MANY-SORTED GAGGLES

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1. Gaggles

K. Bimbó and J.M. Dunn (2008).

Let L be a lattice, and let $o : L^n \mapsto L$ be an operation ($n \geq 1$).

We say that o has *the distribution type* $d_1, \ldots, d_n \to d_{n+1}$, where each $d_i \in \{\vee, \wedge\}$, if, for any $1 \leq i \leq n$:

$$o(a_1, \ldots, a_i d_i b_i, \ldots, a_n) =$$
$$o(a_1, \ldots, a_i, \ldots, a_n) d_{n+1} o(a_1, \ldots, b_i, \ldots, a_n)$$

Example

Let $L = (L, \vee, \wedge, \cdot, \backslash, /, 1)$ be a residuated lattice. The residuation laws

$$a \cdot b \leq c \text{ iff } b \leq a\backslash c \text{ iff } a \leq c/b$$

determine the following distributions types:

$$\cdot : \vee, \vee \to \vee, \;\; \backslash : \vee, \wedge \to \wedge, \;\; / : \wedge, \vee \to \wedge$$

Distribution types determine *tonicity types* $(t_1, \ldots, t_n)$, where each $t_i \in \{\uparrow, \downarrow\}$. An operation o of the tonicity type $(t_1, \ldots, t_n)$ is isotone (resp. antitone) in $i-$th argument, if $t_i = \uparrow$ (resp. $t_i = \downarrow$). For the above example:

$\cdot : (\uparrow, \uparrow), \quad \backslash : (\downarrow, \uparrow), \quad / : (\uparrow, \downarrow)$

If L contains bounds $\bot, \top$, then the $\bot, \top-$laws *associated* with distribution types are as follows:

$$o : d_1, \ldots, \vee, \ldots, d_n \to \vee \quad o(a_1, \ldots, \bot, \ldots, a_n) = \bot$$

$$o : d_1, \ldots, \wedge, \ldots, d_n \to \vee \quad o(a_1, \ldots, \top, \ldots, a_n) = \bot$$

$$o : d_1, \ldots, \vee, \ldots, d_n \to \wedge \quad o(a_1, \ldots, \bot, \ldots, a_n) = \top$$

$$o : d_1, \ldots, \wedge, \ldots, d_n \to \wedge \quad o(a_1, \ldots, \top, \ldots, a_n) = \top$$

For the example:

$$\bot \cdot a = \bot = a \cdot \bot, \;\; a\backslash\top = \top = \top/a, \;\; \bot\backslash a = \top = a/\bot$$

Two operations $o_1, o_2 : L^n \mapsto L$ can be connected by one of the following *residuation laws*.

(r1) $o_1(a_1, \ldots, b[i], \ldots, a_n) \leq c$ iff $b \leq o_2(a_1, \ldots, c[i], \ldots, a_n)$

(r2) $o_1(a_1, \ldots, b[i], \ldots, a_n) \leq c$ iff $o_2(a_1, \ldots, c[i], \ldots, a_n) \leq b$

(r3) $c \leq o_1(a_1, \ldots, b[i], \ldots, a_n)$ iff $b \leq o_2(a_1, \ldots, c[i], \ldots, a_n)$

Here $b[i]$ (resp. $c[i]$) means that b (resp. c) is the $i-$th argument.

According to (Bimbó and Dunn 2008), *a multi-gaggle* is an ordered algebra L supplied with operations $(o_j)_{j \in J}$, having distribution (tonicity) types; in some cases, one also assumes the associated $\bot, \top-$laws. *A gaggle* is a multi-gaggle such that any two operations o_i, o_j can be connected by the reflexive and transitive closure of the relation of 'being connected by a residuation law'.

The ordered algebras can be boolean algebras (boolean gaggles), lattices (nondistributive gaggles), distributive lattices (distributive gaggles), posets (partial gaggles), and others.

It seems reasonable to modify the above definitions. One fixes a set $\mathcal{R}$ of residuation laws for (names of) some operations o_j. A multi-gaggle can be formally defined as a triple $(L, \Omega, \mathcal{R})$ such that L is an ordered algebra, $\Omega = (o_j)_{j \in J}$ is a family of operations on L, and all laws from $\mathcal{R}$ are satisfied.

A gaggle is a multi-gaggle $(L, \Omega, \mathcal{R})$ such that any two operations from Ω are connected by the reflexive and transitive closure of the relation of ‘being connected by a law from $\mathcal{R}$’.

If the underlying ordered algebras (semi-lattices, at least) form a variety, then the class of all (multi-)gaggles based on these algebras with a fixed similarity type of Ω and a fixed $\mathcal{R}$ is a variety.

According to (Buszkowski 2011), *a residuated algebra* is an ordered algebra L with operations $(o_j)_{j\in J}$ and their residual operations o_j/i, $1 \leq i \leq n(j)$, satisfying (r1) for $o_1 = o_j$ and $o_2 = o_j/i$, for any $1 \leq i \leq n(j)$. It means: each *basic* operation o_j admits $n(j)$ residual operations o_j/i, satisfying (r1).

Clearly residuated algebras are certain multi-gaggles.

A residuated lattice is an example of a residuated algebra with the basic operation $\cdot$ and its residual operations $\cdot/1 = /$, $\cdot/2 = \backslash$.

In a residuated algebra, each basic operation o has the distribution type $\vee, \ldots, \vee \to \vee$, and its residual o/i has the type $\vee, \ldots, \wedge[i], \ldots, \vee \to \wedge$, and the associated $\bot, \top-$laws are fulfilled (if the bounds exist); for posets, tonicity types are used instead.

In general, each law (r1)-(r3) determines d_i and d_{n+1} in the distribution types of o_1, o_2, but the other d_j remain unspecified. Nonetheless, the distribution type of o_1 determines that of o_2, and conversely.

2. Many-sorted gaggles

We define *a many-sorted multi-gaggle* as a family of ordered algebras $(L_s)_{s \in S}$ supplied with operations $(o_j)_{j \in J}$ such that each operation $o_j : L_{j(1)} \times \cdots \times L_{j(n(j))} \mapsto L_{j(n(j)+1)}$, where all $j(i) \in S$, has a distribution-tonicity type (optionally: some $\bot, \top-$laws are satisfied). We also assume that a fixed set $\mathcal{R}$ of residuation laws must be satisfied (see below).

For instance, if all L_s are boolean algebras, then we use distribution types and assume $\bot, \top-$laws, and similarly for bounded distributive lattices. If all L_s are posets, then we use tonicity types and do not assume $\bot, \top-$laws.

We admit the case that some L_s are boolean algebras, some other are (bounded) distributive lattices, and the remaining ones are posets. Then, distribution-tonicity types are involved, as e.g. $\uparrow, \vee \rightarrow \wedge$; an operation o of this type is isotone in the first argument and satisfies: $o(a, b \vee b') = o(a, b) \wedge o(a, b')$.

Many-sorted residuation laws:

(R1) $o_1 : L_1 \times \cdots \times L_n \mapsto L$, $o_2 : L_1 \times \cdots \times L[i] \times \cdots \times L_n \mapsto L_i$

$o_1(a_1, \ldots, b[i], \ldots, a_n) \leq_L c$ iff $b \leq_{L_i} o_2(a_1, \ldots, c[i], \ldots, a_n)$

(R2) o_1, o_2 as above.

$o_1(a_1, \ldots, b[i], \ldots, a_n) \leq_L c$ iff $o_2(a_1, \ldots, c[i], \ldots, a_n) \leq_{L_i} b$

(R3) o_1, o_2 as above.

$c \leq_L o_1(a_1, \ldots, b[i], \ldots, a_n)$ iff $b \leq_{L_i} o_2(a_1, \ldots, c[i], \ldots, a_n)$

We define *a many-sorted gaggle* (based on $\mathcal{R}$) as a many-sorted multi-gaggle (based on $\mathcal{R}$) such that any two operations o_i, o_j are connected by the reflexive and transitive closure of the relation of 'being connected by a law from $\mathcal{R}$'.

Many notions and results for (multi-)gaggles can easily be generalized for the many-sorted versions.

3. Example: canonical embeddings

Given a frame (W, R) such that $R \subseteq W^{n+1}$, one defines an infinitely additive operation $o^R : P(W)^n \mapsto P(W)$:

$o^R(X_1, \ldots, X_n) = \{\gamma \in W : (\exists \alpha_1 \in X_1, \ldots, \alpha_n \in X_n)R(\alpha_1, \ldots, \alpha_n, \gamma)\}$

The operation o^R admits n residual operations $(o^R/i)(X_1, \ldots, X_n) =$

$= \{\alpha_i \in W : (\forall j \neq i)(\forall \alpha_j \in X_j)(\forall \gamma \in W)(R(\alpha_1, \ldots, \alpha_n, \gamma) \Rightarrow \gamma \in X_i)\}$,

which satisfy (r1).

The Jónsson-Tarski theorem states that every BAO A (a boolean algebra with a family of additive operations, satisfying $\bot-$laws) is embeddable in the canonical BAO $P(W)$; here $W = F(A)$ is the set of all ultrafilters in A, and an additive operation o from A determines the operation o^R as above, where the canonical accessibility relation R is defined as follows:

$R(\alpha_1, \ldots, \alpha_n, \gamma)$ iff $o^P(\alpha_1, \ldots, \alpha_n) \subseteq \gamma$.

Here $o^P(X_1, \ldots, X_n) = \{o(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}$ is the powerset operation on $P(A)$ determined by o. The canonical embedding $v : A \mapsto P(F(A))$ is given by: $v(a) = \{\gamma \in F(A) : a \in \gamma\}$.

In (Bimbó and Dunn 2008) this theorem is generalized for different (multi-)gaggles, e.g. boolean, distributive, partial. The authors prove that the canonical embedding preserves operations having arbitrary distribution (tonicity) types and preserves residuals (precisely, if a residuation law holds in A, then it holds in $P(F(A))$ for the corresponding operations).

For boolean gaggles, $F(A)$ consists of ultrafilters on A; for distributive lattices, $F(A)$ consists of prime filters on A, and for posets, $F(A)$ consists of all upsets.

This task can be performed in a more uniform and elegant way, if one applies many-sorted gaggles. Furthermore, analogous results can be obtained for many-sorted (multi-)gaggles.

Many-sorted frames are tuples $(W_1, \ldots, W_{n+1}, R)$ such that $R \subseteq W_1 \times \cdots \times W_{n+1}$.

The additive operation $o^R : P(W_1) \times \cdots \times P(W_n) \mapsto P(W_{n+1})$ can be defined as above, and similarly for the residual operations (o^R/i), $1 \leq i \leq n$.

Clearly the family $(P(W_i))_{1 \leq i \leq n+1}$ with operations $o^R, o^R/1, \ldots, o^R/n$ and the set of the corresponding residuation laws (R1) is a many-sorted gaggle (residuated algebra).

Let $A = ((A_s)_{s \in S}, (o^a_j)_{j \in J})$ and $B = ((B_s)_{s \in S}, (o^b_j)_{j \in J})$ be two many-sorted multi-gaggles of the same similarity type (the set of residuation laws is not essential). *A homomorphism* $h : A \mapsto B$ is a family $(h_s)_{s \in S}$ of homomorphisms $h_s : A_s \mapsto B_s$ which preserves all operations o_j. Precisely, if $o^a_j : A_{s_1} \times \cdots \times A_{s_n} \mapsto A_s$ (hence $o^b_j : B_{s_1} \times \cdots \times B_{s_n} \mapsto B_s$) then:

$h_s(o^a_j(a_1, \ldots, a_n)) = o^b_j(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$, for all $a_i \in A_{s_i}$.

We consider a special case of a bounded distributive lattice L with an operation $\cdot$ and its residuals $\backslash$, $/$, satisfying:

$a \cdot b \leq c$ iff $b \leq a \backslash c$ iii $a \leq c/b$.

We have $v(a \cdot b) = v(a) \cdot^{R(\cdot)} v(b)$, which can be proved like the Jónsson-Tarski theorem. Recall the proof of $\subseteq$. Assume $\gamma \in v(a \cdot b)$. Then $a \cdot b \in \gamma$. The operation $\cdot$ is additive (type $\vee, \vee \to \vee$), hence $a^{\uparrow} \cdot^{P} b^{\uparrow} \subseteq \gamma$. Using the maximality principle, one finds prime filters α, β such that $a^{\uparrow} \subseteq \alpha$, $b^{\uparrow} \subseteq \beta$ and $\alpha \cdot^{P} \beta \subseteq \gamma$. Consequently, there exist $\alpha \in v(a), \beta \in v(b)$ such that $R(\cdot)(\alpha, \beta, \gamma)$, which yields $\gamma \in v(a) \cdot^{R(\cdot)} v(b)$. The proof of the converse inclusion is easier.

We want to prove $v(c/b) = v(c)/' v(b)$. Here $/'$ is the (left) residual operation for $\cdot^{R(\cdot)}$. We cannot copy the above argument, since $/ : \wedge, \vee \to \wedge$. We transform $/$ into an additive operation $/_1 : L^{\partial} \times L \mapsto L^{\partial}$ such that $/ = /_1$ as functions. L^{∂} denotes the lattice dual to L. $/_1$ satisfies $\bot-$laws.

By copying the above argument, one shows:

$v_{L^\partial}(c/b) = v_{L^\partial}(c)/_1^{R(/_1)} v_L(b)$

Clearly $v_{L^\partial}(a) = \{\gamma \in F(L^\partial) : a \in \gamma\}$ consists of all prime ideals of L which contain a.

For $X \subseteq L$, we define $\sim X = L - X$.

For $W \subseteq P(L)$, we define $W^\sim = \{\sim X : X \in W\}$.

For $U \subseteq F(L)$, we define $U^* = \sim_{F(L^\partial)} (U^\sim) = (\sim_{F(L)} U)^\sim$.

For $V \subseteq F(L^\partial)$, we define ${}^*V = \sim_{F(L)} (V^\sim) = (\sim_{F(L^\partial)} V)^\sim$.

Here $\sim_{F(L)} U = F(L) - U$, $\sim_{F(L^\partial)} V = F(L^\partial) - V$.

Clearly $(-)^* : P(F(L)) \mapsto P(F(L^\partial))$, ${}^*(-) : P(F(L^\partial)) \mapsto P(F(L))$. We have ${}^*(U^*) = U$, $({}^*V)^* = V$.

Also $v_{L^\partial}(a) = v_L(a)^*$, which yields:

$v_L(c/b) = {}^*(v_L(c)^* /_1^{R(/_1)} v_L(b))$

It remains to show that $U_1/'U_2 = {}^*(U_1^* /_1^{R(/_1)} U_2)$, for any $U_1, U_2 \subseteq F(L)$. For $\gamma \in F(L)$, we have:

$\gamma \in {}^*(U_1^* /_1^{R(/_1)} U_2)$ iff
$\sim \gamma \notin U_1^* /_1^{R(/_1)} U_2$ iff
$\neg(\exists \alpha \in F(L^\partial), \beta \in F(L))(\alpha \in U_1^* \,\&\, \beta \in U_2 \,\&\, R(/_1)(\alpha, \beta, \sim \gamma))$ iff
$\neg(\exists \alpha \in F(L^\partial), \beta \in F(L))(\alpha \in U_1^* \,\&\, \beta \in U_2 \,\&\, \alpha /_1^P \beta \subseteq \sim \gamma)$ iff
$(\forall \alpha \in F(L^\partial), \beta \in F(L))(\beta \in U_2 \,\&\, \alpha /_1^P \beta \subseteq \sim \gamma \Rightarrow \alpha \notin U_1^*)$ iff
$(\forall \alpha \in F(L^\partial), \beta \in F(L))(\beta \in U_2 \,\&\, \alpha /_1^P \beta \subseteq \sim \gamma \Rightarrow \sim \alpha \in U_1)$ iff
$(\forall \alpha, \beta \in F(L))(\beta \in U_2 \,\&\, (\sim \alpha) /_1^P \beta \subseteq \sim \gamma \Rightarrow \alpha \in U_1)$ iff
$\gamma \in U_1/'U_2$,

since

$(\sim \alpha) /_1^P \beta \subseteq \sim \gamma$ iff $\gamma \cdot^P \beta \subseteq \alpha$ iff $R(\cdot)(\gamma, \beta, \alpha)$

$(\sim\alpha)/_1^P\beta \subseteq \sim\gamma$ iff
$(\forall x, y, z \in L)(x \in \sim\alpha \,\&\, y \in \beta \,\&\, x/y \leq_{L^\partial} z \Rightarrow z \in \sim\gamma)$ iff
$(\forall x, y, z \in L)(z \in \gamma \,\&\, y \in \beta \,\&\, z \leq_L x/y \Rightarrow x \in \alpha)$ iff
$(\forall x, y, z \in L)(z \in \gamma \,\&\, y \in \beta \,\&\, z \cdot y \leq_L x \Rightarrow x \in \alpha)$ iff
$\gamma \cdot^P \beta \subseteq \alpha$

In this way one shows that the canonical embedding preserves residual operations satisfying (r1). One generalizes this result for many-sorted multi-gaggles in an obvious way. Other residuation laws can be reduced to (R1) by analogous transformations. For instance, (R2) is reduced to (R1), if one replaces L_i with L_i^∂.

For many-sorted boolean multi-gaggles, these transformations can be performed easier, using the internal boolean negations. For instance, an operation $o : \wedge, \vee \to \wedge$ is transformed into an additive operation $o_1 : \vee, \vee \to \vee$ by setting $o_1(a, b) = -o(-a, b)$.

4. Many-sorted residuation logics

GL Generalized Lambek Calculus; (W.B. 2011) and earlier papers.

Groupoid Logic - with one binary operation and its residuals, satisfying (r1).

Formulae: variables, $o(A_1, \ldots, A_n)$, $(o/i)(A_1, \ldots, A_n)$, for $1 \leq i \leq n$; optionally: $A \vee B$, $A \wedge B$, $-A$, $\bot$, $\top$.

Formula-structures: formulae, $(X_1, \ldots, X_n)_o$.

Sequents: $X \Rightarrow A$, where X is a structure and A is a formula.

Axioms: (Id) $A \Rightarrow A$, (0) $()_o \Rightarrow o$.

Rules:

$$(o \Rightarrow)\ \frac{X[(A_1, \ldots, A_n)_o] \Rightarrow A}{X[o(A_1, \ldots, A_n)] \Rightarrow A}$$

$$(\Rightarrow o)\ \frac{X_1 \Rightarrow A_1; \ldots; X_n \Rightarrow A_n}{(X_1, \ldots, X_n)_o \Rightarrow o(A_1, \ldots, A_n)}$$

$$(o/i \Rightarrow)\ \frac{X[A_i] \Rightarrow B; (Y_j \Rightarrow A_j)_{j \neq i}}{X[(Y_1, \ldots, (o/i)(A_1, \ldots, A_n), \ldots, Y_n)_o] \Rightarrow B}$$

$$(\Rightarrow o/i)\ \frac{(A_1, \ldots, X, \ldots, A_n)_o \Rightarrow A_i}{X \Rightarrow (o/i)(A_1, \ldots, A_n)}$$

$$(\vee \Rightarrow)\ \frac{X[A] \Rightarrow C;\ X[B] \Rightarrow C}{X[A \vee B] \Rightarrow C}$$

$$(\Rightarrow \vee)\ \frac{X \Rightarrow A_i}{X \Rightarrow A_1 \vee A_2},\ \text{for } i = 1 \text{ and } i = 2$$

$$(\wedge \Rightarrow)\ \frac{X[A_i] \Rightarrow B}{X[A_1 \wedge A_2] \Rightarrow B},\ \text{for } i = 1 \text{ and } i = 2$$

$$(\Rightarrow \wedge)\ \frac{X \Rightarrow A;\ X \Rightarrow B}{X \Rightarrow A \wedge B}$$

$$(\bot \Rightarrow)\ X[\bot] \Rightarrow A,\ (\Rightarrow \top)\ X \Rightarrow \top$$

$$(\mathrm{CUT})\ \frac{X[A] \Rightarrow B;\ Y \Rightarrow A}{X[Y] \Rightarrow B}$$

GL is strongly complete w.r.t. the class of residuated algebras.

Distributive **GL** (**DGL**) admits the distributive axiom:

$$(\mathrm{D})\ A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$$

Boolean **GL** (**BGL**) admits (D) and the axioms:

$$(-1)\ A \wedge (-A) \Rightarrow \bot,\ (-2)\ \top \Rightarrow A \vee (-A)$$

GL admits cut-elimination. It does not hold for **DGL**, **BGL**, although cut-free sequent systems for these logics exist (they employ more involved structures).

Many-sorted **GL** will be denoted by **msGL**. We fix a set of basic operation symbols.

We fix a (finite) set S of types. We assign types from S to variables $p : s$ and function types to operation symbols $o : s_1, \ldots, s_n \rightarrow s$.

The types od operations o and o/i must satisfy the condition: $o : s_1, \ldots, s_n \rightarrow s$ iff $o/i : s_1, \ldots, s[i], \ldots, s_n \rightarrow s_i$.

Now, types are assigned to formulae and formula-structures.

If $A_1 : s_1, \ldots, A_n : s_n$ and $o : s_1, \ldots, s_n \rightarrow s$ then $o(A_1, \ldots, A_n) : s$.

If $A : s$ and $B : s$ then $A \vee_s B : s$, $A \wedge_s B : s$ and $-_s A : s$. We may assume $\bot_s : s$ and $\top_s : s$.

If $X_1 : s_1, \ldots, X_n : s_n$ and $o : s_1, \ldots, s_n \rightarrow s$ then $(X_1, \ldots, X_n)_o : s$.

Sequents are of the form $X \Rightarrow^s A$, where $X : s$ and $A : s$.

The axioms and the rules are the same as for **GL**, restricted to correctly typed formulae and sequents.

msGL is strongly complete with respect to many-sorted residuated algebras. It admits cut-elimination. The consequence relation is decidable; it is undecidable, if at least one binary operation is associative. Without lattice operations the consequence relation of **msGL** is polynomial. The proofs are as in (W.B. 2005), (W.B. 2011) for **GL**.

With lattice (boolean) operations the consequence relation is PSPACE-hard (if at least one o is binary). The pure **msGL** (without (D)) is PSPACE.

Horčik and Terui (2011) prove the PSPACE-hardness of many substructural logics with a binary associative operation $\cdot$ and its residuals \, /, but the proof essentially uses associativity. Without associativity, I can only prove the PSPACE-hardness of the consequence relation.

An interpolation lemma, proved for **GL** without lattice, **DGL** and **BGL** in (W.B. 2005, 2011), also holds for **msGL**.

INTERPOLATION LEMMA (W.B. 2005, 2011). Let Φ be a finite set of assumptions (certain sequents). Let T be a finite set of formulae, containing all formulae occurring in Φ and $X \Rightarrow A$ and being closed under subformulae. If $\Phi \vdash X[Y] \Rightarrow A$ then there is $D \in \overline{T}$ such that $\Phi \vdash Y \Rightarrow D$ and $\Phi \vdash X[D] \Rightarrow A$.

Without lattice operations $\overline{T} = T$; with lattice operations, $\overline{T}$ is the closure of T under $\vee, \wedge$ (if $-$ is present, it is also closed under $-$).

For **GL** without lattice, **DGL**, **BGL**, the set $\overline{T}$ is finite up to the deductive equivalence. The interpolation lemma implies the finiteness of the family of basic closed sets $[X[\ ], B]$ used in some standard constructions of models (by nuclear completion). This yields the strong finite model property of these systems, and consequently, the finite embeddability property of the corresponding classes of algebras (Farulewski 2008, W.B. 2011).

The same results can be obtained for many-sorted residuated algebras and many-sorted multi-gaggles.

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