What should a logic of vagueness be useful for?

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The future of MFL

We have the (non-exclusive) choice of ...

- further expanding the success of fuzzy logic
  as a branch of mathematical logic, or
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Let's opt for the second way —

and let's have a look back.

The beginnings: a central figure and a central notion

INFORMATION AND CONTROL 8, 338-353 (1965)

Fuzzy Sets*

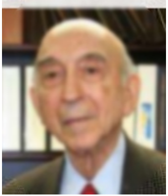
L. A. ZADEH

Department of Electrical Engineering and Electronics Research Laboratory, University of California, Berkeley, California

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.

I. INTRODUCTION

More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc., as its members, and clearly excludes such objects as rocks, fluids



Our problem is to formalise reasoning about vague notions.

More than one level of granularity

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The human body temperature can be specified

- by an expression like *low*, *normal*, *slightly elevated*, *fever*,
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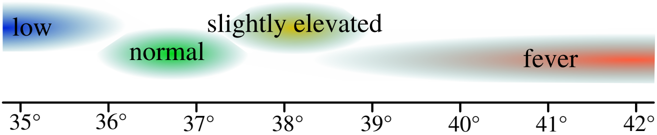

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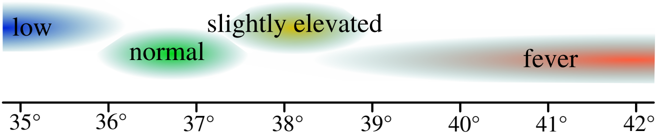

These notions are bound to different levels of granularity.

A notion bound to a refinable level of granularity is called vague.

Merging two levels of granularity

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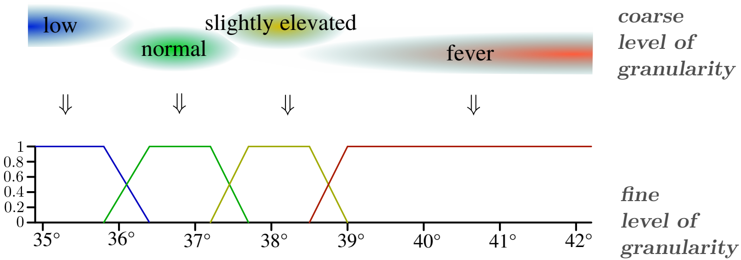

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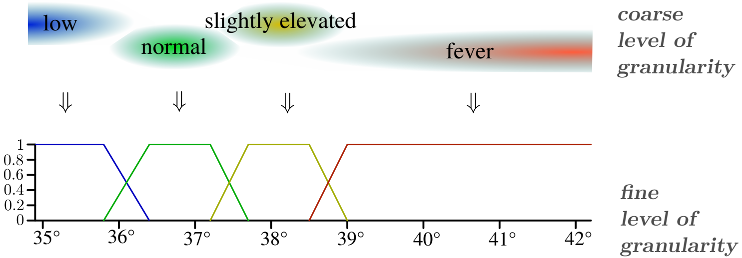

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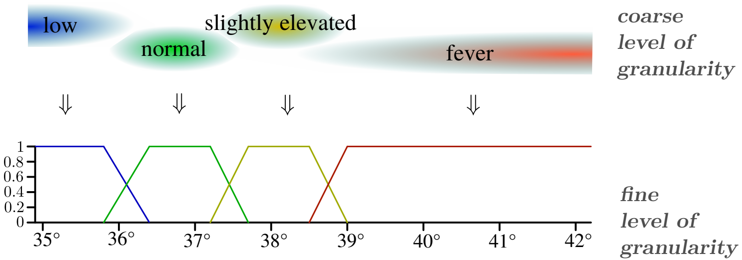

The idea is to "lift up" coarse notions into the fine level.

Fuzzy sets provide a (somewhat pragmatic) means of "embedding" the coarse level into a finer level.

The resuling “multi-level” model

The practical procedure

- We choose a universe W according to the finest level of granularity that we want to take into account.
- We model vague properties by fuzzy sets $u\colon W \to [0,1]$.

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- We choose a universe W according to the finest level of granularity that we want to take into account.
- We model vague properties by fuzzy sets $u\colon W \to [0,1]$.
- Logical combinations then correspond to operations on fuzzy sets. We define these operations pointwise.
- We may use $\wedge$ (minimum), $\vee$ (maximum), $\sim$ (standard negation).

Evaluation: the positive side

Embedding the coarse level into the fine level makes notions artificially more precise than they actually are. Hence the choice of fuzzy sets always involves arbitrariness.

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Luckily, there are experiments supporting certain "design choices".

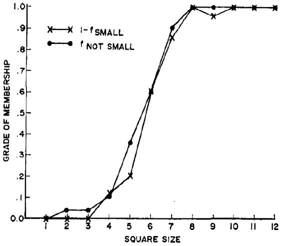

FIGURE 2. Membership function for *not small.*

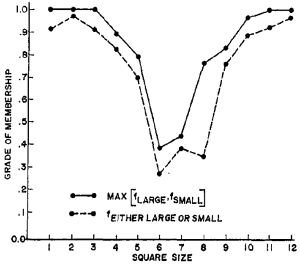

FIGURE 13. Membership function for *either large or small.*

H.M. Hersh, A. Caramazza, A fuzzy set approach to modifiers and vagueness in natural language, J. Exp. Psychology General 1976.

Evaluation: the negative side

We must live with the following adversities.

- Any non-Zadeh choice of operations seems (for the present purpose) not to be justifiable.

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- Any non-Zadeh choice of operations seems (for the present purpose) not to be justifiable.
- In particular, logic calls for an implication connective. Easy to define, easy to explain, but hard to justify.
- Using a universe of all fuzzy sets blurs the levels of granularity.

As regards the connectives

Proposition 1

Statements of truth-functional fuzzy logic would be easier interpretable if

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F. Bou, À. García-Cerdaña, V. Verdú,
On two fragments with negation and without implication of the logic of residuated lattices, Arch. Math. Logic 2006.

As regards fuzzy sets and granularity

Proposition 2

A logic of fuzzy sets might be seen as a logic related to vagueness if

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V. Novák, I. Perfilieva, J. Močkoř,
Mathematical principles of fuzzy logic, 1999.

Logic of prototypes

The transition from “coarse” to “fine” amounts to a standardisation of the formation of fuzzy sets.

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- We endow our universe W with a similarity relation s.
- We decide about the prototypes $P \subseteq W$ of a vague property φ.

We model φ by

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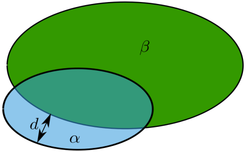
Most remarkable consequence

The "vertical" viewpoint on fuzzy sets is replaced by the "horizontal" viewpoint.

Digression: Approximate Reasoning à la Ruspini

LAE, the *Logic of Approximate Entailment* uses graded implications $\alpha \overset{d}{\Rightarrow} \beta$, interpreted in similarity spaces:



Side remark

LAE and related calculi offer a considerable potential in logical respects.

Ll. Godo, R.O. Rodríguez, Logical approaches to fuzzy similarity-based reasoning: an overview, 2008.

A logic of prototypes and counterexamples

The formation of fuzzy sets is more appropriately standardised as follows.

- We endow our universe W with a metric d.
- We decide about the prototypes $P^+ \subseteq W$ as well as the counterexamples $P^- \subseteq W$ of a vague property φ.

We model φ by

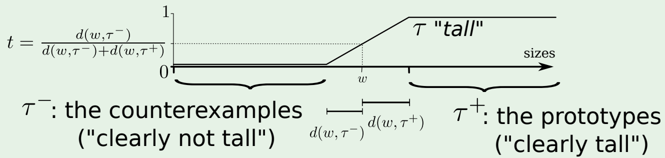
$$u\colon W \to [0,1], \quad w \mapsto \frac{d(w,P^-)}{d(w,P^+) + d(w,P^-)},$$

that is, by "linear interpolation".

V. Novák, A comprehensive theory of trichotomous evaluative linguistic expressions, *Fuzzy Sets Syst.* 2008.

Th.V., A. Zamansky, Reasoning with graded information: the case of diagnostic rating scales in healthcare, *Fuzzy Sets Syst.* 2015.

A logic of prototypes and counterexamples



Benefits:

- Transparent transition from “coarse” to “fine”.
- Easy switching between granularities.

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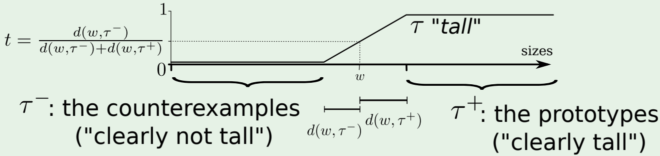

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Problem:

- Although a logic of fuzzy sets, the approach is **not compatible with the idea of truth-functionality**.
- Defining a logic at all is a challenge.

Vagueness again

We want to put fuzzy logic on firm conceptual grounds?

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What about asking ourselves:

What do we actually want?

What is the prototypical example of a conclusion
that a fuzzy logic should be able to reproduce?

First possible field of application

The medical decision support system

MONI (K.-P. Adlassnig, Vienna)

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applies fuzzy logic to rules like:

If

- the patient does not have fever ($\geq 38.0°$) or urinary urgency or frequency or dysuria or suprapubic tenderness
- the patient has had a positive urine culture ($\geq 10^5$ microorganisms/cm^3 of ≤ 2 species)
- the patient had an indwelling urinary catheter within 7 days before the culture

then possibly *asymptomatic bacteriuria*.

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In practice, the fine level of granularity is of primary relevance.

Second possible field of application

This is an Aristotelian syllogism:

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This is a generalised Aristotelian syllogism:

$$\frac{\text{Most } X \text{ are } M \quad \text{All } M \text{ are } Y}{\text{Many } X \text{ are } Y}.$$

V. Novák, A formal theory of intermediate quantifiers, *Fuzzy Sets Syst.* 2008.

Th.V., Vagueness: where degree-based approaches are useful, and where we can do without, *Soft Computing* 2012.

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- Reproducing the argument by means of a fuzzy logic is possible.
- However, the reasoning does in general not involve the fine level.
  We may also reproduce the argument by modelling the coarse level only.

The theory TSR

Consider the language $\subseteq, \overset{\text{few}}{\subset}, \overset{\text{many}}{\subset}, \overset{\text{most}}{\subset}, \overset{\text{n.a.}}{\subset}, \sim; \cap, \cup, \setminus; \emptyset$.

Let TSR extend the theory of generalised Boolean algebras by:

Axioms for size, where $A \preccurlyeq B$ is $\exists C((A \sim C) \wedge (C \subseteq B))$:

$$A \sim A \qquad (A \sim B) \rightarrow (B \sim A) \qquad (A \sim B) \wedge (B \sim C) \rightarrow (A \sim C)$$
$$(A \preccurlyeq B) \vee (B \preccurlyeq A) \qquad (A \sim \emptyset) \leftrightarrow (A = \emptyset) \qquad (A \sim B) \wedge (A \subseteq B) \rightarrow (A = B)$$
$$(A \sim C) \wedge (B \sim D) \wedge (A \cap B = \emptyset) \wedge (C \cap D = \emptyset) \rightarrow (A \cup B \sim C \cup D)$$

General axioms for proportions:

$$(A \overset{\star}{\subset} B) \rightarrow (\emptyset \subset A) \wedge (A \subseteq B)$$
$$(A \overset{\star}{\subset} C) \wedge (B \subseteq C) \wedge (A \sim B) \rightarrow (B \overset{\star}{\subset} C),$$

where $\overset{\star}{\subset}$ is one of $\overset{\text{few}}{\subset}, \overset{\text{many}}{\subset}, \overset{\text{most}}{\subset}, \overset{\text{n.a.}}{\subset}$

The logic TSR

Axioms for "few":

$$(\emptyset \subset A) \wedge (A \subseteq B) \wedge (B \overset{\text{few}}{\subset} C) \;\rightarrow\; (A \overset{\text{few}}{\subset} C)$$

$$(A \overset{\text{few}}{\subset} B) \wedge (B \subseteq C) \;\rightarrow\; (A \overset{\text{few}}{\subset} C)$$

Axioms for "many":

$$(A \overset{\text{many}}{\subset} C) \wedge (A \subseteq B) \wedge (B \subseteq C) \;\rightarrow\; (B \overset{\text{many}}{\subset} C)$$

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$$(A \overset{\text{many}}{\subset} B) \;\rightarrow\; \neg(A \overset{\text{few}}{\subset} B)$$

$$(B \overset{\text{many}}{\subset} C) \wedge (B \subset C) \;\rightarrow\; \exists A((A \subset B) \wedge (A \overset{\text{few}}{\subset} C))$$

Axioms for "most", where $A \prec B$ is $\exists C((A \sim C) \wedge (C \subset B))$:

$$(A \overset{\text{most}}{\subset} B) \;\leftrightarrow\; (A \subseteq B) \wedge (B \backslash A \prec A) \qquad (A \overset{\text{most}}{\subset} B) \;\rightarrow\; (A \overset{\text{many}}{\subset} B)$$

Axioms for "nearly all":

$$(A \overset{\text{n.a.}}{\subset} B) \;\leftrightarrow\; (B \backslash A \overset{\text{few}}{\subset} B)$$

The logic TSR: example

$$\frac{\text{Most } X \text{ are } M \quad \text{All } M \text{ are } Y}{\text{Many } X \text{ are } Y}$$

Lemma

From TSR we derive

$$\frac{M \cap X \overset{\text{most}}{\subset} X \quad M \subseteq Y}{Y \cap X \overset{\text{many}}{\subset} X}.$$

To formalise generalised Aristotelian syllogisms, we can – although this is uncommon – restrict to a coarse model.

Third possible field of application

Let us return to medicine.

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Here is a fictitious report on an examination of the stomach:

PROCEDURES PERFORMED: Esophagogastroduodenoscopy with biopsies and colonoscopy.

ESOPHAGOGASTRODUODENOSCOPY

PROCEDURE IN DETAIL: The patient was given topical benzocaine spray and placed on the left lateral decubitus position. Following the administration of appropriate anesthesia, a diagnostic gastroscope was advanced under direct vision to the second portion of the duodenum without difficulty. The Z-line was regular and noted at approximately 38 cm from the incisors. Examination of the esophagus otherwise was endoscopically unremarkable. Patchy erythematous gastropathy was visualized, particularly in the distal portion of the stomach. No ulcers however were visualized. The stomach appeared to distend normally. Retroflexed views in the stomach did not reveal additional abnormalities. Multiple biopsies were obtained secondary to the findings above. Examination of the duodenum was endoscopically unremarkable though, secondary to the patient's history, biopsies were obtained for histology. No additional findings were noted as the scope was slowly withdrawn.

IMPRESSION: Erythematous gastropathy.

Third possible field of application

(...) The Z-line was regular and noted at approximately 38 cm from the incisors. (...)
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Task: Represent this report such that the question "Has a suspicion of gastropathy been reported?" can be answered in an automated way.

Comments

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- A fine level is not even applicable.
- Spatio-temporal aspects are predominant, but not the only ones.
- Efficient methods of representation are not available.

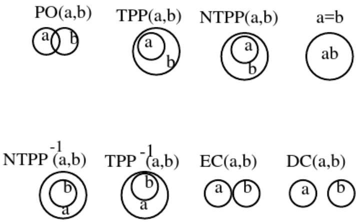
Qualitative spatio-temporal reasoning

There is an abundance of literature on the qualitative representation of facts in space and time.

Relation	Symbol	Symbol for Inverse	Pictoral Example
X *before* Y	<	>	XXX YYY
X *equal* Y	=	=	XXX YYY
X *meets* Y	m	mi	XXXYYY
X *overlaps* Y	o	oi	XXX YYY
X *during* Y	d	di	XXX YYYYYY
X starts Y	s	si	XXX YYYYY
X finishes Y	f	fi	XXX YYYYY

FIGURE 2. The Thirteen Possible Relationships.

Allen's time interval relations



the RCC8 spatial region relations

Qualitative spatio-temporal reasoning

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Qualitative spatio-temporal reasoning

- There is a huge demand for tools supporting the representation of spatio-temporal facts in a qualitative way.
- The aspect of vagueness is no academic feature here.
  The use of fine-grained models is inappropriate,
  if possible at all.
- Whether the qualitative-reasoning community has coped with vagueness better than fuzzy logic, is doubtful.
  But an exchange of experiences might be useful.

G. Ligozat, J. Renz, What Is a Qualitative Calculus? A General Framework, LNCS, 2004.

R. Hirsch, M. Jackson, T. Kowalski, T. Niven, Algebraic foundations for qualitative calculi and networks, draft.

Towards fuzzy logic of a different sort

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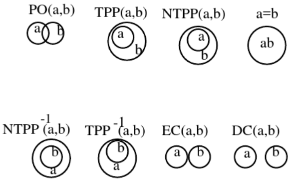

Proposal 3

We could

- review approaches to "qualitative reasoning" from a logical angle.

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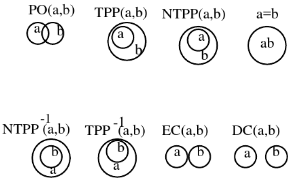

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- review approaches to "qualitative reasoning" from a logical angle.
- check to which extent many-valued logics could be useful or are successfully avoided in this field.