Cancellative residuated lattices arising on 2-generated submonoids of natural numbers

Rostislav Horčík

Institute of Computer Science
Academy of Sciences of the Czech Republic

Lattice-Valued Logic and its Applications
31st Linz Seminar
February 9–13, 2010

Motivation

- There are only two atoms in the subvariety lattice of cancellative residuated lattices (Galatos 2005).

Motivation

- There are only two atoms in the subvariety lattice of cancellative residuated lattices (Galatos 2005).
- The first one is the variety $\mathcal{V}(\mathbf{Z})$ of Abelian ℓ-groups and the second one is the variety $\mathcal{V}(\mathbf{Z}^{-})$ of negative cones of Abelian ℓ-groups.

Motivation

- There are only two atoms in the subvariety lattice of cancellative residuated lattices (Galatos 2005).
- The first one is the variety $\mathcal{V}(\mathbf{Z})$ of Abelian ℓ-groups and the second one is the variety $\mathcal{V}(\mathbf{Z}^-)$ of negative cones of Abelian ℓ-groups.
- What is above $\mathcal{V}(\mathbf{Z}^-)$? It is known that $\mathcal{V}(\mathbf{Z})$ has uncountably many covers (Holand 1994) and therefore $\mathcal{V}(\mathbf{Z}^-)$ as well.

Motivation

- There are only two atoms in the subvariety lattice of cancellative residuated lattices (Galatos 2005).
- The first one is the variety $\mathcal{V}(\mathbf{Z})$ of Abelian ℓ-groups and the second one is the variety $\mathcal{V}(\mathbf{Z}^-)$ of negative cones of Abelian ℓ-groups.
- What is above $\mathcal{V}(\mathbf{Z}^-)$? It is known that $\mathcal{V}(\mathbf{Z})$ has uncountably many covers (Holand 1994) and therefore $\mathcal{V}(\mathbf{Z}^-)$ as well.
- The question is what happens if we restrict the covers to satisfy other well-known identities like commutativity, integrality, representability.

Motivation (cont.)

- In order to obtain algebras generating covers of $\mathcal{V}(\mathbf{Z}^-)$,
  I considered several 2-generated submonoids of $\mathbf{Z}^-$ and found countably many commutative, integral and representable covers of $\mathcal{V}(\mathbf{Z}^-)$.

Motivation (cont.)

- In order to obtain algebras generating covers of $\mathcal{V}(\mathbf{Z}^-)$, I considered several 2-generated submonoids of $\mathbf{Z}^-$ and found countably many commutative, integral and representable covers of $\mathcal{V}(\mathbf{Z}^-)$.
- In this talk I will consider all varieties generated by any 2-generated submonoid of $\mathbf{Z}^-$ and describe which of them cover $\mathcal{V}(\mathbf{Z}^-)$.

Motivation (cont.)

- In order to obtain algebras generating covers of $\mathcal{V}(\mathbf{Z}^-)$, I considered several 2-generated submonoids of $\mathbf{Z}^-$ and found countably many commutative, integral and representable covers of $\mathcal{V}(\mathbf{Z}^-)$.

- In this talk I will consider all varieties generated by any 2-generated submonoid of $\mathbf{Z}^-$ and describe which of them cover $\mathcal{V}(\mathbf{Z}^-)$.

- Moreover, I will characterize the inclusion relation among these varieties (i.e., the order in the subvariety lattice).

Motivation (cont.)

- In order to obtain algebras generating covers of $\mathcal{V}(\mathbf{Z}^-)$, I considered several 2-generated submonoids of $\mathbf{Z}^-$ and found countably many commutative, integral and representable covers of $\mathcal{V}(\mathbf{Z}^-)$.
- In this talk I will consider all varieties generated by any 2-generated submonoid of $\mathbf{Z}^-$ and describe which of them cover $\mathcal{V}(\mathbf{Z}^-)$.
- Moreover, I will characterize the inclusion relation among these varieties (i.e., the order in the subvariety lattice).
- In order to apply some facts from elementar number theory, I will work in the dual (term-wise equivalent) setting and consider 2-generated sumonoids of natural numbers.

Algebras of interest

- A dual integral commutative residuated lattice (dICRL) is an algebra $\mathbf{A} = (A, \wedge, \vee, +, \dot{-}, 0)$, where the following conditions are satisfied:
  - $(A, +, 0)$ is a commutative monoid,
  - $(A, \wedge, \vee, 0)$ is a lattice with a bottom element 0,
  - $x + y \geq z$ iff $x \geq z \dot{-} y$.

Algebras of interest

- A dual integral commutative residuated lattice (dICRL) is an algebra $\mathbf{A} = (A, \wedge, \vee, +, \dot{-}, 0)$, where the following conditions are satisfied:
  - $(A, +, 0)$ is a commutative monoid,
  - $(A, \wedge, \vee, 0)$ is a lattice with a bottom element 0,
  - $x + y \geq z$ iff $x \geq z \dot{-} y$.

- A dICRL is cancellative if $+$ is cancellative; $(x + y) \dot{-} y = x$.

Algebras of interest

- A dual integral commutative residuated lattice (dICRL) is an algebra $\mathbf{A} = (A, \wedge, \vee, +, \dot{-}, 0)$, where the following conditions are satisfied:
  - $(A, +, 0)$ is a commutative monoid,
  - $(A, \wedge, \vee, 0)$ is a lattice with a bottom element 0,
  - $x + y \geq z$ iff $x \geq z \dot{-} y$.
- A dICRL is cancellative if $+$ is cancellative; $(x + y) \dot{-} y = x$.
- A dICRL is divisible if it satisfies $x \vee y = (y \dot{-} x) + x$.

Algebras of interest

- A dual integral commutative residuated lattice (dICRL) is an algebra $\mathbf{A} = (A, \wedge, \vee, +, \dot{-}, 0)$, where the following conditions are satisfied:
  - $(A, +, 0)$ is a commutative monoid,
  - $(A, \wedge, \vee, 0)$ is a lattice with a bottom element 0,
  - $x + y \geq z$ iff $x \geq z \dot{-} y$.
- A dICRL is cancellative if $+$ is cancellative; $(x + y) \dot{-} y = x$.
- A dICRL is divisible if it satisfies $x \vee y = (y \dot{-} x) + x$.
- $\mathbf{N} = \langle \mathbb{N}, \min, \max, +, \dot{-}, 0 \rangle$, where $y \dot{-} x = \max(y - x, 0)$, is a cancellative divisible dICRL.

Algebras of interest

- A dual integral commutative residuated lattice (dICRL) is an algebra $\mathbf{A} = (A, \wedge, \vee, +, \dot{-}, 0)$, where the following conditions are satisfied:
  - $(A, +, 0)$ is a commutative monoid,
  - $(A, \wedge, \vee, 0)$ is a lattice with a bottom element 0,
  - $x + y \geq z$ iff $x \geq z \dot{-} y$.
- A dICRL is cancellative if $+$ is cancellative; $(x + y) \dot{-} y = x$.
- A dICRL is divisible if it satisfies $x \vee y = (y \dot{-} x) + x$.
- $\mathbf{N} = \langle \mathbb{N}, \min, \max, +, \dot{-}, 0 \rangle$, where $y \dot{-} x = \max(y - x, 0)$, is a cancellative divisible dICRL.
- $\mathbf{N}$ is the term-wise equivalent dual of $\mathbf{Z}^{-}$.

Algebras of interest

- A dual integral commutative residuated lattice (dICRL) is an algebra $\mathbf{A} = (A, \wedge, \vee, +, \dot{-}, 0)$, where the following conditions are satisfied:
  - $(A, +, 0)$ is a commutative monoid,
  - $(A, \wedge, \vee, 0)$ is a lattice with a bottom element 0,
  - $x + y \geq z$ iff $x \geq z \dot{-} y$.

- A dICRL is cancellative if $+$ is cancellative; $(x + y) \dot{-} y = x$.
- A dICRL is divisible if it satisfies $x \vee y = (y \dot{-} x) + x$.
- $\mathbf{N} = \langle \mathbb{N}, \min, \max, +, \dot{-}, 0 \rangle$, where $y \dot{-} x = \max(y - x, 0)$, is a cancellative divisible dICRL.
- $\mathbf{N}$ is the term-wise equivalent dual of $\mathbf{Z}^{-}$.

Theorem (Blok, Ferreirim)

Let $\mathbf{A}$ *be a totally ordered cancellative dICRL. Then* $\mathbf{A} \in \mathcal{V}(\mathbf{N})$ *iff* $\mathbf{A}$ *is divisible.*

Submonoids of $\mathbb{N}$

- Let $a_1, \ldots, a_k \in \mathbb{N}$. Then $M(a_1, \ldots, a_k) = \{\sum_{i=1}^{k} n_i a_i \mid n_i \in \mathbb{N}\}$ denotes the subuniverse of $\langle \mathbb{N}, +, 0 \rangle$ generated by $\{a_1, \ldots, a_k\}$.

Submonoids of $\mathbb{N}$

- Let $a_1, \ldots, a_k \in \mathbb{N}$. Then $M(a_1, \ldots, a_k) = \{\sum_{i=1}^{k} n_i a_i \mid n_i \in \mathbb{N}\}$ denotes the subuniverse of $\langle \mathbb{N}, +, 0 \rangle$ generated by $\{a_1, \ldots, a_k\}$.
- If $a_1, \ldots, a_n$ are coprime, then $\mathbb{N} \setminus M(a_1, \ldots, a_k)$ is always finite and the greatest number which does not belong to $M(a_1, \ldots, a_k)$ is called the Frobenius number. This number is usually denoted $g(a_1, \ldots, a_k)$.

Submonoids of $\mathbb{N}$

- Let $a_1, \ldots, a_k \in \mathbb{N}$. Then $M(a_1, \ldots, a_k) = \{\sum_{i=1}^{k} n_i a_i \mid n_i \in \mathbb{N}\}$ denotes the subuniverse of $\langle \mathbb{N}, +, 0 \rangle$ generated by $\{a_1, \ldots, a_k\}$.
- If $a_1, \ldots, a_n$ are coprime, then $\mathbb{N} \setminus M(a_1, \ldots, a_k)$ is always finite and the greatest number which does not belong to $M(a_1, \ldots, a_k)$ is called the Frobenius number. This number is usually denoted $g(a_1, \ldots, a_k)$.
- For $k = 2$ Sylvester found in 1884 an explicit expression $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$.

Submonoids of $\mathbb{N}$

- Let $a_1, \ldots, a_k \in \mathbb{N}$. Then $M(a_1, \ldots, a_k) = \{\sum_{i=1}^{k} n_i a_i \mid n_i \in \mathbb{N}\}$ denotes the subuniverse of $\langle \mathbb{N}, +, 0 \rangle$ generated by $\{a_1, \ldots, a_k\}$.
- If $a_1, \ldots, a_n$ are coprime, then $\mathbb{N} \setminus M(a_1, \ldots, a_k)$ is always finite and the greatest number which does not belong to $M(a_1, \ldots, a_k)$ is called the Frobenius number. This number is usually denoted $g(a_1, \ldots, a_k)$.
- For $k = 2$ Sylvester found in 1884 an explicit expression $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$.
- If $a_1, \ldots, a_k$ are not coprime and $d = \gcd(a_1, \ldots, a_k)$, then for any $b > g(a_1/d, \ldots, a_k/d)$ we have $bd \in M(a_1, \ldots, a_k)$.

dICRLs arising from $M(a, b)$

- Let $\langle a, b\rangle \in \mathbb{N}^2$. Then $\mathbf{M}(a, b) = (M(a, b), \min, \max, +, \dot{-}, 0)$ is a simple cancellative dICRL, where

$$x \dot{-} y = \min\{z \in M(a, b) \mid z \geq x - y\}\,,$$

i.e., $x \dot{-} y = x - y$ if $x - y \in M(a, b)$ and $x \dot{-} y > x - y$ otherwise.

dICRLs arising from $M(a, b)$

- Let $\langle a, b\rangle \in \mathbb{N}^2$. Then $\mathbf{M}(a, b) = (M(a, b), \min, \max, +, \dot{-}, 0)$ is a simple cancellative dICRL, where

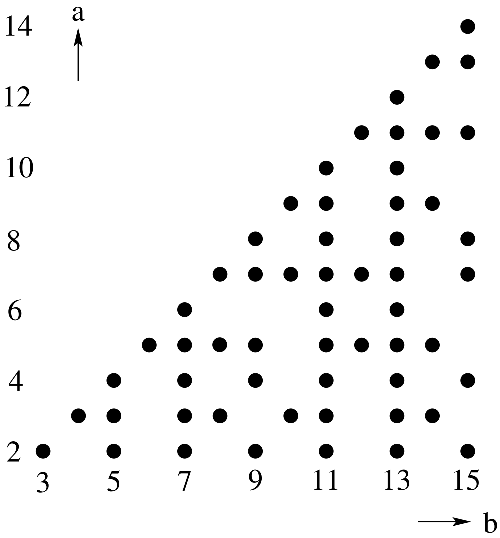
$$x \dot{-} y = \min\{z \in M(a, b) \mid z \geq x - y\}\,,$$

i.e., $x \dot{-} y = x - y$ if $x - y \in M(a, b)$ and $x \dot{-} y > x - y$ otherwise.

- We will consider varieties $\mathcal{V}(\mathbf{M}(a, b))$ for $1 < a < b$, a, b coprime.

$$Gen = \{\langle a, b\rangle \in \mathbb{N}^2 \mid 1 < a < b,\ a, b \text{ coprime}\}\,.$$

Generators – the set *Gen*



Is mapping $\langle a, b\rangle \in Gen \mapsto \mathcal{V}(\mathbf{M}(a, b))$ bijection?

- We want to show that different tuples from *Gen* generate different varieties.

Is mapping $\langle a, b \rangle \in Gen \mapsto \mathcal{V}(\mathbf{M}(a, b))$ bijection?

- We want to show that different tuples from *Gen* generate different varieties.
- This can be done for tuples with different first components by using the following identities for $k \in \mathbb{N}$:

$$kx \vee ky = (ky \dot{-} kx) + kx \,,$$

which we call *k*-divisibility.

Is mapping $\langle a, b\rangle \in Gen \mapsto \mathcal{V}(\mathbf{M}(a, b))$ bijection?

- We want to show that different tuples from *Gen* generate different varieties.
- This can be done for tuples with different first components by using the following identities for $k \in \mathbb{N}$:

$$kx \vee ky = (ky \dot{-} kx) + kx\,,$$

which we call *k*-divisibility.

Lemma

Let $\langle a, b\rangle \in Gen$*. Then* $\mathbf{M}(a, b)$ *satisfies the a-divisibility. In addition,* $\mathbf{M}(a, b)$ *does not satisfy k-divisibility for any* $k \in [1, a - 1]$*.*

Is mapping $\langle a, b\rangle \in Gen \mapsto \mathcal{V}(\mathbf{M}(a, b))$ bijection?

- In order to separate the varieties generated by tuples with the same first component, we introduce the following identities for each $n \in \mathbb{N}$:

$$z \wedge \Big(((y \dot{-} x) + x) \dot{-} (x \vee y)\Big) \leq (x + nz) \dot{-} y\,. \qquad (1n)$$

Is mapping $\langle a, b\rangle \in Gen \mapsto \mathcal{V}(\mathbf{M}(a, b))$ bijection?

- In order to separate the varieties generated by tuples with the same first component, we introduce the following identities for each $n \in \mathbb{N}$:

$$z \wedge \Big(((y \dot{-} x) + x) \dot{-} (x \vee y)\Big) \leq (x + nz) \dot{-} y\,. \qquad (1n)$$

Lemma

Let $\langle a, b\rangle \in Gen$. Then $\mathbf{M}(a, b)$ satisfies the identity $(1n)$ for $n = \lceil g(a, b)/a\rceil$.

Is mapping $\langle a, b\rangle \in Gen \mapsto \mathcal{V}(\mathbf{M}(a, b))$ bijection?

- In order to separate the varieties generated by tuples with the same first component, we introduce the following identities for each $n \in \mathbb{N}$:

$$z \wedge \Big(((y \dot{-} x) + x) \dot{-} (x \vee y)\Big) \leq (x + nz) \dot{-} y\,. \qquad (1n)$$

Lemma

Let $\langle a, b\rangle \in Gen$. Then $\mathbf{M}(a, b)$ satisfies the identity $(1n)$ for $n = \lceil g(a, b)/a\rceil$.

Lemma

Let $\langle a, b\rangle, \langle a, c\rangle \in Gen$ and $b < c$. Then $\mathbf{M}(a, c)$ does not satisfy the identity $(1n)$ for $n = \lceil g(a, b)/a\rceil$.

Is mapping $\langle a, b \rangle \in Gen \mapsto \mathcal{V}(\mathbf{M}(a, b))$ bijection?

Summary

Let $\langle a, b \rangle, \langle c, d \rangle \in Gen$ such that $\langle a, b \rangle \neq \langle c, d \rangle$. Then

Is mapping $\langle a, b\rangle \in Gen \mapsto \mathcal{V}(\mathbf{M}(a, b))$ bijection?

Summary

Let $\langle a, b\rangle, \langle c, d\rangle \in Gen$ such that $\langle a, b\rangle \neq \langle c, d\rangle$. Then

1. $\mathcal{V}(\mathbf{M}(a, b)) \neq \mathcal{V}(\mathbf{M}(c, d))$,

Is mapping $\langle a, b\rangle \in Gen \mapsto \mathcal{V}(\mathbf{M}(a, b))$ bijection?

Summary

Let $\langle a, b\rangle, \langle c, d\rangle \in Gen$ such that $\langle a, b\rangle \neq \langle c, d\rangle$. Then

1. $\mathcal{V}(\mathbf{M}(a, b)) \neq \mathcal{V}(\mathbf{M}(c, d))$,
2. $\mathcal{V}(\mathbf{N}) \subsetneq \mathcal{V}(\mathbf{M}(a, b))$.

Subalgebras of $\mathbf{M}(a, b)$

Lemma

Let $a, b \in \mathbb{N}$. Then each divisible nontrivial subalgebra $\mathbf{B}$ *of* $\mathbf{M}(a, b)$ *is isomorphic to* $\mathbf{N}$.

- Thus we are interested only in subalgebras which are not divisible.

Subalgebras of $\mathbf{M}(a, b)$

Lemma

Let $a, b \in \mathbb{N}$. Then each divisible nontrivial subalgebra $\mathbf{B}$ of $\mathbf{M}(a, b)$ is isomorphic to $\mathbf{N}$.

- Thus we are interested only in subalgebras which are not divisible.

Lemma

Let $\langle a, b \rangle \in Gen$ and $\mathbf{B}$ a subalgebra of $\mathbf{M}(a, b)$. If $\mathbf{B}$ is not divisible then $a \in B$ and $nb \in B$ for some $n \in [1, a-1]$.

- Thus if we want to show that an algebra $\mathbf{M}(a, b)$ does not contain non-divisible subalgebras, it suffices to show that $\{a, nb\}$ for some $n \in [1, a-1]$ generates $\mathbf{M}(a, b)$.

Subalgebras of $\mathbf{M}(a, b)$

- Let $\rho_a(b)$ be the remainder on division of b by a.

Subalgebras of $\mathbf{M}(a, b)$

- Let $\rho_a(b)$ be the remainder on division of b by a.
- We will split our discussion into two cases:

1. The case $\rho_a(b) = 1$ when it is possible to have non-divisible subalgebras.

Subalgebras of $\mathbf{M}(a, b)$

- Let $\rho_a(b)$ be the remainder on division of b by a.
- We will split our discussion into two cases:

1. The case $\rho_a(b) = 1$ when it is possible to have non-divisible subalgebras.
2. The case $\rho_a(b) \neq 1$ when there are only divisible subalgebras.

Case $\rho_a(b) = 1$ (example)

- Consider the algebra $\mathbf{M}(4,5)$ and its submonoid $M(4,10)$.

Case $\rho_a(b) = 1$ (example)

- Consider the algebra $\mathbf{M}(4,5)$ and its submonoid $M(4,10)$.
- Then $\mathbb{N} \setminus M(4,5) = C_1 \cup C_2 \cup C_3$ where

$$C_1 = \{1,2,3\}\,, \quad C_2 = \{6,7\}\,, \quad C_3 = \{11\}\,.$$

Case $\rho_a(b) = 1$ (example)

- Consider the algebra $\mathbf{M}(4,5)$ and its submonoid $M(4,10)$.
- Then $\mathbb{N} \setminus M(4,5) = C_1 \cup C_2 \cup C_3$ where

$$C_1 = \{1,2,3\}\,, \quad C_2 = \{6,7\}\,, \quad C_3 = \{11\}\,.$$

- The sets C_i's are convex subsets of missing elements in $M(4,5)$.

Case $\rho_a(b) = 1$ (example)

- Consider the algebra $\mathbf{M}(4,5)$ and its submonoid $M(4,10)$.
- Then $\mathbb{N} \setminus M(4,5) = C_1 \cup C_2 \cup C_3$ where

$$C_1 = \{1,2,3\}\,, \quad C_2 = \{6,7\}\,, \quad C_3 = \{11\}\,.$$

- The sets C_i's are convex subsets of missing elements in $M(4,5)$.
- Observe that $1 + \max C_i \in M(4,10)$ for each $i \in \{1,2,3\}$.

Case $\rho_a(b) = 1$ (example)

- Consider the algebra $\mathbf{M}(4,5)$ and its submonoid $M(4,10)$.
- Then $\mathbb{N} \setminus M(4,5) = C_1 \cup C_2 \cup C_3$ where

$$C_1 = \{1,2,3\}\,, \quad C_2 = \{6,7\}\,, \quad C_3 = \{11\}\,.$$

- The sets C_i's are convex subsets of missing elements in $M(4,5)$.
- Observe that $1 + \max C_i \in M(4,10)$ for each $i \in \{1,2,3\}$.
- Whenever $y - x \notin M(4,5)$ then
  $y \dot{-} x = \min\{z \in M(4,5) \mid z \geq y - x\} = 1 + \max C_i \in M(4,10)$.

Case $\rho_a(b) = 1$ (example)

- Consider the algebra $\mathbf{M}(4,5)$ and its submonoid $M(4,10)$.
- Then $\mathbb{N} \setminus M(4,5) = C_1 \cup C_2 \cup C_3$ where

$$C_1 = \{1,2,3\}\,, \quad C_2 = \{6,7\}\,, \quad C_3 = \{11\}\,.$$

- The sets C_i's are convex subsets of missing elements in $M(4,5)$.
- Observe that $1 + \max\, C_i \in M(4,10)$ for each $i \in \{1,2,3\}$.
- Whenever $y - x \notin M(4,5)$ then
  $y \dot{-} x = \min\{z \in M(4,5) \mid z \geq y - x\} = 1 + \max\, C_i \in M(4,10)$.
- Thus it is impossible (using $\dot{-}$) to produce an element which does not belong to $M(4,10)$.

Case $\rho_a(b) = 1$ (example)

- Consider the algebra $\mathbf{M}(4,5)$ and its submonoid $M(4,10)$.
- Then $\mathbb{N} \setminus M(4,5) = C_1 \cup C_2 \cup C_3$ where

$$C_1 = \{1,2,3\}\,, \quad C_2 = \{6,7\}\,, \quad C_3 = \{11\}\,.$$

- The sets C_i's are convex subsets of missing elements in $M(4,5)$.
- Observe that $1 + \max\, C_i \in M(4,10)$ for each $i \in \{1,2,3\}$.
- Whenever $y - x \notin M(4,5)$ then
  $y \mathbin{\dot{-}} x = \min\{z \in M(4,5) \mid z \geq y - x\} = 1 + \max\, C_i \in M(4,10)$.
- Thus it is impossible (using $\dot{-}$) to produce an element which does not belong to $M(4,10)$.
- Consequently, $\mathbf{M}(4,10)$ is a subalgebra isomorphic to $\mathbf{M}(2,5)$.

Case $\rho_a(b) = 1$

Lemma

Let $\langle a, b\rangle \in$ *Gen such that* $\rho_a(b) = 1$*. Then for any* $x, y \in M(a, b)$ *such that* $y - x \notin M(a, b)$ *we have* $y \dot{-} x = ka$ *for* $k = \lceil (y - x)/a \rceil$*.*

Case $\rho_a(b) = 1$

Lemma

Let $\langle a, b\rangle \in Gen$ such that $\rho_a(b) = 1$. Then for any $x, y \in M(a, b)$ such that $y - x \notin M(a, b)$ we have $y \mathbin{\dot{-}} x = ka$ for $k = \lceil (y - x)/a \rceil$.

Theorem

Let $\langle a, b\rangle \in Gen$ such that $\rho_a(b) = 1$. For each divisor d of a there is a nontrivial subalgebra of $\mathbf{M}(a, b)$ isomorphic to $\mathbf{M}(a/d, b)$ and each nontrivial subalgebra of $\mathbf{M}(a, b)$ is isomorphic to $\mathbf{M}(a/d, b)$ for a divisor d of a.

Especially, if a is prime then $\mathbf{M}(a, b)$ has only divisible subalgebras.

Case $\rho_a(b) \neq 1$ (example)

- Consider $\mathbf{M}(9, 11)$ and subalgebra $\mathbf{B}$ generated by $\{9, 33\}$, i.e. $M(9, 33) \subseteq B$.

Case $\rho_a(b) \neq 1$ (example)

- Consider $\mathbf{M}(9, 11)$ and subalgebra $\mathbf{B}$ generated by $\{9, 33\}$, i.e. $M(9, 33) \subseteq B$.
- We know that $3k \in M(9, 33)$ for sufficiently large $k \in \mathbb{N}$ since $3 = \gcd(9, 33)$.

Case $\rho_a(b) \neq 1$ (example)

- Consider $\mathbf{M}(9, 11)$ and subalgebra $\mathbf{B}$ generated by $\{9, 33\}$, i.e. $M(9, 33) \subseteq B$.
- We know that $3k \in M(9, 33)$ for sufficiently large $k \in \mathbb{N}$ since $3 = \gcd(9, 33)$.
- Thus we can find $x, y \in M(9, 33)$ such that $x - y = 3k$ for any $k \in \mathbb{N}$, e.g. $k = 7$.

Case $\rho_a(b) \neq 1$ (example)

- Consider $\mathbf{M}(9, 11)$ and subalgebra $\mathbf{B}$ generated by $\{9, 33\}$, i.e. $M(9, 33) \subseteq B$.
- We know that $3k \in M(9, 33)$ for sufficiently large $k \in \mathbb{N}$ since $3 = \gcd(9, 33)$.
- Thus we can find $x, y \in M(9, 33)$ such that $x - y = 3k$ for any $k \in \mathbb{N}$, e.g. $k = 7$.
- Consequently, $21 = x - y < x \dot{-} y = 22 \in B$, i.e. $M(9, 22) \subseteq B$.

Case $\rho_a(b) \neq 1$ (example)

- Consider $\mathbf{M}(9, 11)$ and subalgebra $\mathbf{B}$ generated by $\{9, 33\}$, i.e. $M(9, 33) \subseteq B$.
- We know that $3k \in M(9, 33)$ for sufficiently large $k \in \mathbb{N}$ since $3 = \gcd(9, 33)$.
- Thus we can find $x, y \in M(9, 33)$ such that $x - y = 3k$ for any $k \in \mathbb{N}$, e.g. $k = 7$.
- Consequently, $21 = x - y < x \dot{-} y = 22 \in B$, i.e. $M(9, 22) \subseteq B$.
- We have $1k \in M(9, 22)$ for sufficiently large $k \in \mathbb{N}$ since $1 = \gcd(9, 22)$.

Case $\rho_a(b) \neq 1$ (example)

- Consider $\mathbf{M}(9,11)$ and subalgebra $\mathbf{B}$ generated by $\{9,33\}$, i.e. $M(9,33) \subseteq B$.
- We know that $3k \in M(9,33)$ for sufficiently large $k \in \mathbb{N}$ since $3 = \gcd(9,33)$.
- Thus we can find $x, y \in M(9,33)$ such that $x - y = 3k$ for any $k \in \mathbb{N}$, e.g. $k = 7$.
- Consequently, $21 = x - y < x \dot{-} y = 22 \in B$, i.e. $M(9,22) \subseteq B$.
- We have $1k \in M(9,22)$ for sufficiently large $k \in \mathbb{N}$ since $1 = \gcd(9,22)$.
- Thus we can find $x, y \in M(9,22)$ such that $x - y = k$ for any $k \in \mathbb{N}$, e.g. $k = 10$.

Case $\rho_a(b) \neq 1$ (example)

- Consider $\mathbf{M}(9,11)$ and subalgebra $\mathbf{B}$ generated by $\{9,33\}$, i.e. $M(9,33) \subseteq B$.
- We know that $3k \in M(9,33)$ for sufficiently large $k \in \mathbb{N}$ since $3 = \gcd(9,33)$.
- Thus we can find $x, y \in M(9,33)$ such that $x - y = 3k$ for any $k \in \mathbb{N}$, e.g. $k = 7$.
- Consequently, $21 = x - y < x \dot{-} y = 22 \in B$, i.e. $M(9,22) \subseteq B$.
- We have $1k \in M(9,22)$ for sufficiently large $k \in \mathbb{N}$ since $1 = \gcd(9,22)$.
- Thus we can find $x, y \in M(9,22)$ such that $x - y = k$ for any $k \in \mathbb{N}$, e.g. $k = 10$.
- Consequently, $10 = x - y < x \dot{-} y = 11 \in B$, i.e. $B = M(9,11)$.

Case $\rho_a(b) \neq 1$

Lemma

Let $\langle a, b \rangle \in$ *Gen,* $\rho_a(b) \neq 1$*, and* $k \in [1, a-1]$*. Then* $\{a, kb\}$ *generates* $\mathbf{M}(a, b)$*.*

Case $\rho_a(b) \neq 1$

Lemma

Let $\langle a, b\rangle \in$ *Gen,* $\rho_a(b) \neq 1$*, and* $k \in [1, a-1]$*. Then* $\{a, kb\}$ *generates* $\mathbf{M}(a, b)$*.*

Theorem

Let $\langle a, b\rangle \in$ *Gen. If a is prime or* $\rho_a(b) \neq 1$ *then each nontrivial proper subalgebra of* $\mathbf{M}(a, b)$ *is isomorphic to* $\mathbf{N}$*.*

Subalgebras of an ultrapower

Lemma

Let $\mathbf{B}$ *be a subdirectly irreducible algebra in* $\mathcal{V}(\mathbf{M}(a,b))$ *for* $\langle a,b\rangle \in Gen$. *Then* $\mathbf{B} \in \mathcal{V}(\mathbf{N})$ *or* $\mathbf{B} \in \mathsf{ISP_U}(\mathbf{M}(a,b))$.

Subalgebras of an ultrapower

Lemma

Let $\mathbf{B}$ *be a subdirectly irreducible algebra in* $\mathcal{V}(\mathbf{M}(a,b))$ *for* $\langle a,b\rangle \in$ *Gen. Then* $\mathbf{B} \in \mathcal{V}(\mathbf{N})$ *or* $\mathbf{B} \in \mathrm{ISP_U}(\mathbf{M}(a,b))$.

Lemma

Let $\langle a,b\rangle \in$ *Gen and let* $\mathbf{B}$ *be a subalgebra of an ultrapower* $\mathbf{M}(a,b)^I/U$. *If* $\mathbf{B} \notin \mathcal{V}(\mathbf{N})$ *(i.e., not divisible) then* $a, nb \in B$ *for some* $n \in [1, a-1]$.

Main results

Theorem

Let $\langle a, b\rangle \in Gen$. *Then*

$$\mathcal{V}(\mathbf{M}(a,b)) \text{ is a cover of } \mathcal{V}(\mathbf{N}) \text{ iff } a \text{ is prime or } \rho_a(b) \neq 1.$$

Main results

Theorem

Let $\langle a, b\rangle \in Gen$. *Then*

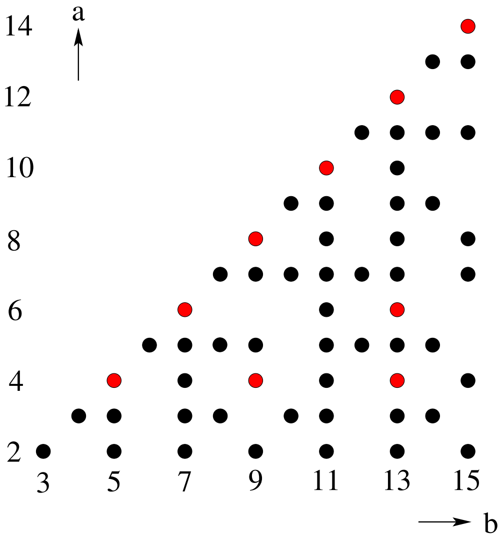
$$\mathcal{V}(\mathbf{M}(a,b)) \text{ is a cover of } \mathcal{V}(\mathbf{N}) \text{ iff } a \text{ is prime or } \rho_a(b) \neq 1.$$

Theorem

Let $\langle a, b\rangle, \langle c, d\rangle \in Gen$ *such that* $\rho_a(b) = \rho_c(d) = 1$. *Then*

$$\mathcal{V}(\mathbf{M}(c,d)) \subseteq \mathcal{V}(\mathbf{M}(a,b)) \text{ iff } c|a \text{ and } d = b.$$

Covers



Order for $b = 13$