What should a logic of vagueness be useful for?

Thomas Vetterlein

Department of Knowledge-Based Mathematical Systems,
Johannes Kepler University (Linz)

16 June 2016
We have the (non-exclusive) choice of ...

- further expanding the success of fuzzy logic as a branch of mathematical logic, or
- broadening the scope of fuzzy logic in order to comply with its original aims.
The future of MFL

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- broadening the scope of fuzzy logic in order to comply with its original aims.

Let’s opt for the second way — and let’s have a look back.
The beginnings: a central figure and a central notion

Fuzzy Sets*
L. A. Zadeh
Department of Electrical Engineering and Electronics Research Laboratory,
University of California, Berkeley, California

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.

1. INTRODUCTION

More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc., as its members, and clearly excludes such objects as rocks, fluids.

Our problem is to formalise reasoning about vague notions.
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The human body temperature can be specified

- by an expression like \textit{low, normal, slightly elevated, fever},
- by a rational number $T$. 

\begin{center}
\begin{tabular}{cccccccc}
35° & 36° & 37° & 38° & 39° & 40° & 41° & 42° \\
low & normal & slightly elevated & fever
\end{tabular}
\end{center}
More than one level of granularity

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- by a rational number $T$.

These notions are bound to different levels of granularity.

A notion bound to a refinable level of granularity is called vague.
Merging two levels of granularity

What if we want to use both coarse and fine notions? Fuzzy sets are a natural choice.
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The idea is to "lift up" coarse notions into the fine level. Fuzzy sets provide a (somewhat pragmatic) means of "embedding" the coarse level into a finer level.
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The resulting "multi-level" model

The practical procedure

- We choose a universe $W$ according to the finest level of granularity that we want to take into account.
- We model vague properties by fuzzy sets $u: W \rightarrow [0, 1]$. Logical combinations then correspond to operations on fuzzy sets. We define these operations pointwise.

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Luckily, there are experiments supporting certain “design choices”.

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- Using a universe of all fuzzy sets blurs the levels of granularity.
As regards the connectives

**Proposition 1**

Statements of truth-functional fuzzy logic would be easier interpretable if

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- and $\rightarrow$ occurred only on the outer-most level.
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F. Bou, À. García-Cerdaña, V. Verdú,
On two fragments with negation and without implication of the logic of residuated lattices, Arch. Math. Logic 2006.
Proposition 2

A logic of fuzzy sets might be seen as a logic related to vagueness if

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Logic of prototypes

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- We endow our universe $W$ with a similarity relation $s$.
- We decide about the prototypes $P \subseteq W$ of a vague property $\varphi$.

We model $\varphi$ by

$$u : W \to [0, 1], \ w \mapsto s(w, P),$$

that is, by means of the “distance to the prototypes”.

Most remarkable consequence

The “vertical” viewpoint on fuzzy sets is replaced by the “horizontal” viewpoint.
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LAE, the *Logic of Approximate Entailment* uses graded implications $\alpha \overset{d}{\Rightarrow} \beta$, interpreted in similarity spaces:

![Diagram of similarity spaces](image)

**Side remark**

LAE and related calculi offer a considerable potential in logical respects.

The formation of fuzzy sets is more appropriately standardised as follows.

- We endow our universe $W$ with a metric $d$.
- We decide about the prototypes $P^+ \subseteq W$ as well as the counterexamples $P^- \subseteq W$ of a vague property $\varphi$.

We model $\varphi$ by

$$u: W \to [0, 1], \quad w \mapsto \frac{d(w, P^-)}{d(w, P^+) + d(w, P^-)},$$

that is, by “linear interpolation”.


A logic of prototypes and counterexamples

\[ t = \frac{d(w, \tau^-)}{d(w, \tau^-) + d(w, \tau^+)} \]

\( \tau^- \): the counterexamples
("clearly not tall")

\( \tau^+ \): the prototypes
("clearly tall")

Sizes: "tall"
: the prototypes
("clearly tall")
: the counterexamples
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Benefits:

- Transparent transition from “coarse” to “fine”.
- Easy switching between granularities.
A logic of prototypes and counterexamples

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Problem:

- Although a logic of fuzzy sets, the approach is not compatible with the idea of truth-functionality.
- Defining a logic at all is a challenge.
We want to put fuzzy logic on firm conceptual grounds?
Vagueness again

We want to put fuzzy logic on firm conceptual grounds?
What about asking ourselves:

What do we actually want?

What is the prototypical example of a conclusion that a fuzzy logic should be able to reproduce?
The medical decision support system

**MONI (K.-P. Adlassnig, Vienna)**
The medical decision support system **MONI** (K.-P. Adlassnig, Vienna) applies fuzzy logic to rules like:

If

- the patient does not have fever ($\geq 38.0^\circ$) or urinary urgency or frequency or dysuria or suprapubic tenderness
- the patient has had a positive urine culture ($\geq 10^5$ microorganisms/cm$^3$ of $\leq 2$ species)
- the patient had an indwelling urinary catheter within 7 days before the culture

then possibly *asymptomatic* bacteriuria.
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In practice, the fine level of granularity is of primary relevance.
This is an **Aristotelian syllogism**:

$$\text{No } X \text{ are } M \quad \text{All } Y \text{ are } M \quad \frac{}{\text{No } X \text{ are } Y}.$$
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\[
\begin{align*}
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\hline
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\end{align*}
\]

This is a **generalised Aristotelian syllogism**:

\[
\begin{align*}
\text{Most } X \text{ are } M \quad \text{All } M \text{ are } Y \\
\hline
\text{Many } X \text{ are } Y
\end{align*}
\]


Th.V., Vagueness: where degree-based approaches are useful, and where we can do without, *Soft Computing* 2012.
Most $X$ are $M$  All $M$ are $Y$
\[
\frac{\text{Many } X \text{ are } Y}{.}
\]

- Reproducing the argument by means of a fuzzy logic is possible.
Most $X$ are $M$ \quad \text{All } M \text{ are } Y
\hline
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- Reproducing the argument by means of a fuzzy logic is possible.
- However, the reasoning does in general not involve the fine level.
Most $X$ are $M$  \hspace{1cm} \text{All } M \text{ are } Y
\rule{0pt}{0.5em}
\hspace{1cm} \text{Many } X \text{ are } Y$.

- Reproducing the argument by means of a fuzzy logic is possible.
- However, the reasoning does in general not involve the fine level.
  We may also reproduce the argument by modelling the coarse level only.
The theory TSR

Consider the language $\subseteq, \subset, \subset, \subset, \subset, \sim; \cap, \cup, \setminus; \emptyset$.

Let TSR extend the theory of generalised Boolean algebras by:

**Axioms for size, where $A \preccurlyeq B$ is $\exists C((A \sim C) \land (C \subseteq B))$:**

- $A \sim A$  
- $(A \sim B) \rightarrow (B \sim A)$  
- $(A \sim B) \land (B \sim C) \rightarrow (A \sim C)$  
- $(A \preccurlyeq B) \lor (B \preccurlyeq A)$  
- $(A \sim \emptyset) \leftrightarrow (A = \emptyset)$  
- $(A \sim B) \land (A \subseteq B) \rightarrow (A = B)$  
- $(A \sim C) \land (B \sim D) \land (A \cap B = \emptyset) \land (C \cap D = \emptyset) \rightarrow (A \cup B \sim C \cup D)$

**General axioms for proportions:**

- $(A \preccurlyeq B) \rightarrow (\emptyset \subset A) \land (A \subseteq B)$
- $(A \preccurlyeq C) \land (B \subseteq C) \land (A \sim B) \rightarrow (B \preccurlyeq C)$,

where $\preccurlyeq$ is one of $\subseteq, \subset, \subset, \subset, \subset$. 

where $\subset$ is one of $\subseteq, \subset, \subset, \subset, \subset$
Axioms for “few”:

\[
(\emptyset \subset A) \land (A \subseteq B) \land (B^{\text{few}} \subset C) \rightarrow (A^{\text{few}} \subset C)
\]

\[
(A^{\text{few}} \subset B) \land (B \subseteq C) \rightarrow (A^{\text{few}} \subset C)
\]

Axioms for “many”:

\[
(A^{\text{many}} \subset C) \land (A \subseteq B) \land (B \subseteq C) \rightarrow (B^{\text{many}} \subset C)
\]

\[
(A^{\text{many}} \subset C) \land (A \subseteq B) \land (B \subseteq C) \rightarrow (A^{\text{many}} \subset B)
\]

\[
(A^{\text{many}} \subset B) \rightarrow \neg(A^{\text{few}} \subset B)
\]

\[
(B^{\text{many}} \subset C) \land (B \subset C) \rightarrow \exists A ((A \subset B) \land (A^{\text{few}} \subset C))
\]

Axioms for “most”, where \( A \prec B \) is \( \exists C ((A \sim C) \land (C \subset B)) \):

\[
(A^{\text{most}} \subset B) \iff (A \subseteq B) \land (B \setminus A \prec A) \quad (A^{\text{most}} \subset B) \rightarrow (A^{\text{many}} \subset B)
\]

Axioms for “nearly all”:

\[
(A^{\text{n.a.}} \subset B) \iff (B \setminus A^{\text{few}} \subset B)
\]
The logic TSR: example

\[
\begin{align*}
\text{Most } X \text{ are } M & \quad \text{All } M \text{ are } Y \\
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\]

Lemma

From TSR we derive

\[
\frac{M \cap X \overset{\text{most}}{\subset} X \quad M \subseteq Y}{Y \cap X \overset{\text{many}}{\subset} X}.
\]

To formalise generalised Aristotelian syllogisms, we can – although this is uncommon – restrict to a coarse model.
Let us return to medicine.
Third possible field of application

Let us return to medicine.

Here is a fictitious report on an examination of the stomach:

**PROCEDURES PERFORMED:** Esophagogastroduodenoscopy with biopsies and colonoscopy.

**ESOPHAGOGASTRODUODENOSCOPY**

PROCEDURE IN DETAIL: The patient was given topical benzocaine spray and placed on the left lateral decubitus position. Following the administration of appropriate anesthesia, a diagnostic gastroscope was advanced under direct vision to the second portion of the duodenum without difficulty. The Z-line was regular and noted at approximately 38 cm from the incisors. Examination of the esophagus otherwise was endoscopically unremarkable. Patchy erythematous gastropathy was visualized, particularly in the distal portion of the stomach. No ulcers however were visualized. The stomach appeared to distend normally. Retroflexed views in the stomach did not reveal additional abnormalities. Multiple biopsies were obtained secondary to the findings above. Examination of the duodenum was endoscopically unremarkable though, secondary to the patient's history, biopsies were obtained for histology. No additional findings were noted as the scope was slowly withdrawn.

**IMPRESSION:** Erythematous gastropathy.
(...) The Z-line was regular and noted at approximately 38 cm from the incisors. (....)
Patchy erythematous gastropathy was visualized, particularly in the distal portion of the stomach. No ulcers however were visualized. The stomach appeared to distend normally. (....)
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Task: Represent this report such that the question “Has a suspicion of gastropathy been reported?” can be answered in an automated way.
(...) The Z-line was regular and noted at approximately 38 cm from the incisors. (...
Patchy erythematous gastropathy was visualized, particularly in the distal portion of the stomach. No ulcers however were visualized. The stomach appeared to distend normally. (...)

Varying coarse levels of granularity are involved. A fine level is not even applicable. Spatio-temporal aspects are predominant, but not the only ones. Efficient methods of representation are not available.
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There is an abundance of literature on the qualitative representation of facts in space and time.

<table>
<thead>
<tr>
<th>Relation</th>
<th>Symbol</th>
<th>Symbol for Inverse</th>
<th>Pictorial Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>X before Y</td>
<td>&lt;</td>
<td>&gt;</td>
<td>XXX YYYY</td>
</tr>
<tr>
<td>X equal Y</td>
<td>=</td>
<td>=</td>
<td>XXX YYYY</td>
</tr>
<tr>
<td>X meets Y</td>
<td>m</td>
<td>mi</td>
<td>XXXYYYY</td>
</tr>
<tr>
<td>X overlaps Y</td>
<td>o</td>
<td>oi</td>
<td>XXX YYYY</td>
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<tr>
<td>X during Y</td>
<td>d</td>
<td>di</td>
<td>XXXYYYY</td>
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<tr>
<td>X starts Y</td>
<td>s</td>
<td>si</td>
<td>XXX YYYY</td>
</tr>
<tr>
<td>X finishes Y</td>
<td>f</td>
<td>fi</td>
<td>XXX YYYY</td>
</tr>
</tbody>
</table>

**FIGURE 2. The Thirteen Possible Relationships.**

Allen’s time interval relations and the RCC8 spatial region relations.
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• There is a huge demand for tools supporting the representation of spatio-temporal facts in a qualitative way.
• The aspect of vagueness is no academic feature here. The use of fine-grained models is inappropriate, if possible at all.
• Whether the qualitative-reasoning community has coped with vagueness better than fuzzy logic, is doubtful. But an exchange of experiences might be useful.


R. Hirsch, M. Jackson, T. Kowalski, T. Niven, Algebraic foundations for qualitative calculi and networks, draft.
Proposal 3

We could

- review approaches to “qualitative reasoning” from a logical angle.
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- review approaches to “qualitative reasoning” from a logical angle.
- check to which extent many-valued logics could be useful or are successfully avoided in this field.