Future of MFL: Fuzzy Natural Logic and Alternative Set Theory

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The research in MFL has been mainly focused on its metamathematics

Repeated proclaim:

MFL should serve as a logic providing:

- Tools for the development of the mathematical model of vagueness
- Tools for various kinds of applications that have to cope with the latter

This goal is not sufficiently carried out!

The general properties of MFL are already known

Let us focus on the inside of few specific fuzzy logics
Develop program solving special problems
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Two of many possible directions

- Fuzzy Natural Logic
- Fuzzy logic in the frame of the Alternative Set Theory
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The concept of natural logic
George Lakoff: *Linguistics and Natural Logic, Synthese 22, 1970*

**Goals**

- to express all concepts capable of being expressed in natural language
- to characterize all the valid inferences that can be made in natural language
- to mesh with adequate linguistic descriptions of all natural languages

Natural logic is a collection of terms and rules that come with natural language and allow us to reason and argue in it.

**Hypothesis (Lakoff)**

Natural language employs a relatively small finite number of atomic predicates that are used in forming sentences. They are related to each other by meaning-postulates — *axioms*, e.g.,

\[
\text{REQUIRE}(x, y) \Rightarrow \text{PERMIT}(x, y)
\]

that do not vary from language to language.
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### The concept of Fuzzy Natural Logic

#### Paradigm of FNL

1. Extend the concept of natural logic to cope with vagueness
2. Develop FNL as a mathematical theory

*Extension of Mathematical Fuzzy Logic; application of Fuzzy Type Theory*

Fuzzy natural logic is a mathematical theory that provides models of terms and rules that come with natural language and allow us to reason and argue in it. At the same time, the theory copes with vagueness of natural language semantics.

(Similar concept was initiated in 1995 by V. Novák under the name *Fuzzy logic broader sense (FLb)*)
The concept of Fuzzy Natural Logic

**Paradigm of FNL**

(i) Extend the concept of natural logic to cope with vagueness

(ii) Develop FNL as a *mathematical theory*

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FNL and semantics of natural language

Sources for the development of FNL

- Results of classical linguistics
- Logical analysis of concepts and semantics of natural language
  *Transparent Intensional Logic* (P. Tichý, P. Materna)
- Montague grammar
  *English as a Formal Language*
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Current constituents of FNL

- Theory of evaluative linguistic expressions
- Theory of fuzzy/linguistic IF-THEN rules and logical inference (*Perception-based Logical Deduction*)
- Theory of fuzzy generalized and intermediate quantifiers including generalized Aristotle syllogisms and square of opposition
- Examples of formalization of special commonsense reasoning cases
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Formal logical analysis of concepts and natural language expressions requires higher-order logic — type theory.

Why fuzzy type theory

- It is a constituent of Mathematical Fuzzy Logic, well established with sound mathematical properties.
- Enables to include model of vagueness in the developed mathematical models.
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Higher-order fuzzy logic — Fuzzy Type Theory

Formal logical analysis of concepts and natural language expressions requires higher-order logic — *type theory*.

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Fuzzy Type Theory

Generalization of classical **type theory**

Syntax of FTT is an extended lambda calculus:

- more logical axioms
- many-valued semantics

**Main fuzzy type theories**

IMTL, Łukasiewicz, EQ-algebra
Fuzzy Type Theory

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Basic concepts

Types

Elementary types: $o$ (truth values), $\epsilon$ (objects)

Composed types: $\beta\alpha$

Formulas have types: $A_\alpha \in Form_\alpha$,

$A_\alpha \equiv B_\alpha \in Form_o$

$\lambda x_\alpha C_\beta \in Form_{\beta\alpha}$

$\Delta_{oo}A_o \in Form_o$

Formulas of type $o$ are propositions

Interpretation of formulas $A_{\beta\alpha}$ are functions $M_\alpha \rightarrow M_\beta$
Semantics of FTT

Frame

\[ M = \langle (M_\alpha, =_\alpha)_{\alpha \in \text{Types}}, \mathcal{E} \Delta \rangle \]

Fuzzy equality \( =_\alpha : M_\alpha \times M_\alpha \rightarrow L \)

\[ [x =_\alpha x] = 1 \] (reflexivity)
\[ [x =_\alpha y] = [y =_\alpha x] \] (symmetry)
\[ [x =_\alpha y] \otimes [y =_\alpha z] \leq [x =_\alpha z] \] (transitivity)
Generalized completeness (Henkin style)

**Theorem**

(a) A theory $T$ of fuzzy type theory is consistent iff it has a general model $M$.

(b) For every theory $T$ of fuzzy type theory and a formula $A_o$

$$T \vdash A_o \iff T \models A_o.$$ 

FTT has a lot of interesting properties and great explication power
Natural language in FNL

Standard Łukasiewicz $MV_\Delta$-algebra

Two important classes of natural language expressions

- Evaluative linguistic expressions
- Intermediate quantifiers
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Evaluative linguistic expressions

**Example**

very short, rather strong, more or less medium, roughly big, extremely big, very intelligent, significantly important, etc.

- Special expressions of natural language using which people evaluate phenomena and processes that they see around
- They are permanently used in any speech, description of any process, decision situation, characterization of surrounding objects; to bring new information, people must to evaluate

We construct a special theory $T^{Ev}$ in the language of FTT formalizing 6 general characteristics of the semantics of evaluative expressions
Evaluative linguistic expressions

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• Special expressions of natural language using which people evaluate phenomena and processes that they see around
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We construct a special theory $T^{Ev}$ in the language of FTT formalizing 6 general characteristics of the semantics of evaluative expressions
Axioms of $T^{\text{Ev}}$

(EV1) $(\exists z)\Delta(\neg z \equiv z)$

(EV2) $(\bot \equiv w^{-1}\bot_w) \land (\mathrel{\mathord{\top}} \equiv w^{-1}\mathrel{\mathord{\top}}_w) \land (\mathrel{\mathord{\top}} \equiv w^{-1}\mathrel{\mathord{\top}}_w)$

(EV3) $t \sim t$

(EV4) $t \sim u \equiv u \sim t$

(EV5) $t \sim u \& u \sim z \cdot \Rightarrow t \sim z$

(EV6) $\neg(\bot \sim \mathrel{\mathord{\top}})$

(EV7) $\Delta(((t \Rightarrow u) \& (u \Rightarrow z)) \Rightarrow \cdot t \sim z \Rightarrow t \sim u$

(EV8) $t \equiv t' \& z \equiv z' \Rightarrow \cdot t \sim z \Rightarrow t' \sim z'$

(EV9) $(\exists u)\mathrel{\mathord{\hat{\gamma}}}(\bot \sim u) \land (\exists u)\mathrel{\mathord{\hat{\gamma}}}(\mathrel{\mathord{\top}} \sim u) \land (\exists u)\mathrel{\mathord{\hat{\gamma}}}(\mathrel{\mathord{\top}} \sim u)$

(EV10) $NatHedge \mathrel{\mathord{\check{\nu}}} \& (\exists \nu)(\exists \nu')(\text{Hedge } \nu \& \text{Hedge } \nu' \& (\nu_1 \preceq \mathrel{\mathord{\check{\nu}}} \land \mathrel{\mathord{\check{\nu}}} \preceq \nu_2))$

(EV11) $(\forall z)((\gamma\mathrel{\mathord{\check{\nu}}}(LH z)) \lor (\gamma\mathrel{\mathord{\check{\nu}}}(MH z)) \lor (\gamma\mathrel{\mathord{\check{\nu}}}(RH z)))$
Relevant characteristics — Context

(A) Nonempty, linearly ordered and bounded scale, three distinguished limit points: *left bound*, *right bound*, and a *central point*

Context $w_{\alpha_0}$ $\mathcal{M}_p(w_{\alpha_0}) = w : [0, 1] \rightarrow M$:

\[
\begin{align*}
w(0) &= v_L \\
w(0.5) &= v_S \\
w(1) &= v_R
\end{align*}
\]

(1) left bound
(2) central point
(3) right bound

Set of contexts $W = \{ w \mid w : [0, 1] \rightarrow M \}$

Crisp linear ordering $\leq_w$ in each context $w$
Relevant characteristics — Context

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Set of contexts \( W = \{ w \mid w : [0, 1] \rightarrow M \} \)

Crisp linear ordering \( \leq_w \) in each context \( w \)
Relevant characteristics — Intension

(B) Function from the set of contexts into a set of fuzzy sets

\[
\text{Int}(\mathcal{A}) = \lambda w \lambda x (Aw)x \quad M(\text{Int}(\mathcal{A})) : W \rightarrow \mathcal{F}(w([0, 1]))
\]
Relevant characteristics — Intension

(B) Function from the set of contexts into a set of fuzzy sets

\[ \text{Int}(\mathcal{A}) = \lambda w \ \lambda x (A_w)x \]

\[ \mathcal{M}(\text{Int}(\mathcal{A})) : W \rightarrow \mathcal{F}(w([0, 1])) \]

Scheme of intension

\[ v_L \rightarrow v_S \rightarrow v_R \]

\[ \vdots \]

\[ v_L \rightarrow v_S \rightarrow v_R \]
**Horizon**

(C) Each of the limit points is a starting point of some *horizon* running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond)

**Three horizons**

\[
\begin{align*}
LH(a) &= [0 \sim a], & LH(wx) &= [v_L \approx_w x] \\
MH(a) &= [0.5 \sim a], & MH(wx) &= [v_S \approx_w x] \\
RH(a) &= [1 \sim a], & RH(wx) &= [v_R \approx_w x]
\end{align*}
\]
(C) Each of the limit points is a starting point of some horizon running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond)

Three horizons

\[ LH(a) = [0 \sim a], \quad LH(w \times) = [v_L \approx_w x] \]
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Relevant characteristics — horizon

(D) Each horizon is represented by a special fuzzy set determined by a reasoning analogous to that leading to the sorites paradox.

Sorites paradox
One grain does not make a heap. Adding one grain to what is not yet a heap does not make a heap. Consequently, there are no heaps.
Relevant characteristics — horizon

**(D)** Each horizon is represented by a special fuzzy set determined by a reasoning analogous to that leading to the sorites paradox.

**Sorites paradox**

One grain does not make a heap. Adding one grain to what is not yet a heap does not make a heap. Consequently, there are no heaps.
Construction of extensions of evaluative expressions

Extension of evaluative expression is delineated by shifting of the horizon using linguistic hedge.

Hedges — shifts of horizon

\[ \nu : [0, 1] \rightarrow [0, 1] \]
Construction of extensions of evaluative expressions
Falakros/sorites paradox in evaluative expressions

**Theorem (⟨linguistic hedge⟩ small)**

- **Zero is “(very) small” in each context**
  \[ \vdash (\forall w)((Smv)w 0) \]

- **In each context there is \( p \) which surely is not “(very) small”**
  \[ \vdash (\forall w)(\exists p)(\Delta \neg (Smv)w p) \]

- **In each context there is no \( n \) which is surely small and \( n + 1 \) surely is not small**
  \[ \vdash (\forall w)\neg (\exists n)(\Delta (Smv)w n \& \Delta \neg (Smv)w(n + 1)) \]

- **In each context: if \( n \) is small then it is almost true that \( n + 1 \) is also small**
  \[ \vdash (\forall w)(\forall n)((Smv)w n \Rightarrow At((Smv)w (n + 1)) \]

(At(\( A \)) is measured by \( (Smv)w n \Rightarrow (Smv)w (n + 1) \)

In each context there is no last surely small \( \times \) and no first surely big \( \times \)
Intermediate quantifiers


### Quantifiers in natural language

Words (expressions) that precede and modify nouns; tell us how many or how much. They specify quantity of specimens in the domain of discourse having a certain property.

### Example

*All, Most, Almost all, Few, Many, Some, No*

Most women in the party are well dressed

Few students passed exam

*Intermediate quantifiers form an important subclass of generalized (possibly fuzzy) quantifiers*
Intermediate quantifiers


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*Intermediate quantifiers form an important subclass of generalized (possibly fuzzy) quantifiers*
Semantics of intermediate quantifiers

**Main idea**

They are classical quantifiers $\forall$ and $\exists$ taken over a *smaller* class of elements. Its size is determined using an appropriate evaluative expression.

Classical logic: *No substantiation why and how the range of quantification should be made smaller*
Semantics of intermediate quantifiers

Main idea

They are classical quantifiers $\forall$ and $\exists$ taken over a smaller class of elements. Its size is determined using an appropriate evaluative expression.

Classical logic: No substantiation why and how the range of quantification should be made smaller
Formal theory of intermediate quantifiers

\[ T^{IQ} = T^{Ev} + 4 \text{ special axioms} \]

“\langle Quantifier\rangle B’s are A”

\[
(Q_{Ev}^\forall x)(B, A) := (\exists z)((\Delta(z \subseteq B)) \land
\begin{array}{l}
\text{“the greatest” part of } B’s \\
(\forall x)(z x \Rightarrow Ax)) \land
\end{array}
\begin{array}{l}
\text{each of } B’s \text{ has } A \\
Ev((\mu B)z)) \land
\end{array}
\text{size of } z \text{ is evaluated by } Ev
\]

\[ Ev — \text{ extension of a certain evaluative expression} \]

\[ \text{(big, very big, small, etc.)} \]
Special intermediate quantifiers

**P:** Almost all $B$ are $A$ := $Q_{B_i \text{Ex}}(B, A)$

$(\exists z)(((\Delta(z \subseteq B)) \& (\forall x)(zx \Rightarrow Ax)) \land (Bi \text{ Ex})(\mu B)z))$

**B:** Almost all $B$ are not $A$ := $Q_{B_i \text{Ex}}(B, \neg A)$

**T:** Most $B$ are $A$ := $Q_{B_i \text{Ve}}(B, A)$

**D:** Most $B$ are not $A$ := $Q_{B_i \text{Ve}}(B, \neg A)$

**K:** Many $B$ are $A$ := $Q_{\neg(Sm \nu)}(B, A)$

**G:** Many $B$ are not $A$ := $Q_{\neg(Sm \nu)}(B, \neg A)$

**F:** Few $B$ are $A$ := $Q_{Sm \text{Ve}}(B, A)$

**H:** Few $B$ are not $A$ := $Q_{Sm \text{Ve}}(B, \neg A)$
Special intermediate quantifiers

Classical quantifiers

A: All $B$ are $A$ := $Q_{B\Delta}^\forall(B, A) \equiv (\forall x)(Bx \Rightarrow Ax)$,

E: No $B$ are $A$ := $Q_{B\Delta}^\forall(B, \neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax)$,

I: Some $B$ are $A$ := $Q_{B\Delta}^\exists(B, A) \equiv (\exists x)(Bx \land Ax)$,

O: Some $B$ are not $A$ := $Q_{B\Delta}^\exists(B, \neg A) \equiv (\exists x)(Bx \land \neg Ax)$. 
105 valid generalized Aristotle’s syllogisms

**Figure I**

<table>
<thead>
<tr>
<th>Q₁</th>
<th>M is Y</th>
</tr>
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<tbody>
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<td>Q₂</td>
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**Figure II**

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**Example (ATT-I)**

All women (M) are well dressed (Y)
Most people in the party (X) are women (M)

(∀x)(M x ⇒ Y x)  
(Q⁻⁻⁻⁻Bi Ve x)(X, M)

Most people in the party (X) are well dressed (Y)

(Q⁻⁻⁻⁻Bi Ve x)(X, Y)

**Example (ETO-II)**

No lazy people (Y) pass exam (M)
Most students (X) pass exam (M)

¬(∃x)(Yx ∧ Mx)  
(Q⁻⁻⁻⁻⁻⁻Bi Ve x)(X, M)

Some students (X) are not lazy people (Y)

(∃x)(X ∧ ¬Y)
105 valid generalized Aristotle’s syllogisms

**Figure I**

\[ Q_1 \ M \text{ is } Y \]

\[ Q_2 \ X \text{ is } M \]

\[ Q_3 \ X \text{ is } Y \]

**Figure II**

\[ Q_1 \ Y \text{ is } M \]

\[ Q_2 \ X \text{ is } M \]

\[ Q_3 \ X \text{ is } Y \]

---

**Example (ATT-I)**

All women (\(M\)) are well dressed (\(Y\))

Most people in the party (\(X\)) are women (\(M\))

Most people in the party (\(X\)) are well dressed (\(Y\))

\[(\forall x)(M \ x \Rightarrow Y \ x)\]

\[(Q^\forall_{Bi \ Ve} x)(X, M)\]

\[(Q^\forall_{Bi \ Ve} x)(X, Y)\]

---

**Example (ETO-II)**

No lazy people (\(Y\)) pass exam (\(M\))

Most students (\(X\)) pass exam (\(M\))

Some students (\(X\)) are not lazy people (\(Y\))

\[\neg(\exists x)(Y \ x \wedge M \ x)\]

\[(Q^\forall_{Bi \ Ve} x)(X, M)\]

\[(\exists x)(X \wedge \neg Y)\]
105 valid generalized Aristotle’s syllogisms

**Figure III**

\[
\begin{align*}
Q_1 & \; M \text{ is } Y \\
Q_2 & \; M \text{ is } X \\
\hline
Q_3 & \; X \text{ is } Y
\end{align*}
\]

**Figure IV**

\[
\begin{align*}
Q_1 & \; Y \text{ is } M \\
Q_2 & \; M \text{ is } X \\
\hline
Q_3 & \; X \text{ is } Y
\end{align*}
\]

**Example (PPI-III)**

Almost all old people (\(M\)) are ill (\(Y\))

\[(\forall x)(\exists! x)(Mx, Yx)\]

Almost all old people (\(M\)) have gray hair (\(X\))

\[(\forall x)(\exists! x)(Mx, Xx)\]

Some people with gray (\(X\)) hair are ill (\(Y\))

\[(\exists x)(Xx \land Yx)\]

**Example (TAI-IV)**

Most shares with downward trend (\(Y\)) are from car industry (\(M\))

\[(\exists x)(\forall y)(Yx, Mx)\]

All shares of car industry (\(M\)) are important (\(X\))

\[(\forall x)(Mx, Xx)\]

Some important shares (\(X\)) have downward trend (\(Y\))

\[(\exists x)(Xx \land Yx)\]
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Example (PPI-III)

Almost all old people (\( M \)) are ill (\( Y \))

\( (Q_{\text{Bi}} \forall \text{Ex}x)(Mx, Yx) \)

Almost all old people (\( M \)) have gray hair (\( X \))

\( (Q_{\text{Bi}} \forall \text{Ex}x)(Mx, Xx) \)

Some people with gray (\( X \)) hair are ill (\( Y \))

\( (\exists x)(Xx \land Yx) \)

Example (TAI-IV)

Most shares with downward trend (\( Y \)) are from car industry (\( M \))

\( (Q_{\text{Bi}} \forall \text{Ve}x)(Yx, Mx) \)

All shares of car industry (\( M \)) are important (\( X \))

\( (\forall x)(Mx, Xx) \)

Some important shares (\( X \)) have downward trend (\( Y \))

\( (\exists x)(Xx \land Yx) \)
Square of opposition — basic relations

$P_1, P_2 \in Form_\circ$ are:

(i) **contraries** if $T \vdash \neg(P_1 \& P_2)$.

$P_1$ and $P_2$ cannot be both true but can be both false

(ii) **sub-contraries** if $T \vdash P_1 \nabla P_2$.

**weak sub-contraries** if $T \vdash \Upsilon(P_1 \lor P_2)$

$P_1$ and $P_2$ cannot be both false but can be both true

(iii) $P_1, P_2 \in Form_\circ$ are **contradictories** if

$$T \vdash \neg(\Delta P_1 \& \Delta P_2)$$

as well as $T \vdash \Delta P_1 \nabla \Delta P_2$.

$P_1$ and $P_2$ cannot be both true as well as both false

(iv) The formula $P_2$ is a **subaltern** of $P_1$ in $T$ if $T \vdash P_1 \Rightarrow P_2$

($P_1$ a **superaltern** of $P_2$)
Generalized 5-square of opposition

*A: All B are A
*P: Almost all B are A
*T: Most B are A
*K: Many B are A
I: Some B are A

*E: No B are A
*B: Almost all B are not A
*D: Most B are not A
*G: Many B are not A
O: Some B are not A

(universal)
(predominant)
(majority)
(common)
(particular)
Negation in FNL

Topic—focus articulation:
Each sentence is divided into topic (known information) and focus (new information)

Focus is negated

Example
It is not true, that:

(i) JOHN reads Jane’s paper
    Somebody else reads it

(ii) John READS JANE’S PAPER
    John does something else

(iii) John reads JANE’S PAPER
    John reads some other paper

(iv) John reads Jane’s PAPER
    John reads Jane’s book (lying on the same table)
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Each sentence is divided into **topic** (known information) and **focus** (new information)

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Negation in FNL

The law of double negation is in NL fulfilled (and so, in FNL as well)

We do not think (Th) that John did not read this paper (R(J, p))

\[ \neg Th(\neg R(J, p)) \equiv Th(R(J, p)) \]

However:

It is not clear (Cl) whether John did not read Jane’s paper (R(J, p))

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Consequence: We suspect John to read Jane’s paper
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Negation in FNL

\[ \text{not unhappy} \not\equiv \text{happy} \]

This is not violation of double negation
Negation in FNL

not unhappy $\not\equiv$ happy

This is not violation of double negation

unhappy is antonym of happy
Fuzzy/linguistic IF-THEN rules and logical inference

Fuzzy/Linguistic IF-THEN rule

\[ \text{IF } X \text{ is } A \text{ THEN } Y \text{ is } B \]

Conditional sentence of natural language
Example: IF X is small THEN Y is extremely strong

Linguistic description

\[ \begin{align*}
\text{IF } X \text{ is } A_1 \text{ THEN } Y \text{ is } B_1 \\
\text{IF } X \text{ is } A_2 \text{ THEN } Y \text{ is } B_2 \\
\text{..................} \\
\text{IF } X \text{ is } A_m \text{ THEN } Y \text{ is } B_m
\end{align*} \]

Text describing one’s behavior in some (decision) situation
Perception-based logical deduction

Imitates human way of reasoning

Crossing strategy when approaching intersection:

- $R_1 := \text{IF Distance is very small THEN Acceleration is very big}$
- $R_2 := \text{IF Distance is small THEN Brake is big}$
- $R_3 := \text{IF Distance is medium or big THEN Brake is zero}$

- Gives general rules for driver’s behavior independently on the concrete place
- People are able to follow them in an arbitrary signalized intersection

We can “distinguish” between the rules despite the fact that their meaning is vague
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We can “distinguish” between the rules despite the fact that their meaning is vague
MFL and AST?

**Suggestion:**
Switch the development of MFL to a new mathematical frame based on the Alternative Set Theory by P. Vopěnka

**Different understanding to infinity**

Vopěnka: “Mathematics uses infinity whenever it faces vagueness!”
Example

Consider the number $10^{120}$

Imagine counting each second $10^{12}$ atoms

Counting $10^{120}$ atoms by counting each second $10^{12}$ of them would take $10^{100}$ years!!

Visible universe has about $10^{80}$ atoms and exists less than $10^{11}$ years!
Source of Natural Infinity

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God’s view

Ha, Ha, why are you boring me with such ridiculously small numbers! Where is INFINITY!?
God’s view

Ha, Ha, why are you boring me with such ridiculously small numbers! Where is INFINITY!?

We cannot imagine even half of this way!

Something is wrong!
God’s view

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We cannot imagine even half of this way! Something is wrong!
Big numbers behave as infinite

Basic property

\[ \alpha \approx \alpha + 1 \]

10001 people slept “whole night” on 10000 beds!
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Alternative Set Theory

- An attempt at developing a new mathematics on the basis of criticism of classical one
  - Take useful principles and replace others by more natural ones
  - Make mathematics closer to human perception of reality
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Fundamental concepts

Actualizability $\leftrightarrow$ Potentiality

Class
Actualized grouping of objects

$X = \{ x \mid \varphi(x) \}$

Set
Sharp class
- All its elements can be put on a list
- There exists $\subseteq$ according to which a set has the first and the last elements

From classical point of view every set is classically finite
Fundamental concepts

Actualizability ↔ Potentiality

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*From classical point of view every set is classically finite*
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Is every part \( Y \subseteq x \) necessarily a set?

**Finite set**
defined very sharply; no part of it can be unsharp; transparent

\[
\text{Fin}(x) \iff (\forall Y \subseteq x) \text{Set}(Y)
\]

**Horizon**
- threshold terminating our view of the world,
- the world continues beyond the horizon,
- part of the world before it is determined nonsharply,
- not fixed, moving around the world
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A class $X$ is a semiset if there is a set $a$ such that $X \subseteq a$

**Theorem**

Let $a$ be an infinite set and $\leq$ its linear ordering. Put

$$X = \{ x \in a \mid \{ y \in a \mid y \leq x \} \text{ is a finite set} \}.$$  

Then $X$ is a proper semiset.
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Indiscernibility relation

\[ x \equiv y \quad \text{iff} \quad \langle x, y \rangle \in \bigcap \{ R_n \mid n \in \mathbb{FN} \} \]

sequence of still sharper criteria \( R_n \)

Indiscernibility of rational numbers

\[ x \dot{=} y \quad \text{iff} \quad |x - y| < \frac{1}{n}; \quad n \in \mathbb{FN} \]

\[ x \in \text{IS} \quad \text{iff} \quad x \dot{=} 0 \quad \text{infinitely small} \]

\[ x \in \text{IL} \quad \text{iff} \quad \frac{1}{x} \in \text{IS} \quad \text{infinitely large} \]

Figure

\( X \) is a figure if \( x \in X \) and \( y \equiv x \) implies \( y \in X \)
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Figure

\( X \) is a figure if \( x \in X \) and \( y \equiv x \) implies \( y \in X \)
Some types of classes

Countable

\( X \approx \mathbb{N} \)

Uncountable

\( X \approx \alpha, \) infinite, not countable

Real

there is \( \equiv \) such that \( X \) is a figure

Imaginary

not real, e.g., \( \Omega \) is imaginary

Imaginary classes are rare

Prolongation axiom

To each countable function \( F \) there exists a set function \( f \) such that

\[ F \subseteq f \]
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**Fuzzy approach**

Indiscernibility $\equiv$ sharpened to

$$\tilde{\equiv} = \bigcap \{ R_\beta \mid \beta \leq \gamma \}$$

The intensity of our “effort” to discern the objects $x$ and $y$ — a number $\alpha$ of

$$\langle x, y \rangle \in R_\beta, \quad \beta \in \alpha$$

**Degree of equality**

$$x \equiv_\nu y \quad \text{iff} \quad \nu = \frac{\alpha}{\gamma}.$$

**Theorem**

(a) $x \equiv_1 x$.

(b) $x \equiv_\nu y$ implies $y \equiv_\nu x$.

(c) $x \equiv_{\nu_1} y$ and $y \equiv_{\nu_2} z$ implies $x \equiv_{\nu_3} z$ where $\nu_1 \otimes \nu_2 \leq \nu_3$
Fuzzy sets

Membership degree of \( x \) in \( X \)

\[
X^F(x) = \bigvee \{ \nu \mid (\exists y \in Y)(x \equiv \nu y) \}
\]

Measure of the greatest intensity of our “effort” to discern \( x \) from elements of the kernel \( Y \).

**\( X^F \)**

A fuzzy set approximating the real class \( X \)
Fuzzy sets

Theorem

(a) \[ X_1^F(x) \otimes X_2^F(x) \preceq (X_1 \cap X_2)^F(x) \preceq X_1^F(x) \land X_2^F(x) \]

(b) \[ X_1^F(x) \lor X_2^F(x) \preceq (X_1 \cup X_2)^F(x) \preceq X_1^F(x) \oplus X_2^F(x) \]

(c) \[ (\overline{X})^F(x) = (V - X)^F(x) \doteq 1 - X^F(x) \]
Conclusions

- MFL should focus on the detailed development of some well selected logics.
- MFL should help to develop the concept of Fuzzy Natural Logic

Open problems:
- How surface structures can be transformed into logical formulas representing their meaning
- Formalization of negation
- Formalization of hedging
- Formalization of presupposition and its consequence
- Formalization of the meaning of verbs; meaning of sentences

- MFL could be developed on the basis of AST

Introducing degrees into AST
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References


Many papers in Commentationes Mathematicae Universitatis Carolinae (CMUC)
References

Holčapek, M., *Monadic L-fuzzy quantifiers of the type* \(\langle 1^n, 1 \rangle\), *FSS* 159(2008), 1811-1835.


Thank you for your attention