Fuzzy Logic

7. Further lines of research and open problems

Petr Cintula

Institute of Computer Science,
Czech Academy of Sciences, Prague, Czech Republic

www.cs.cas.cz/cintula/mfl-tuw

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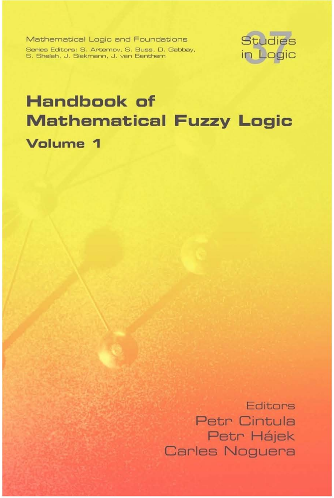

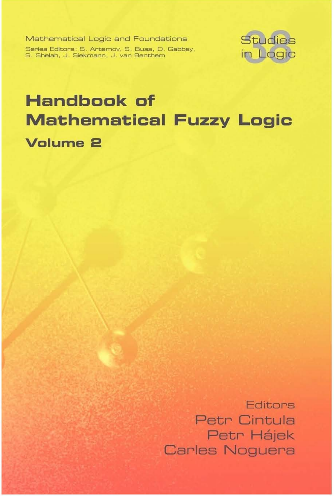

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Volume 1

Volume 2

An even more general approach

Why should we stop at SL^{ℓ}?

fuzzy logics = logics of chains $\Rightarrow$ general theory of semilinear logics

Necessary ingredients:

- An order relation on all algebras (so, in particular, we have chains)
- An implication $\to$ s.t. for every $a, b \in A$, $a \leq b$ iff $a \to^{\boldsymbol{b}}$ is true in $\boldsymbol{A}$
- The implication gives a congruence w.r.t. all connectives (so, we can do the Lindenbaum–Tarski construction)

Using Abstract Algebraic Logic we can develop a theory of weakly implicative semilinear logics.

Calculus for $\mathrm{FL_{ew}}$: structural rules

A sequent is a pair $\Gamma \Rightarrow \Delta$ where Γ is a multiset of formulas and Δ is a formula or the empty set.

The calculus has the following axiom and the structural rules:

(ID) $$\frac{}{\varphi \Rightarrow \varphi}$$

(Cut) $$\frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Delta \Rightarrow \chi}{\Gamma, \Delta \Rightarrow \chi}$$

(W-L) $$\frac{\Gamma \Rightarrow \chi}{\varphi, \Gamma \Rightarrow \chi}$$

(W-R) $$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}$$

Calculus for $\mathrm{FL_{ew}}$: operational rules

(∧-L) $\dfrac{\varphi, \Gamma \Rightarrow \chi}{\varphi \wedge \psi, \Gamma \Rightarrow \chi}$, ditto ψ

(∧-R) $\dfrac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}$

(&-L) $\dfrac{\varphi, \psi, \Gamma \Rightarrow \chi}{\varphi \,\&\, \psi, \Gamma \Rightarrow \chi}$

(&-R) $\dfrac{\Gamma \Rightarrow \varphi \qquad \Delta \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \varphi \,\&\, \psi}$

(∨-L) $\dfrac{\varphi, \Gamma \Rightarrow \chi \qquad \psi, \Gamma \Rightarrow \chi}{\varphi \vee \psi, \Gamma \Rightarrow \chi}$

(∨-R) $\dfrac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}$, ditto ψ

(→-L) $\dfrac{\Gamma \Rightarrow \varphi \qquad \psi, \Delta \Rightarrow \chi}{\varphi \to \psi, \Gamma, \Delta \Rightarrow \chi}$

(→-R) $\dfrac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}$

(¬-L) $\dfrac{\Gamma \Rightarrow \varphi}{\neg\varphi, \Gamma \Rightarrow}$

(¬-R) $\dfrac{\varphi, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\varphi}$

From sequents to hypersequents

A hypersequent is a multiset of sequents. We add hypersequent context $\mathcal{G}$ to all rules:

(ID) $\dfrac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}$ (∨-R) $\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \vee \psi}$, ditto ψ

What we need is Avron's communication rule

(COM) $\dfrac{\mathcal{G} \mid \Gamma_1, \Pi_1 \Rightarrow \chi_1 \qquad \mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow \chi_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \chi_1 \mid \Pi_1, \Pi_2 \Rightarrow \chi_2}$

Characterizations of completeness properties

Let L be core semilinear logic and $\mathbb{K}$ a class of L-chains.

Theorem 7.1 (Characterization of strong $\mathbb{K}$-completeness)

1. *For each $T \cup \{\varphi\}$ holds: $T \vdash_{\mathrm{L}} \varphi$ iff $T \models_{\mathbb{K}} \varphi$.*
2. $\mathbb{L} = \mathbf{ISP}_{\sigma\text{-}f}(\mathbb{K})$.
3. *Each countable L-chain is embeddable into some member of $\mathbb{K}$.*

Theorem 7.2 (Characterization of finite strong $\mathbb{K}$-completeness)

1. *For each finite $T \cup \{\varphi\}$ holds: $T \vdash_{\mathrm{L}} \varphi$ iff $T \models_{\mathbb{K}} \varphi$*
2. $\mathbb{L} = \mathbf{Q}(\mathbb{K})$, *i.e., $\mathbb{K}$ generates $\mathbb{L}$ as a quasivariety.*
3. *Each countable L-chain is embeddable into some ultrapower of $\mathbb{K}$.*
4. *Each finite subset of an L-chain is partially embeddable into an element of $\mathbb{K}$.*

Completeness properties

Let L be a core semilinear logic and $\mathbb{K}$ a class of L-chains.

Definition 7.3

- L has the $\mathrm{S\mathbb{K}C}$ if:

$$\text{for every } \Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}},\ \Gamma \vdash_{\mathrm{L}} \varphi \text{ iff } \Gamma \models_{\mathbb{K}} \varphi$$

- L has the $\mathrm{FS\mathbb{K}C}$ if:

$$\text{for every finite } \Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}},\ \Gamma \vdash_{\mathrm{L}} \varphi \text{ iff } \Gamma \models_{\mathbb{K}} \varphi$$

Distinguished semantics

Typical instances: $\mathbb{K} \in \{\mathcal{R}, \mathcal{Q}, \mathcal{F}\}$ (real, rational, finite-chain semantics).

Theorem 7.4 (Strong finite-chain completeness)

1. L *enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$,
2. *all* L*-chains are finite,*
3. *there exists* $n \in \mathrm{N}$ *such each* L*-chain has at most* n *elements,*
4. *there exists* $n \in \mathrm{N}$ *such that* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

Theorem 7.5 (Relation of Rational and Real completeness)

1. L *has the* $\mathrm{FS}\mathcal{Q}\mathrm{C}$ *iff it has the* $\mathrm{S}\mathcal{Q}\mathrm{C}$.
2. *If* L *has the* $\mathcal{R}\mathrm{C}$, *then it has the* $\mathcal{Q}\mathrm{C}$.
3. *If* L *has the* $\mathrm{FS}\mathcal{R}\mathrm{C}$, *then it has the* $\mathrm{S}\mathcal{Q}\mathrm{C}$.

Known results and open problems

Logic	$\mathcal{SRC}$	$\mathrm{FS}\mathcal{RC}$	$\mathrm{S}\mathcal{QC}$	$\mathrm{FS}\mathcal{QC}$	$\mathrm{FS}\mathcal{FC}$
FL^{ℓ}	No	No	No	No	No
$\mathrm{FL}^{\ell}_{\mathrm{c}}$	No	No	No	No	?
$\mathrm{FL}^{\ell}_{\mathrm{e}} = \mathrm{UL}$	Yes	Yes	Yes	Yes	No
$\mathrm{FL}^{\ell}_{\mathrm{w}} = \mathrm{psMTL}^{r}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{FL}^{\ell}_{\mathrm{ew}} = \mathrm{MTL}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{FL}^{\ell}_{\mathrm{ec}}$	?	?	?	?	?
$\mathrm{FL}^{\ell}_{\mathrm{wc}} = \mathrm{G}$	Yes	Yes	Yes	Yes	Yes

Problem 7.6

Solve the missing cases.

Known results and open problems

Logic	$\mathcal{SRC}$	$\mathrm{F}\mathcal{SRC}$	$\mathcal{SQC}$	$\mathrm{F}\mathcal{SQC}$	$\mathrm{FS}\mathcal{FC}$
InFL^{ℓ}	No	No	No	No	No
$\mathrm{InFL}^{\ell}_{\mathrm{c}}$	No	No	No	No	?
$\mathrm{InFL}^{\ell}_{\mathrm{e}} = \mathrm{IUL}$	?	?	?	?	No
$\mathrm{InFL}^{\ell}_{\mathrm{w}}$	Yes	Yes	Yes	Yes	?
$\mathrm{InFL}^{\ell}_{\mathrm{ew}} = \mathrm{IMTL}$	Yes	Yes	Yes	Yes	Yes
$\mathrm{InFL}^{\ell}_{\mathrm{ec}}$	?	?	?	?	?
$\mathrm{InFL}^{\ell}_{\mathrm{wc}} = \mathrm{CL}$	No	No	No	No	Yes

Problem 7.7

Solve the missing cases.

Known results in non-associative logics

Logic	$S\mathcal{R}C$	$FS\mathcal{R}C$	$S\mathcal{Q}C$	$FS\mathcal{Q}C$	$FS\mathcal{F}C$
SL^{ℓ}	Yes	Yes	Yes	Yes	Yes
SL^{ℓ}_{c}	Yes	Yes	Yes	Yes	Yes
SL^{ℓ}_{e}	Yes	Yes	Yes	Yes	Yes
SL^{ℓ}_{w}	Yes	Yes	Yes	Yes	Yes
SL^{ℓ}_{ew}	Yes	Yes	Yes	Yes	Yes
SL^{ℓ}_{ec}	Yes	Yes	Yes	Yes	Yes
$SL^{\ell}_{wc} = G$	Yes	Yes	Yes	Yes	Yes

More interesting questions (no one addressed yet)

Logic	$\mathrm{S}\mathcal{R}\mathrm{C}$	$\mathrm{FS}\mathcal{R}\mathrm{C}$	$\mathrm{S}\mathcal{Q}\mathrm{C}$	$\mathrm{FS}\mathcal{Q}\mathrm{C}$	$\mathrm{FS}\mathcal{F}\mathrm{C}$
InSL^{ℓ}	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{c}}$	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{e}}$	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{w}}$	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{ew}}$	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{ec}}$	?	?	?	?	?
$\mathrm{InSL}^{\ell}_{\mathrm{wc}}$ = CL	No	No	No	No	Yes

An extensive research field . . .

- based on the structural description of HL-chains
- classification and axiomatization of subvarieties
- amalgamation, interpolation, and Beth properties
- completions theory
- etc.

We have heard a lot about it already . . .

If you want to know more, read:

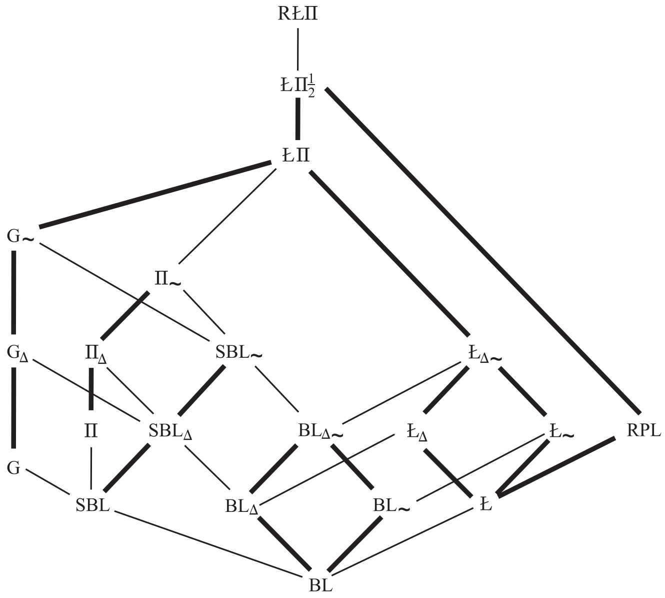
D. Mundici. Advanced Łukasiewicz calculus and MV-algebras.
Trends in Logic, Vol. 35 Springer, New York, 2011.

We have heard a lot about it already . . .

If you want to know more, read:

anything from Vienna school: M. Baaz, N. Preining, C. Fermüller, R. Zach, etc.

A plethora of results not only about . . .



Basic notions

We fix a logic L which is standard complete w.r.t. $[0,1]_{\mathrm{L}}$.

Definition 7.8

Function $f\colon [0,1]^n \to [0,1]$ is *represented* by formula φ of logic L if $e(\varphi) = f(e(v_1), e(v_2), \dots, e(v_m))$ for each $[0,1]_{\mathrm{L}}$-evaluation e.

Definition 7.9

Functional representation of logic L is a class of functions from any power of $[0,1]$ into $[0,1]$ s.t. each $C \in \mathcal{C}$ is represented by some formula φ and vice-versa (i.e., for each φ there is $C \in \mathcal{C}$ represented by φ).

An overview

Łukasiewicz logic:

Operations: truncated sum $\min\{1, x+y\}$ and involutive negation $1-x$
Functions: continuous piece-wise linear functions with integer coeff.

$$f(x_1,\ldots,x_n) = a_1x_1 + \ldots + a_nx_n + a_0\,, \quad a_i \in \mathbf{Z}$$

Ext.	**Added operations**	**Functions**
P	multiplication	polynomial
RP	rational constants	rational shift ($a_0 \in \mathbf{Q}$)
$\triangle$	$\triangle(x) = 1 \quad \text{if } x = 1$ $\triangle(x) = 0 \quad \text{if } x < 1$	non-continuity
δ	dividing by integers	rational coefficients ($a_i \in \mathbf{Q}$)
ŁΠ	fractions	fractions of functions

Some known results

Definition 7.10

A subset S of $[0,1]^n$ is Q-*semialgebraic* if it is a Boolean combination of sets of the form

$$\{\langle x_1,\dots,x_n\rangle \in [0,1]^n \mid P(\langle x_1,\dots,x_n\rangle) > 0\}$$

for polynomials P with integer coefficients. If all of the polynomials are linear, then S is *linear* Q-*semialgebraic.*

Logic	Contin.	Domains	Pieces
Ł	yes	linear	linear functions with integer coefficients
$Ł_\triangle$	no	linear	linear functions with integer coefficients
RPL	yes	linear	linear integer coefficients and a rational shift
δŁ	yes	linear	linear rational coefficients
PŁ$'$	yes	?	??? Pierce-Birkhoff conjecture ???
PŁ$'_\triangle$	no	all	polynomials with integer coefficients
ŁΠ	no	all	fractions of polynomials with integer coeff.
Ł$\Pi\frac{1}{2}$	no	all	as above plus $f[\{0,1\}^n] \subseteq \{0,1\}$

Known results (see Handbook ch. X)

Logic L	THM(L)	CONS(L)	expansion by rational constants
HL	**coNP**-c.	**coNP**-c.	–
Ł	**coNP**-c.	**coNP**-c.	**coNP**-c.
L ⊃Ł	**coNP**-c.	**coNP**-c.	–
G	**coNP**-c.	**coNP**-c.	**coNP**-c.
Π	**coNP**-c.	**coNP**-c.	∈ **PSPACE**
L(∗) ⊃ HL	**coNP**-c.	**coNP**-c.	–
$Ł''\frac{1}{2}$	∈ **PSPACE**	∈ **PSPACE**	–
MTL	decidable	decidable	–
IMTL	decidable	decidable	–
ΠMTL	decidable	decidable	–
NM	**coNP**-c.	**coNP**-c.	**coNP**-c.
WNM	**coNP**-c.	**coNP**-c.	–

Problem 7.11

Determine the precise complexity in all cases.

Known results (see Handbook ch. XI)

Logic	$stTAUT_1$	$stSAT_1$	$stTAUT_{pos}$	$stSAT_{pos}$
(I)MTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
WCMTL∀	Σ_1-hard	Π_1-hard	Σ_1-hard	Π_1-hard
ΠMTL∀	Σ_1-hard	Π_1-hard	Σ_1-hard	Π_1-hard
(S)HL∀	Non-arithm.	Non-arithm.	Non-arithm.	Non-arithm.
Ł∀	Π_2-complete	Π_1-complete	Σ_1-complete	Σ_2-complete
Π∀	Non-arithm.	Non-arithm.	Non-arithm.	Non-arithm.
G∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
C_nMTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
C_nIMTL∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
WNM∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete
NM∀	Σ_1-complete	Π_1-complete	Σ_1-complete	Π_1-complete

Problem 7.12

Determine the precise complexity in all cases.

Volume III of the Handbook

- XII Algebraic Semantics: Structure of Chains (Vetterlein)
- XIII Semantic Games for Fuzzy Logics (Fermüller)
- XIV Ulam–Rényi Games Based Semantics for Fuzzy Logics (Cicalese, Montagna)
- XV Fuzzy Logics with Evaluated Syntax (Novák)
- XVI Fuzzy Description Logics (Bobillo, Cerami, Esteva, García-Cerdaña, Peñaloza, Straccia)
- XVII States of MV-algebras (Flaminio, Kroupa)
- XVIII Fuzzy Logics in Theories of Vagueness (Smith)

(edited by Cintula, Fermüller, and Noguera)

Those would deserve a Handbook chapter in some of the future volumes, but are not ready yet . . .

- Model Theory of Fuzzy Logics
- Model Theory in Fuzzy Logics
- Fuzzy Modal Logics
- Duality Theory
- Fragments of Fuzzy Logics
- Higher-Order Fuzzy logics
- Fuzzy Set Theories
- Fuzzy Arithmetics

Conclusions

- MFL is a well-developed field, with a genuine agenda of Mathematical logic: axiomatization, completeness, proof theory, computational complexity, model theory, etc.
- All these areas are active, mathematically deep and pose challenging open problems.
- The objects studied by MFL are semilinear logics, i.e. logics of chains.
- Graduality in the semantics (linearly ordered truth-values) is a flexible tool amenable for many interesting applications.
- MFL, its extensions and applications has still a long way to go, the best is yet to come, and so there are plenty of topics for potential Ph.D. theses.

Bibliography

- Libor Běhounek, Ondrej Majer. A semantics for counterfactuals based on fuzzy logic. In M. Peliš, V. Punčochař (eds.), The Logica Yearbook 2010, pp. 25–41, College Publications, 2011.
- Libor Běhounek. Fuzzy logics interpreted as logics of resources. In M. Peliš (ed.): The Logica Yearbook 2008, pp. 9–21, College Publications 2009.
- Libor Běhounek, Petr Cintula. From fuzzy logic to fuzzy mathematics: a methodological manifesto. *Fuzzy Sets and Systems* 157(5): 642–646, 2006.
- Roberto Cignoli, Itala M.L. D'Ottaviano and Daniele Mundici, *Algebraic Foundations of Many-Valued Reasoning*, Trends in Logic, vol. 7, Kluwer, Dordrecht, 1999.

Bibliography

- Petr Cintula, Francesc Esteva, Joan Gispert, Lluís Godo, Franco Montagna and Carles Noguera. Distinguished Algebraic Semantics For T-Norm Based Fuzzy Logics: Methods and Algebraic Equivalencies, *Annals of Pure and Applied Logic* 160(1): 53–81, 2009.
- Petr Cintula, Petr Hájek, Carles Noguera (eds). Handbook of Mathematical Fuzzy Logic. Studies in Logic, Mathematical Logic and Foundations, vol. 37 and 38, London, College Publications, 2011.
- Petr Cintula, Rostislav Horčík and Carles Noguera. Non-Associative Substructural Logics and their Semilinear Extensions: Axiomatization and Completeness Properties, *The Review of Symbolic Logic* 6(3):394–423, 2013.
- Michael Dummett. A propositional calculus with denumerable matrix. Journal of Symbolic Logic, 24:97–106, 1959.

Bibliography

- Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Studies in Logic and the Foundations of Mathematics, vol. 151, Amsterdam, Elsevier, 2007.
- Joseph Amadee Goguen. The logic of inexact concepts. Synthese, 19:325–373, 1969.
- Siegfried Gottwald. A Treatise on Many-Valued Logics, volume 9 of Studies in Logic and Computation. Research Studies Press, Baldock, 2001.
- Petr Hájek. Metamathematics of Fuzzy Logic, volume 4 of Trends in Logic. Kluwer, Dordrecht, 1998.
- Jan Łukasiewicz and Alfred Tarski. Untersuchungen über den Aussagenkalkül. Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, 23:30–50, 1930.

Bibliography

- George Metcalfe, Nicola Olivetti and Dov M. Gabbay, *Proof Theory for Fuzzy Logics*, Applied Logic Series, vol. 36, Springer, 2008.
- Daniele Mundici, *Advanced Łukasiewicz Calculus and MV-Algebras*, Springer, New York, 2011.
- Vilém Novák, Irina Perfilieva, and Jiří Močkoř. Mathematical Principles of Fuzzy Logic. Kluwer, Dordrecht, 2000.
- Lotfi A. Zadeh. Fuzzy sets. Information and Control, 8:338–353, 1965.