A Weaker Form of Wajsberg's Axiom

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- He showed that $\mathcal{V}$ is not generated by Archimedean members as a quasivariety.
- Namely, the quasi-identity

$$(Q) \qquad (p \to q) \to q = \mathbf{1} \ \Rightarrow p \vee q = \mathbf{1}$$

holds in any Archimedean member but it does not hold in $\mathcal{V}$.

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$$(A_1) \qquad (p \to q) \to q \leq (q \to p) \to p$$

- (Q) is strictly weaker than (A_1). Namely, we will show that it is equivalent to

$$(A_1^2) \qquad ((p \to q) \to q)^2 \leq (q \to p) \to p$$

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Algebras of interest

Definition

An integral commutative residuated lattice (ICRL) is an algebra $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, \mathbf{1})$ where the following conditions are satisfied:

- $(A, \cdot, \mathbf{1})$ is a commutative monoid,
- $(A, \wedge, \vee)$ is a lattice,
- $\mathbf{1}$ is a top element,
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- $xy \leq z$ iff $x \leq y \to z$.

- A totally ordered ICRL is called an ICRC.
- Variety generated by ICRCs is denoted $\mathcal{ICRL}^{\mathcal{C}}$.

Wajsberg hoops

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Theorem

Let $\mathbf{L} \in \mathcal{ICRL}^{\mathcal{C}}$. Then $\mathbf{L}$ is a Wajsberg hoop iff it satisfies (A_1).

Convex subalgebras and congruences

Theorem (Hart, Rafter, Tsinakis)

Let $\mathbf{L}$ *be an ICRL. Then its congruence lattice* $\mathrm{Con}\ \mathbf{L}$ *is isomorphic to the lattice of all convex subalgebras of* $\mathbf{L}$*. The isomorphism is established via the assignments* $\theta \mapsto F_\theta$ *and* $F \mapsto \theta_F$*, where*

$$F_\theta = \{a \in L \mid \langle a, \mathbf{1}\rangle \in \theta\},$$

and

$$\theta_F = \{\langle a, b\rangle \in L \times L \mid a \to b \in F \text{ and } b \to a \in F\}.$$

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Let $(\mathcal{C}_{\mathbf{L}}, \leq)$ *be the chain of all Archimedean classes of an ICRC* **L**. *Then* $\mathcal{C}_{\mathbf{L}}$ *is dually-isomorphic to the chain of all principal filters* $\mathcal{P}_{\mathbf{L}}$.

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$$\phi(C) = F(b)\,, \text{ for any } b \in C\,.$$

For the inverse of ϕ *we have* $\phi^{-1}(F(b)) = F(b) \setminus F$ *where* F *is the predecessor of* $F(b)$.

Lexicographic product

Definition

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$$\langle a, b\rangle \cdot \langle c, d\rangle = \langle a \cdot_A c, b \cdot_B d\rangle \,,$$

$$\langle a, b\rangle \to \langle c, d\rangle = \begin{cases} \langle a \to_A c, \mathbf{1}_B\rangle & \text{if } a \cdot (a \to_A c) <_A c \,, \\ \langle a \to_A c, b \to_B d\rangle & \text{otherwise.} \end{cases} \,.$$

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Proposition

Let $\mathbf{A}$ and $\mathbf{B}$ be cancellative ICRCs. Then $\mathbf{A} \overrightarrow{\times} \mathbf{B}$ is a cancellative ICRC.

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Maxima of congruence classes

Lemma

Let $\mathbf{L} \in \mathcal{ICRL}^{\mathcal{C}}$. *Then*

- *If* $p \to q = q$ *then* $q = \max [q]_{F(p)}$.
- *If* F *is a filter and* $q = \max [q]_F$ *then* $p \to q = q$ *for all* $p \in F$.

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- Obvious since $p \to q \geq q$ and $p \to q \in [q]_F$.

□

1st characterization

Theorem

Let $\mathbf{L}$ *be an ICRC. Then* $\mathbf{L}$ *satisfies* (Q) *iff each* $[x]_F$ *different from* $[\mathbf{1}]_F$ *has no maximum for all nontrivial filters* F*.*

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- As $(p \to q) \to q = \mathbf{1}$, we get $p \to q = q$, i.e. $q = \max\,[q]_{F(p)}$.
- $F(p)$ is nontrivial and $q \notin F(p)$ otherwise $q = \mathbf{1}$.

Almost divisible

Proposition

Let $\mathbf{L}$ be an ICRC satisfying the quasi-identity (Q) and $p, q \in L$ such that $p \geq q$. Then $qs \leq p(p \to q)$ for all $s < \mathbf{1}$.

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- There is $n \in \mathbb{N}$ such that $xs^{n+1} \leq p \to q < xs^n$.
- $pxs^{n+1} \leq p(p \to q) < qs$.
- Thus $pxs^n < q$, i.e. $xs^n \leq p \to q$ (a contradiction).

□

2nd characterization

Theorem

An ICRC $\mathbf{L}$ *satisfies the quasi-identity* (Q) *iff* $\mathbf{L}$ *is either a Wajsberg hoop or there is a minimal nontrivial filter* F *(i.e.,* $\mathbf{L}$ *is subdirectly irreducible) and* $\mathbf{L}/F$ *is a Wajsberg hoop such that each* $[x]_F \neq [\mathbf{1}]_F$ *has no maximum.*

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Corollary

Let $\mathbf{L}$ *be an ICRC. If* $\mathbf{L}$ *is Archimedean, then* $\mathbf{L}$ *satisfies* (Q)*.*

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$$F = \{\langle 0, y\rangle \in \mathbf{R}^2 \mid y \in \mathbf{Z}^-\}\,, \quad G = \{\langle x, y\rangle \in \mathbf{R}^2 \mid x < 0\}\,.$$

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- Each $[\langle x, y\rangle]_F$ has no maximum for $x < 0$.

Co-atom

Lemma

Let $\mathbf{L}$ *be an ICRC satisfying (Q) such that* $\mathbf{L}$ *is not a Wajsberg hoop. Then* $\mathbf{L}$ *has a co-atom, i.e., the set* $L \setminus \{\mathbf{1}\}$ *has a maximum.*

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Corollary

Let **L** *be a subdirectly irreducible ICRC and* θ *its monolith. Then* F_θ *is either a Wajsberg hoop or has a co-atom.*

Axiomatization

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Let **L** *be an ICRC. Then (Q) is valid in* **L** *iff* (A^2_1) *is valid in* **L**.

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Let $\mathbf{L}$ *be an ICRC. Then (Q) is valid in* $\mathbf{L}$ *iff* (A_1^2) *is valid in* $\mathbf{L}$.

Lemma

Let $\mathbf{L} \in \mathcal{Q}$. *Then the following quasi-identity is valid in* $\mathbf{L}$*:*

$$((p \to q) \to q) \vee r = \mathbf{1} \;\Rightarrow\; p \vee q \vee r = \mathbf{1}\,.$$

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Using Cintula's result $\mathcal{Q}$ is generated by chains. Thus

Corollary

$\mathcal{Q} = \mathcal{A}_2^1$.

Generalized identity

Inspired by identity for basic hoops expressing the number of components in the ordinal sum

$$(A_n) \quad \bigwedge_{i=1}^{n}((p_{i-1} \to p_i) \to p_i) \leq \bigvee_{i=0}^{n} p_i,$$

we consider analogous identity in our context

$$(A_n^2) \quad \bigwedge_{i=1}^{n}((p_{i-1} \to p_i) \to p_i)^2 \leq \bigvee_{i=0}^{n} p_i.$$

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Given an ICRC **L** and any identity, there is always a maximal convex subalgebra satisfying the identity.

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Theorem

Let **L** *be an ICRC and* $n \geq 2$. *Then* **L** *belongs to* $\mathcal{A}_n^2$ *iff* $\mathbf{L}/F_M$ *belongs to* $\mathcal{A}_1^2$ *where* F_M *is the maximal convex subalgebra belonging to* $\mathcal{A}_{n-1}^2$.

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Let $\mathbf{L}$ be an ICRC and $n \geq 2$. Then $\mathbf{L}$ belongs to $\mathcal{A}_n^2$ iff $\mathbf{L}/F_M$ belongs to $\mathcal{A}_1^2$ where F_M is the maximal convex subalgebra belonging to $\mathcal{A}_{n-1}^2$.

Corollary

Let $\mathbf{L}$ be an ICRC. If $\mathbf{L}$ consists of at most $n+1$ Archimedean classes then (A_n^2) is valid in $\mathbf{L}$.

Lattice of subvarieties

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The chain of varieties $\mathcal{C}an\mathcal{A}_n^2$ is strictly increasing and its limit is $\mathcal{C}an\mathcal{ICRL}^{\mathcal{C}}$.

Proof.

Let $\mathbf{L}_n$ be the lexicographic product of n copies of $\mathbf{Z}^-$. Then (A_{n-1}^2) is not valid in $\mathbf{L}_n$ but (A_n^2) holds. □

Outline

Number of Archimedean classes

Theorem

Let $\mathcal{K}$ be a class of ICRCs satisfying the following conditions:

1. *$\mathcal{K}$ is closed under homomorphic images.*
2. *Let $\mathbf{L} \in \mathcal{K}$. Then (A^2_1) is valid in $\mathbf{L}$ iff $\mathbf{L}$ is Archimedean.*

Then an algebra $\mathbf{L} \in \mathcal{K}$ satisfies (A^2_n) iff $\mathbf{L}$ contains at most $n+1$ Archimedean classes.

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Corollary

The previous theorem is applicable to the class of complete ICRCs and of k-contractive ICRCs ($x^k = x^{k-1}$).

MTL-algebras

Definition

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1. $(A, \cdot, \rightarrow, \wedge, \vee, \mathbf{1})$ is an ICRL,
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- A totally ordered MTL-algebra is called an MTL-chain.
- The zero-free subreducts of MTL-algebras are exactly representable ICRLs.

C_kMTL-algebras, SMTL-algebras, ΠMTL-algebras

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Let $\mathbf{L}$ *be a nontrivial SMTL-chain (ΠMTL-chain). Then* $\mathbf{L} \cong \mathbf{2} \oplus \mathbf{C}$ *for a (cancellative) ICRC.*

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Conversely, let $\mathbf{L}$ *be a (cancellative) ICRC. Then* $\mathbf{2} \oplus \mathbf{L}$ *is a nontrivial SMTL-chain (ΠMTL-chain).*

Lattice of subvarieties of $\mathcal{MTL}$

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The chain of varieties $\mathcal{MTL} + (A_n^2)$ is strictly increasing and its limit is $\mathcal{MTL}$.

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Let $k \geq 2$. The chain of varieties $\mathcal{C}_k\mathcal{MTL} + (A_n^2)$ is strictly increasing and its limit is $\mathcal{C}_k\mathcal{MTL}$.

Translation from $\mathcal{ICRL}^{\mathcal{C}}$ to $\mathcal{SMTL}$

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Lemma

Let $\mathcal{V}$ be a subvariety of $\mathcal{ICRL}^{\mathcal{C}}$ and $\Gamma \cup \Sigma$ a set of its defining identities. Then $\mathbf{2} \oplus \mathcal{V}$ is axiomatized by $\Delta = \Gamma \cup \{\mathbf{0} \leq x, x \wedge \neg x = \mathbf{0}\} \cup \Sigma_0$, where

$$\Sigma_0 = \{\neg v_1 \vee \cdots \vee \neg v_n \vee \varphi(v_1, \ldots, v_n) = \mathbf{1} \mid \varphi(v_1, \ldots, v_n) = \mathbf{1} \in \Sigma\}.$$

Isomorphism

Lemma

Each subvariety of SMTL-algebras is axiomatized by identities of $\mathcal{SMTL}$ and identities of the form $\varphi(v_1, \ldots, v_n) \vee \bigvee_{i \leq n} \neg v_i = \mathbf{1}$, where φ is a $\mathbf{0}$-free term.

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Corollary

$\mathcal{SMTL} = \mathbf{2} \oplus \mathcal{ICRL}^{\mathcal{C}}$ and $\mathcal{PMTL} = \mathbf{2} \oplus \mathcal{C}an\mathcal{ICRL}^{\mathcal{C}}$.

Subvarieties of $\mathcal{SMTL}$ and $\mathcal{PMTL}$

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The chains of subvarieties $\mathbf{2} \oplus \mathcal{A}_n^2$ *and* $\mathbf{2} \oplus \mathcal{C}an\mathcal{A}_n^2$ *are strictly increasing and their limits are respectively* $\mathcal{SMTL}$ *and* $\mathcal{PMTL}$.

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Proposition

The variety $\mathbf{2} \oplus \mathcal{A}_n^2$ is a subvariety of $\mathcal{SMTL}$ defined by identity (A_{n+1}^2) and $\mathbf{2} \oplus \mathcal{C}an\mathcal{A}_n^2$ is a subvariety of $\mathcal{PMTL}$ defined by the same identity.

Final picture