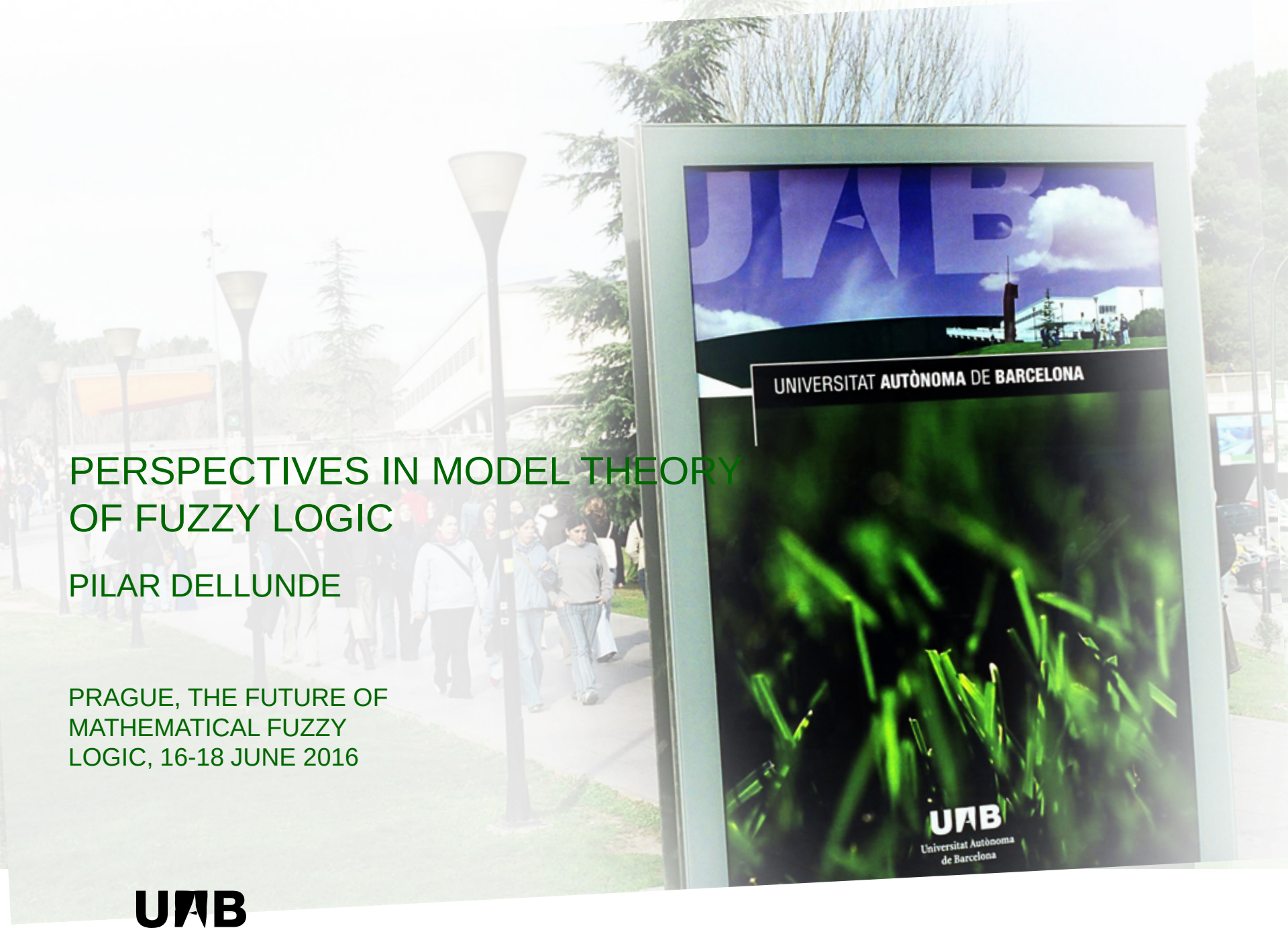


PERSPECTIVES IN MODEL THEORY OF FUZZY LOGIC

PILAR DELLUNDE

PRAGUE, THE FUTURE OF MATHEMATICAL FUZZY LOGIC, 16-18 JUNE 2016

UAB
Universitat Autònoma de Barcelona

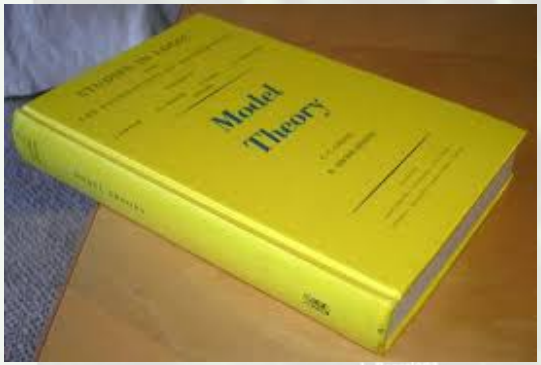
LESSONS LEARNT FROM THE RECENT HISTORY OF CLASSICAL MODEL THEORY.

7 PROPOSALS IN MODEL THEORY OF MFL.

WHAT IS MODEL THEORY?

Model theory is the branch of mathematical logic that studies the construction and classification of structures.

Chang & Keisler (1990):
universal algebra + logic = model theory.



THE BEGINNINGS

1915: Über Möglichkeiten im Relativkalkül, by L. Löwenheim.

1915 to 1935: **decision methods** and finding complete sets of axioms. Examples: theories of (Q,<), (Z,+,<), the field of complex numbers and the ordered field of real numbers (using the method of elimination of quantifiers) (Langford, Presburger, Tarski...)

Definitions of the notions of **truth and definability** in a structure, elementary equivalence and elementary extensions (Tarski)

In the fifties and sixties: ultraproducts, back-and-forth methods, amalgamation properties and non-standard analysis (Los, Fraïssé, Robinson...)

THE BEGINNINGS

- Boundaries of the subject were stablished.
- Study of one structure at a time.
- Feeling of exhaustion started pervading the whole theory:

“Un temps d’arrêt, comme si la machinerie, prête à tourner, ne savait quelle direction prendre.”

Daniel Lascar

SECOND BIRTH OF MODEL THEORY

(Morley, 1965) For every (countable) theory, categoricity at one uncountable cardinal implies categoricity at every other uncountable cardinal.

Topological spaces of types and a rank on types and formulas (Morley rank).

Omega-stability: over a countable set there are at most countably many complete types.

Study of the countable models of uncountably categorical theories (Baldwin, Lachlan, Morley...)

SECOND BIRTH OF MODEL THEORY

1973: Model Theory by C.C. Chang and H. J. Keisler. It has been the book of reference for decades.

In the seventies several branches of Model Theory started having some independent life, like Infinitary Logics, admissible structures, Topological Model Theory, Recursive Model Theory, Abstract Model Theory and Model Theory of Peano Arithmetic...

Dependence on hypotheses and methods from **Set Theory**

STABILITY THEORY

Shelah has led the research in Pure Model Theory for twenty years. He defined stable theories, a class of theories extending the omega-stable ones.

Shelah's ideology was finding a dividing line between theories where a classification of models by some kind of **invariants** can be done and theories with too many models to try to classify them.

Theory of independence and dimension intended to help analyzing and describing the way models are constructed from its basic parts (based on the notion of forking).

STABILITY THEORY

Invariants in modules and algebraically closed fields (stable structures).

Classification Theory,1978:

L'intrusion de Shelah dans ce domain a découragé un certain nombre de bons mathématiciens, qui en sont sortis; d'autres on voulu y survivre; d'autres encore — les inconscients — ont voulu y entrer, et, à défaut de concurrencer Shelah dans son domain favori, du moins en traiter des aspects que Shelah n'avait pas considéré.

Daniel Lascar

STRUCTURES OUT OF SHELAH'S PARADISE

Model theory applied to algebra: ordered fields and valued fields (Macintyre, Cherlin...)

Results combining notions and methods from the two branches: stability and model theory applied to algebra.

Eighties and nineties: **geometrical stability theory** (analysis of the combinatorial geometries, Hrushovski, Zil'ber...) and **o-minimality** (restriction on the definable sets in presence of a linear ordering: only finite unions of intervals and points are definable, Pillay, Wilkie, van der Dries...).

IMPORTANCE OF MODEL THEORY IN MATHEMATICS.

Hrushovski,1996: The Mordell-Lang conjecture for function fields.

Wilkie, 1996: Model completeness results for expansions of the real field by restricted Pfaffian functions and the exponential function. He proves that the real exponential field is o-minimal.

E. Hrushovski, 2001: A new proof of the Manin-Mumford conjecture.


1900
+100
2006
VICTOR HUGO

LEARNED LESSONS, GOOD AND BAD PRACTICES...

Hodges, 2007: *One-off successes of this kind were never a likely way to win converts to model theory from outside. Sometimes model theorists claimed to have better model-theoretic proofs of known results from other branches of mathematics; this was even less calculated to win friends and influence people.*

LEARNED LESSONS, GOOD AND BAD PRACTICES...

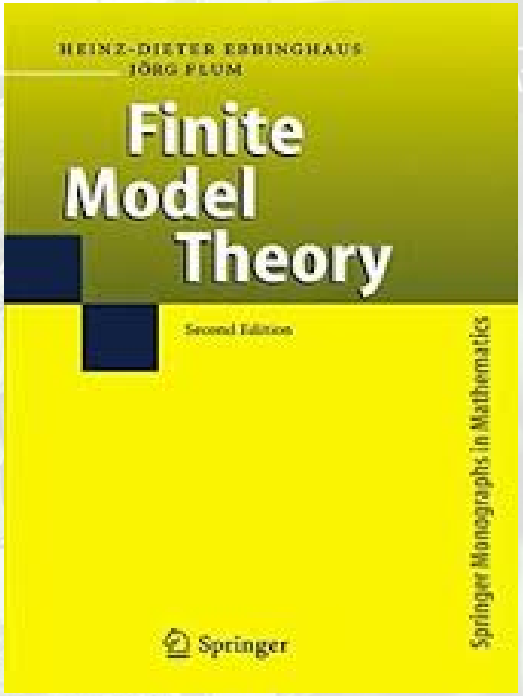
Hodges, 2007: *There was a revealing episode in the late 1960s. Ax gave a brief and neat model-theoretic proof of the 'somewhat unexpected fact that an injective morphism of an algebraic variety into itself is surjective' (Armand Borel's description). Borel and Shimura promptly found proofs not using model theory...*

The outcome seemed to be that the model-theoretic proof was elegant but unnecessary, and Ax's result owed more to his skill as a mathematician than it did to his model-theoretic approach. It seems not to have inspired other geometers or number theorists to learn model theory.

LEARNED LESSONS, GOOD AND BAD PRACTICES...

Hodges, 2007: *The natural ebb and flow of mathematical research throws up much subtler relationships than 'applying area X in area Y '. For example two areas may overlap; questions or methods of common interest form a weak overlap, and a stronger overlap is where the same researchers place themselves in both fields.*

MODEL THEORY AND COMPUTER SCIENCE



FINITE MODEL THEORY

FMT focuses on the study of logics on classes of finite structures. Identify decidable fragments of first-order logic.

Basic notions. *Finite model property:* if a sentence in a class has a model, then it has a finite model. *Finitely valid first-order sentence*: first-order sentence true on all finite models.

Trakhtenbrot's Theorem (1950) There is no effective axiomatization of the set of all finitely valid first-order sentences.

FINITE MODEL THEORY

Other logics have been explored in the context of FMT: fragments of second-order logic, logics with fixed-point operators, infinitary logics, and logics with generalized quantifiers.

Classes of finite structures: finite graphs, finite ordered graphs, planar graphs, finite strings...

Finite model theory benefitted from a continuous interaction with certain areas of computer science, especially computational complexity and database theory.

4 BRANCHES OF FINITE MODEL THEORY

1. **Descriptive complexity:** interplay between computational complexity and logical definability. Fagin's Theorem. Existential second-order logic (ESO) can express NP-complete problems on finite graphs.

2. **Expressive power of logics on finite structures:** what can and what cannot be expressed in various logics on classes of finite structures? Ehrenfeucht-Fraïssé games. None of the following properties is FO-definable in the class of all finite graphs: connectivity, acyclicity, planarity and 2-colorability.

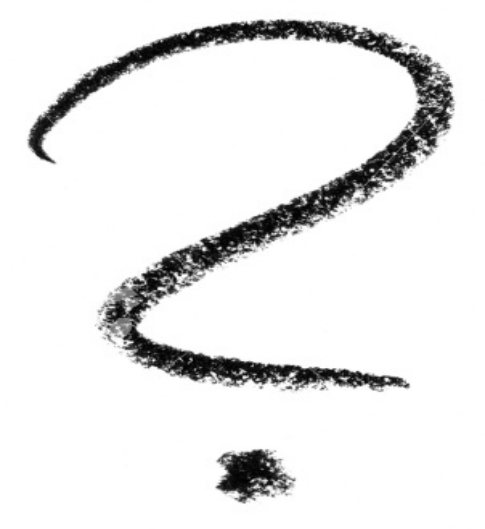
4 BRANCHES OF FINITE MODEL THEORY

3. **Connections between logic and asymptotic combinatorics** (0–1 laws and convergence laws for the asymptotic probabilities of sentences of various logics on classes of finite structures).

4. **Classical model theory in the finite:** do the classical results of model theory hold in the finite? NOT Compactness, Craig Interpolation Theorem. YES: Preservation under homomorphisms.

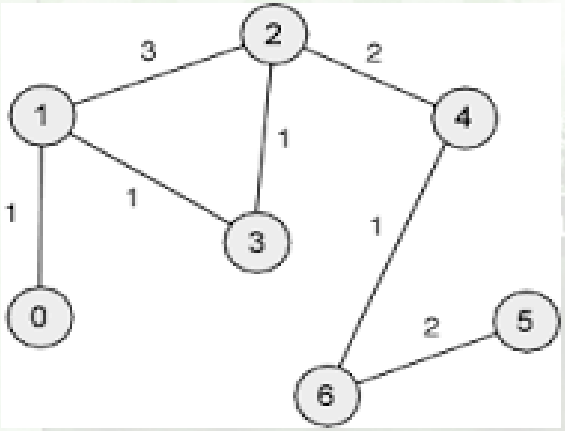
IF MODEL THEORY OF MFL IS THE ANSWER...

WHICH IS THE QUESTION?



WHICH STRUCTURES?

- Preference modelling
- Valued constraint satisfaction
- Weighted graphs
- Similarities
- …



PROPOSAL 1: Create a repository of structures relevant to CS, linguistics... with information about their model-theoretic properties.

GETTING OUT OF OUR COMFORT ZONE

PROPOSAL 2: Collaborative work with other disciplines. Learn the basics of other disciplines and identify the main open problems in the area. Traslation of these open problems to our language. Organize a seminar with researchers of those areas to which we think we could contribute. Attend general CS or AI Conferences...

Rostislav Horcík (seminar March 2016). Example of Valued CSP

Classical Finite Theory: Constraint Satisfaction and Homomorphisms of Finite Models.

GETTING OUT OF OUR COMFORT ZONE: VCSP

Does exists an assignment of values to variables that satisfies all the constraints?

Each constraint in the set C is specified as a pair (v,R) where v, is a tuple of variables from V, of length n, and R is an n-ary relation on the set D.

The assignment of values to variables is then a mapping $h : V \rightarrow D$. Such **a mapping satisfies the constraint** (v,R) if $h(v) \in R$.

GETTING OUT OF OUR COMFORT ZONE: VCSP

<u>Representation of constraint satisfaction problems in terms of the existence of certain homomorphisms between finite structures:</u>

Suppose we are given a constraint satisfaction problem P = (V,D,C). Let $R_1, \ldots, R_l$ list all the relations mentioned in C. We define two structures as follows:

$A = (V, \{ v \mid (v,R_1) \in C\}, \ldots, \{ v \mid (v,R_l) \in C\})$

$B = (D,R_1, \ldots, R_l)$.

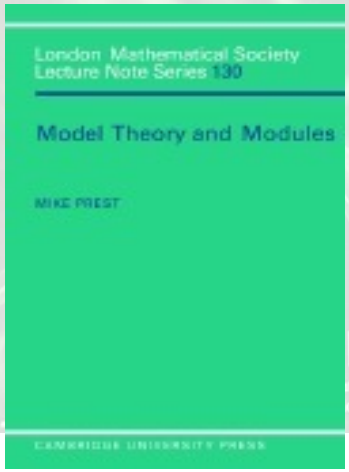
P has a solution $\Leftrightarrow$ there exists a homomorphism $h : A \rightarrow B$.

GETTING OUT OF OUR COMFORT ZONE: VCSP

In formal terms, the existence of a solution can be expressed by a **positive primitive formula** (pp-formula)

pp-formulas: The smallest set of formulas that contain the atomic and it is closed under conjunction and the existential quantifier.

A formula which is preserved under products and homomorphisms is equivalent to a pp-formula.



GETTING OUT OF OUR COMFORT ZONE: VCSP

Valued Constraint Satisfaction Problem (V**CSP**) is a constraint satisfaction problem where constraints can be violated (according a violation degree) and in which preferences among solutions can be expressed.

A wide range of problems are over-constrained (no solution can be found without violating at least one constraint) or have multiple solutions and the goal is to find the solution having minimal cost according to a cost function.

PROPOSAL 3: Study pp-formulas. In general: definibility results, study of types and important fragments of MFL, preservation theorems...

FUNDAMENTAL NOTIONS IN MODEL THEORY

- Isomorphism (strong and weak...)
- Elementary Equivalence (strong and weak...)
- Model in a degree.
- The role of crisp equality. (similarities, reduced models...)

...

PROPOSAL 4: From the experience adquired in the study of different structures, redefine the basic notions of model theory. The fundamental concepts of model theory of MFL can not be obtained only from fuzzifying every classical concept.

REFINEMENTS OF MODEL-THEORETIC TECHNIQUES

- Ultraproducts and reduced products.
- Back-and-forth systems.
- Löwenheim-Skolem Theorems.
- Diagrams

...

PROPOSAL 5: Refine the techniques and obtain new ones in the light of the results obtained using specific classes of structures.

TRANSLATION INTO TWO-SORTED LANGUAGES

Given a fuzzy structure (A,M), we can associate a 2-sorted classical structure A_M. There is a translation between formulas of the two languages.

Formula C translates into E_C:
The truth value of C in (A,M) is a iff A_M is true $E_C(a)$

Can model theory of MFL be reduced to the modelo-theoretic study of these classical 2-sorted structures?

We can express “too many” properties of the algebra of values. The two sorts are “symmetrical”...

PROPOSAL 6: Study the properties of the translation. Study the classical model theory of the algebra of values.

FOCUS ON THE FINITE STRUCTURES

- Definability on classes of finite structures (finite weighted graphs...)
- Model Theory of MFL in the finite: which results of Model Theory of MFL hold in the class of finite structures?
- Expressiveness of certain extensions of Predicate Fuzzy Logics.
- Modal extensions... model checking...

PROPOSAL 7: Classification of the structures of different theories. Big and small models...

THANKS TO PETR, CARLES AND NICK FOR MAKING US TO EXIT OUR COMFORT ZONE...