$$(\text{PRL}_\beta) \quad (\phi \to \psi) \land \overline{1} \lor (\delta \to (\varepsilon \to (\varepsilon \land \delta) \land (\psi \to \phi) \land \overline{1})))$$

$$(\text{PRL}_{\beta'}) \quad (\phi \to \psi) \land \overline{1} \lor (\delta \to (\varepsilon \rightarrow (\delta \land \varepsilon) \land (\psi \to \phi) \land \overline{1})))$$
A linear/standard completeness theorem of $\text{SL}^\ell$

Let us by $\text{SL}^\ell_{\text{std}}$ denote the class of $\text{SL}$-algebras with the domain $[0, 1]$ and the usual order.

**Theorem 5.20 (Standard completeness theorem of $\text{SL}^\ell$)**

The following are equivalent for every set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$:

1. $\Gamma \vdash_{\text{SL}^\ell} \varphi$
2. $\Gamma \models_{\text{SL}^\ell} \varphi$
3. $\Gamma \models_{\text{SL}^\ell_{\text{lin}}} \varphi$
4. $\Gamma \models_{\text{SL}^\ell_{\text{std}}} \varphi$
Is $\text{SL}^\ell$ the new basic fuzzy logic?

We need to show that it is basic in the following two senses:

1. *it cannot be made weaker without losing essential properties and*
2. *it provides a base for the study of all fuzzy logics.*
Is $SL^\ell$ the new basic fuzzy logic?

We need to show that it is basic in the following two senses:
1. it cannot be made weaker without losing essential properties and
2. it provides a base for the study of all fuzzy logics.

And indeed we have seen that
1. $SL^\ell$ is complete w.r.t. a hardly-to-be-made-weaker semantics over real numbers.
Is $\text{SL}^\ell$ the new basic fuzzy logic?

We need to show that it is basic in the following two senses:

1. \textit{it cannot be made weaker without losing essential properties and}
2. \textit{it provides a base for the study of all fuzzy logics.}

And indeed we have seen that

1. $\text{SL}^\ell$ is complete w.r.t. a hardly-to-be-made-weaker semantics over real numbers.
2. Almost all reasonable fuzzy logics expands $\text{SL}^\ell$. The methods to introduce, algebraize, and study $\text{SL}^\ell$ could be utilized for any such logic. We can develop a uniform mathematical theory for MFL based on $\text{SL}^\ell$. 
Is $\mathbb{SL}^\ell$ the new basic fuzzy logic?

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And indeed we have seen that

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fuzzy logics = core semilinear logics