Full Lambek Calculus with Contraction is Undecidable

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&
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Weakening
$$\frac{\ldots, \ldots \Rightarrow B}{\ldots, A, \ldots \Rightarrow B}$$

$$\frac{\ldots \Rightarrow}{\ldots \Rightarrow A}$$

A DRUNKARD'S PROGRESS

TIPSY

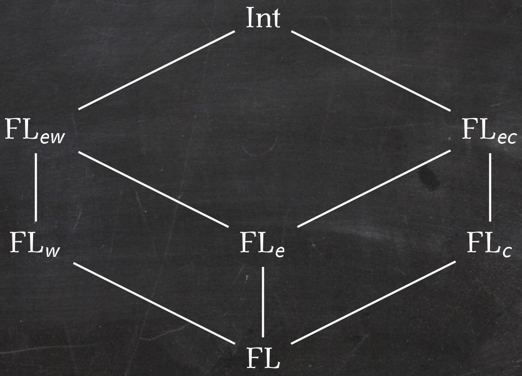
DRUNKEN

LEGLESS

DRUNK

(BY STRICT NAUTICAL STANDARD): immobile

Basic substructural logics



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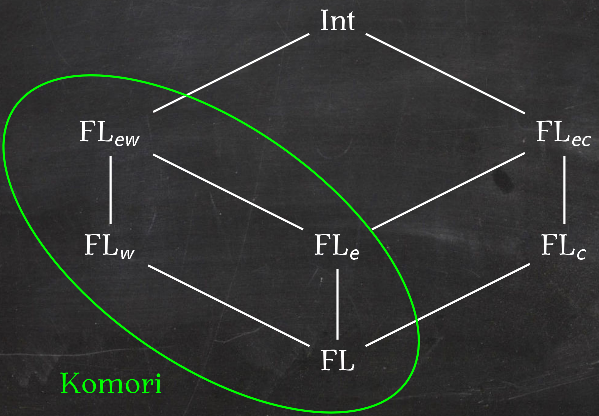

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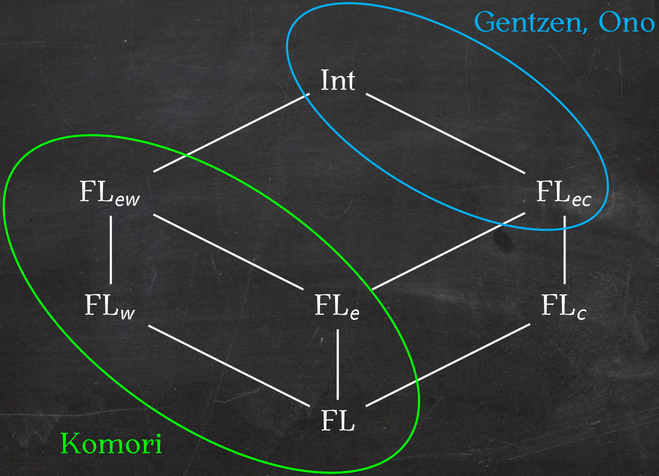

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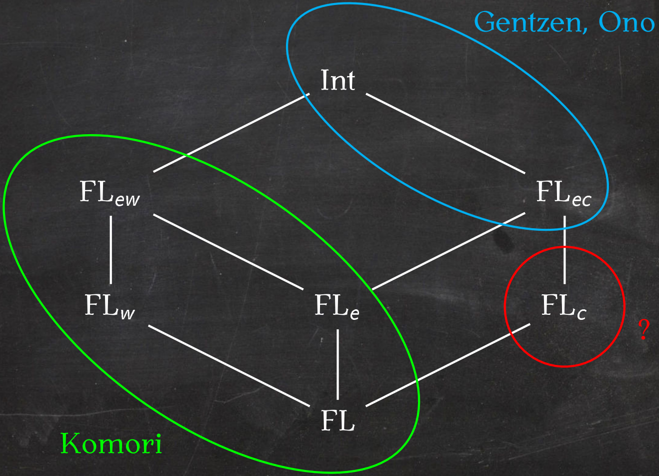

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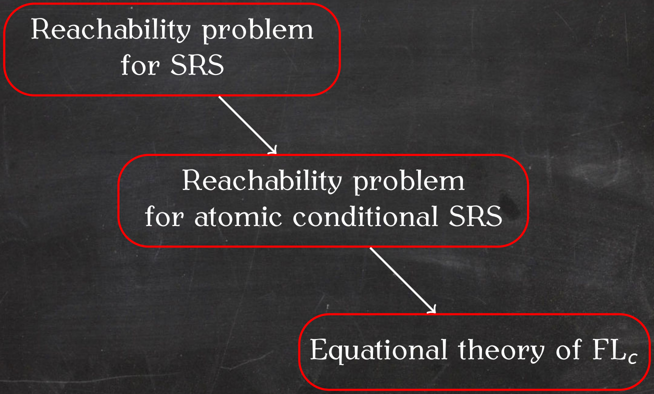
Facts:

1. $\langle A, \wedge, \vee \rangle$ – lattice
2. $\langle A, \cdot, 1 \rangle$ – monoid
3. $a \leq a^2$
4. $a(a \backslash b) \leq b$
5. $a(b \vee c)d = abd \vee acd$

Strategy


Reachability problem for SRS
Reachability problem for atomic conditional SRS
Equational theory of FL_c

String rewriting systems (SRS)

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Theorem [RH]:

There is an SRS $\langle \Sigma, R \rangle$ and $w_0 \in \Sigma^*$ such that

$$L(w_0) = \{w \in \Sigma^* \mid w \to_R^* w_0\}$$

is undecidable and $L(w_0)$ consists only of square-free words.

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If all the rules are of the form $\langle x \to a, L_\ell, L_r \rangle$ for $a \in \Sigma$, we call the CSRS atomic.

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$aba \to_R aa \to_R b$ is simulated by

$$(aba)^\theta \leq ab\theta a\theta \leq ab(ab\backslash a)\theta a\theta \leq a\theta a\theta \leq aa(aa\backslash b) \leq b$$

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Main result

Theorem:

The equational theory of FL_c-algebras is undecidable.

Corollary:

The set of provable formulas in FL_c is undecidable.

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2. We can also eliminate 1 and multiplication.
3. Our undecidability proof can be modified for $x^m \leq x^n$ for $1 \leq m < n$.
4. Algorithmic deduction theorem:

   Let $T \cup \{A\}$ be a finite set of formulae. Then there is an algorithm which produces a formula B (given the input T and A) such that $\vdash_{FL_c} B$ iff $T \vdash_{FL_c} A$.

Thank you!